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## Macroeconomic Scenarios and Reality

Steehouwer, H.

2005

### **document version**

Publisher's PDF, also known as Version of record

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### **citation for published version (APA)**

Steehouwer, H. (2005). *Macroeconomic Scenarios and Reality: A Frequency Domain Approach for Analyzing Historical Time Series and Generating Scenarios for the Future*. Vrije Universiteit.

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Macroeconomic scenarios are an important component of Asset and Liability Management (ALM) models as used by financial institutions around the world such as pension funds and insurance companies to support important strategic policy decisions, for example on the optimal strategic asset allocation. The scenarios are used to model the fundamental uncertainty about the future state of the economy that will effect the outcomes of the policy decisions in terms of the objectives and constraints of the various stakeholders. The exact statistical dynamic properties of the scenarios can have an enormous impact on the outcomes of ALM models and thereby also on the strategic policy decisions that are based on these outcomes. The objective of this book is therefore to contribute to a higher quality of economic scenarios and thereby to a higher quality of the strategic decision making that is based on these scenarios.

The book is divided into four parts. Part I serves as an introduction. Part II provides the necessary technical background. Part III presents almost ninety stylized facts about the empirical behavior of important macroeconomic variables in the Netherlands, the United States and the United Kingdom which are consistently obtained by means of a specific, clear and extensively tested frequency domain methodology. The results of Part III can serve as a complete and consistent reference for all those interested in the empirical behavior of macroeconomic variables. Finally, Part IV tests existing Vector AutoRegressive (VAR) scenario models against the benchmark empirical knowledge from Part III and presents a flexible and transparent frequency domain VAR scenario framework that resolves many of the shortcomings of the existing models. The book is complete and consistent in terms of data, techniques and notation used. It is completely self contained and can be studied without having to resort to much additional literature and can even be used as a (partial) textbook in the fields of autoregressive models, spectral analysis and filtering techniques.

**Hens Steehouwer** (1972) studied econometrics at the Erasmus University of Rotterdam from which he graduated in 1996. Since that time he has been working as a consultant in the field of ALM and scenario analysis at ORTEC. For the last three years he has been heading a separate unit that focuses on ALM for insurance companies. Parallel to his work at ORTEC, he worked on the research described in this book for which he received a Ph.D. degree in economics at the Free University of Amsterdam in 2005. The author gratefully acknowledges the support he received for these purposes from the Bank Nederlandse Gemeenten (BNG) and ORTEC.



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**ORTEC**

ISBN 90-8559-091-4

**HENS  
STEEHOUWER**

**MACROECONOMIC SCENARIOS AND REALITY**  
A FREQUENCY DOMAIN APPROACH FOR ANALYZING HISTORICAL TIME SERIES  
AND GENERATING SCENARIOS FOR THE FUTURE

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**HENS STEEHOUWER**

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A Frequency Domain Approach for Analyzing Historical Time  
Series and Generating Scenarios for the Future

ISBN 90-8559-091-4

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This research was supported by

1. nv Bank Nederlandse Gemeenten, Den Haag, Netherlands
2. ORTEC bv, Gouda, Netherlands

Cover photo: Light spectrum by Alfred Pasieka / Photo Researchers, Inc.

Printed and bound by Optima Grafische Communicatie, Rotterdam, Netherlands

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# Preface and acknowledgements

This book contains an exact copy of my Ph.D. thesis which I worked on from 1997 until 2005. From the very start, my objective has been to work on topics that were not only suitable for a scientific Ph.D. thesis but also had a large amount of direct practical relevance. I was able to achieve such a fruitful combination by working on my research while at the same time working as a consultant in the field of Asset and Liability Management (ALM) and scenario analysis at ORTEC. The objectives of the research bear direct relevance for the important work being done in these fields. Another objective has been, given some assumed level of basic knowledge, to be as complete and consistent as possible in terms of data, techniques and notation used. This has also been done because the central technique that is applied here, spectral analysis, contrary to the natural sciences, is not as well known in the economic sciences as for example the classical time series models. The result is a book that is completely self contained and can be studied without having to resort to much additional literature and even as a (partial) textbook in the fields of autoregressive models, spectral analysis and filtering techniques. Note that this is a substantial difference with a conventional Ph.D. thesis which (understandably) focuses only on the specific new scientific aspects without explaining all the underlying ideas and existing techniques. The setup is as modular as possible to offer readers with different levels of knowledge and different interests an easy access to the for them relevant parts of the research. Part I serves as an introduction. Part II provides the necessary technical background. Part III presents almost ninety stylized facts about the empirical behavior of important macroeconomic variables in the Netherlands, the United States and the United Kingdom which are consistently obtained by means of a specific, clear and extensively tested frequency domain methodology. The results of Part III together with the corresponding appendices D and E can serve as a complete and consistent reference for all those interested in the empirical behavior of macroeconomic variables. Finally, Part IV tests existing Vector AutoRegressive (VAR) scenario models against the benchmark empirical knowledge from Part III and presents a flexible and transparent frequency domain VAR scenario framework that resolves many of the shortcomings of the existing models.

Once you have worked on a project for eight years there are many people to thank. First of all I need to thank Guus Boender, a friend and my coach from the very beginning more than eight years ago through to the actual completion of this book. Guus, thank you very much for everything you have done in terms of stimulating me and creating the possibilities for me to perform this research. I also want to thank my second Ph.D. supervisor Frank den Butter for his fundamental comments on the preliminary versions of this book. He made me think harder about certain issues and thereby certainly increased the quality of the final result. Next I want to thank Geurt Thomas and at a later stage John Reichardt of the BNG for their confidence in me and arranging the support from the BNG. Of course I also thank the companies BNG and ORTEC for providing the actual financial support for so many years. A research project like this cannot be completed in evenings and weekends alone, so you need the facilities to work on it during days of the week. Special thanks go to Roland de Bruijn who, during his time as an assistant at ORTEC, helped me to perform the empirical research for the United States and the United Kingdom and also contributed to the research on the Eigen Value Restricted

VAR models. Roland, thank you, your help probably saved me about a year. Next I want to thank my colleagues Lucas Vermeulen and Henk Hoek for reading and commenting on large sections of preliminary versions of this book. Finally, I want to thank all those people and institutions that took the time over the years to talk with me about my ideas and results at various stages and provide me with feedback and interesting references: André Lucas, Pieter Klaassen, Marius Ooms, Christiaan Heij, the members of the reading committee of my Ph.D. thesis, various people at DNB, ABP and CPB and everyone I have failed to mention explicitly. I welcome all comments on the book from readers. My e-mail address is [hsteehouwer@ortec.nl](mailto:hsteehouwer@ortec.nl).

Hens Steehouwer  
August 2005

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## Reader's Guide

The research presented here resulted in an extensive book of almost eight hundred pages. Partially, because a lot of results have been obtained. Partially, also the result of attempting, given some assumed level of basic knowledge, to be as complete and consistent as possible in terms of data, techniques and notation used. Moreover this has been done because the central technique that is applied here, spectral analysis, contrary to the natural sciences, is not as well known in the economic sciences as for example the classical time series models. The intention is that the results can be studied without having to resort to much additional literature. The setup is as modular as possible to offer readers with different levels of knowledge and different interests an easy access to the for them relevant parts of the research. The twenty one chapters are divided over four parts. Part I serves as an introduction. Part II provides the technical background. Part III and IV contain the work on the two specific research objectives as described in section 1.3. The following table can be used by readers to make a choice about which sections are the most suitable to read, depending on their specific level of expertise and on their specific interests. In the table, the chapters and sections of each of the four parts are attributed to three levels. For Part I, III and IV these levels relate especially to the interests of the reader. For Part II, these levels relate especially to the level of expertise the reader already has. The three levels are:

### 1. *Complete*

The chapters and sections indicated by this first level should be studied if the reader is interested in the full theory and results and if he or she has only a limited background in the related topics. This obvious level contains all chapters of the relevant parts.

### 2. *Intermediate*

The chapters and sections indicated by this second level should be studied if the reader is not interested in the full theory and results and if he or she already has some background in one or more of the related topics. This level contains the full version of all the specific new result as well as all introductions and summaries and the additional information that is considered necessary to get a good understanding of the research.

### 3. *Limited*

The chapters and sections indicated by this third level should be studied if the reader is not interested in the full theory and results and if he or she already has a solid background in one or more of the related topics. This level contains the shortest version of all specific new result as well as all introductions and summaries.

Table of all chapters and sections, attributed to three levels in terms of level of expertise and the specific interests of the reader. The ✓ sign indicates a complete chapter. The numbers indicate specific sections from the table of contents.

Chapter	<i>Part I Introduction</i>			<i>Part II Technical Background</i>			<i>Part III Stylized Facts</i>			<i>Part IV Scenarios and Stylized Facts</i>		
	<i>Complete</i>	<i>Intermediate</i>	<i>Limited</i>	<i>Complete</i>	<i>Intermediate</i>	<i>Limited</i>	<i>Complete</i>	<i>Intermediate</i>	<i>Limited</i>	<i>Complete</i>	<i>Intermediate</i>	<i>Limited</i>
1	✓	✓	1.3									
2	✓	2.2.11 2.3	2.3									
3				✓	3.3 3.4.4 3.6 3.7 3.8	3.3 3.7 3.8						
4				✓	4.5 4.6 4.7 4.8	4.6.2 4.6.3 4.6.4 4.7.1 4.7.7 4.8.1						
5				✓	5.3 5.4 5.5 5.6	5.4 5.5.8						
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10							✓					
11							✓					
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13							✓	13.2.1 13.2.2				
14							✓					
15							✓	15.2.1				
16							✓	16.1.1 16.2.1 16.4				
17							✓	✓	✓			
18										✓	✓	✓
19										✓	19.3	19.3
20										✓	20.1 20.2 20.5	20.1
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## *Part I*

# Introduction

*“The Gross National Product measures everything except that which makes life worthwhile.”*

Robert F. Kennedy (1968)

Part I of this research consists of two chapters. Chapter 1 provides a background on Asset Liability Management (ALM) and Scenario Analysis with respect to the research presented in the following parts. It also gives the two major research objectives. The results on the first objective are presented in Part III, the results on the second objective in Part IV. Chapter 2 positions the research presented here in the more or less standard approach for scientific research as it prevails today. After presenting a general overview of current and past paradigms and techniques used for macroeconomic research, Chapter 2 also motivates the choice for three specific techniques from this list that were used to carry out the research described in Part III and IV. Part II contains the relevant theory and experiments of these three techniques, for each one in a separate chapter.



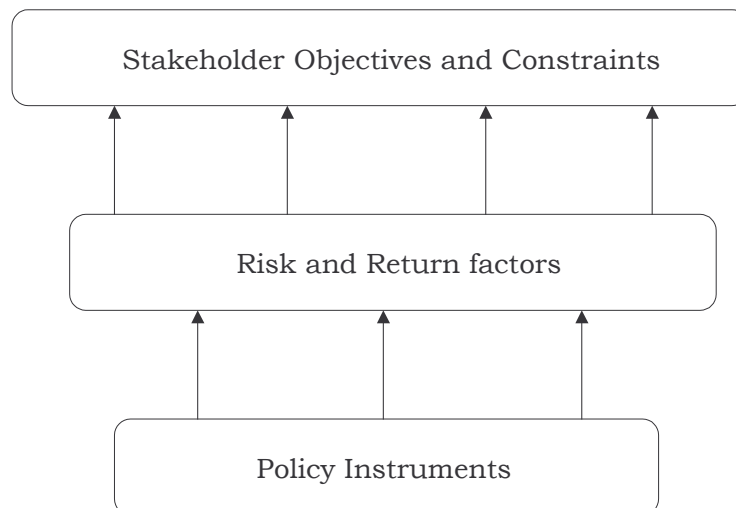
# 1 Background and Relevance

## 1.1 ALM and Scenario Analysis

Nowadays the words Asset and Liability Management (ALM) stand for many types of problems and applications in different fields, both academic and practical. Different types of techniques and models are used to find the solutions and insights to these problems. This first paragraph does not intend to give a full survey of the literature on ALM and the models used. Instead, it describes the general background and relevance of the research presented here. For example Dert (1995), Smink (1995) and Ziemba and Mulvey (1998) can serve as starting references for a more complete overview of ALM and its models.

No matter what type of model or technique is used, at a general level each ALM problem can be formulated in terms of the components shown in Figure 1.1.

Figure 1.1 General ALM problem formulation.



First, a typical ALM problem consists of *objectives and constraints* of one or more *stakeholders* of the economic entity under investigation. For example in the search for optimal life cycle asset allocations in a (Defined Contribution) pension scheme, the only stakeholder is an individual whose objectives could be a high expected pension at retirement but with a sufficiently small risk of falling below some minimum pension level. For a commercial bank or life insurance company trying to determine an optimal duration policy, the stakeholders are the shareholders striving for high and stable returns on the one hand and the regulating authorities, on behalf of the clients and the stability of the financial system, guarding solvency risk on the other. Typically the objectives and constraints will be to some extent conflicting. The sponsor of a (Defined Benefit) pension plan might for example desire a high equity exposure in the asset allocation to lower the contribution rate in the long run. In the absence of a sufficient surplus however the regulating authorities and the beneficiaries might find such an asset allocation unacceptable because of the high level of risk involved.

Second, a typical ALM problem consists of one or more *policy instruments* which the decision maker (for example an individual, a bank's Asset Liability Committee (ALCO) or the board of a pension fund) can use to meet the stakeholders objectives and constraints as best as possible. For the board of a pension fund for example these are the strategic asset allocation, the funding policy and the indexation policy. For a bank's ALCO these include the structure of the balance sheet in general which can be split up in the types and terms of standard assets and liabilities on the one hand and the use of complex derivative instruments on the other.

The third and final component of a typical ALM problem consists of *risk and return factors*. Without the elements risk and return ALM would lose most of its reasons of existence. If the consequences with respect to the stakeholders objectives and constraints of each possible policy would be known beforehand, it would only be a matter of evaluating all possibilities and simply choosing the optimal policy without any risk. Furthermore in a risk free world there would be no compensation for risk (so called *risk premiums*) and there would be no need to decide on the optimal amount of risk to take (not too much and not too little). The determinants of risk and return are all the factors that can cause a policy to turn out good or bad depending on the future development of such a factor. Although these may include specific factors such as the mortality rate of participants in a pension fund or changes in a country's tax regulation for life insurance policy holders, to a (very) large extent these factors consist of general macroeconomic variables such as interest rates, inflations and equity returns.

The concepts of *scenario analysis*, also called *Monte Carlo simulation* or *stochastic simulation*, are often applied in ALM to model the economic risk and return factors. Simply stated, an economic scenario is a possible future evolution of all relevant (uncertain) macroeconomic (and other) variables. Figure 1.2 illustrates how scenario analysis and ALM are interrelated. Instead of one, a large number of scenarios of economic variables is generated, say several hundreds with a horizon of for example fifteen years. Together with the strategic policy under consideration these are fed into a model which states all relations between policy instruments, scenario variables and relevant output measures with respect to the objectives of the stakeholders. Using these relations the model simulates what would happen to the objectives of the stakeholders if the policy under consideration would be applied during the simulation period. The output of the model may for example be the future evolution of the solvency ratio of an insurance company for each of the economic scenarios. It is important to note that the scenarios should be neutral with respect to the objectives and constraints of the various stakeholders. That is, they should not favor one or more of the stakeholders over the others. The scenarios represent one and the same, independent, macroeconomic world in which the economic entity under investigation and its stakeholders need to operate and which they cannot change by themselves<sup>1</sup>. Based on the scenarios, different risk and return measures can be calculated. Examples of such measures are the probability of the solvency ratio falling below the legally required 100% (a risk measure) or the expected return on equity during the next fifteen years (a return measure). In the next step decision makers have to evaluate these risk and return numbers of the policy under consideration and decide whether for example the risk falls within their maximum

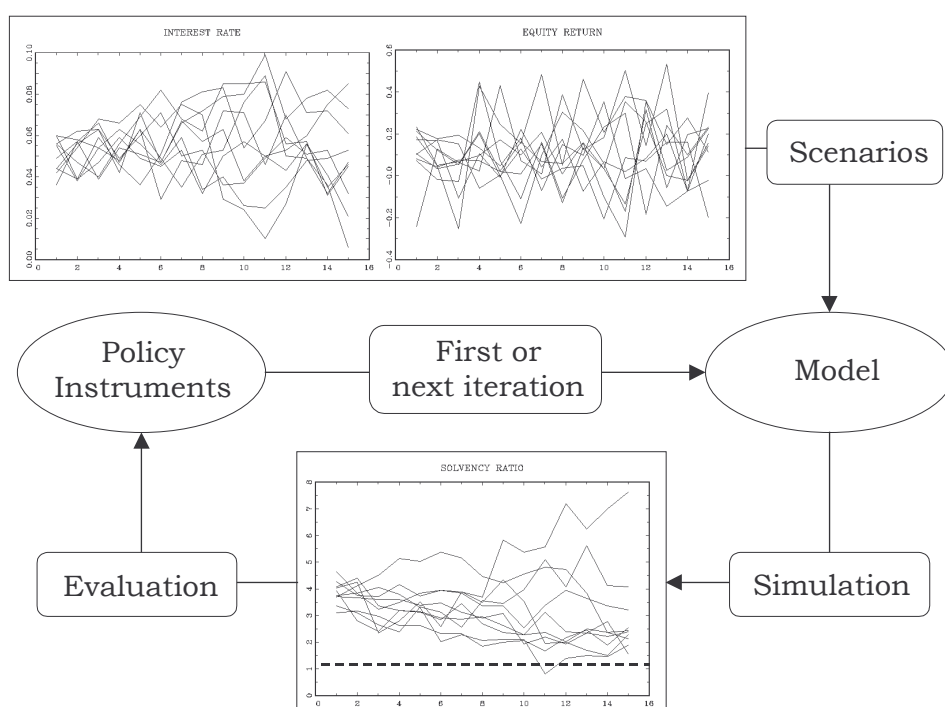
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<sup>1</sup> One exception might be ALM for a local government or central bank which can to some extent change the course of economic life by means of its policy actions.

risk tolerance<sup>2</sup>. If the policy is not satisfactory (for example too much solvency risk), in following iterations alternative policies (for example with a less risky asset allocation) are analyzed (at the cost of a lower expected return). This process goes on until a satisfactory policy is obtained which meets the stakeholders objectives and constraints as best as possible, given the assumptions made with respect to the simulated scenarios.

It is important to note that in practice there is not a one-way flow of information from the models towards the decision makers. The requirements and questions of decision makers that are confronted with the results from the models play an important role in the development and revision of these very models. Ideally, decision makers use the models as a neutral partner for “discussing” the various policy alternatives. See for example Den Butter and Morgan (eds., 2000) and Kooiman (2003).

Figure 1.2 Scenario Analysis and ALM.



Along these lines, ALM and scenario analysis enable decision makers to evaluate and compare the risk and return consequences of different policies. Thereby they can arrive at both more efficient and more effective strategic policies. A policy is called more *efficient* than others if it results in a higher expected return at the same level of risk or, stated otherwise, lower risk at the same level of expected return. A policy is called more *effective* if it makes optimal use of some defined risk budget. Although a policy with risk above this level yields a higher expected return, the associated risk is obviously too much given the risk appetite of the decision makers. A policy with risk below this level on the other hand produces a lower expected return than is strictly possible given the risk appetite of the decision makers. Certainly also other

<sup>2</sup> In general one looks for the policy with the optimal risk and return tradeoff given the risk and return preferences of the stakeholders. Although in most practical ALM applications such a preference function is not made explicit in terms of some appropriate functional form, it is encountered more often in analytical ALM optimization models.

approaches to ALM, for example analytical (optimization) models, can lead to these kind of policy improvements. However, there are three main reasons why scenario analysis is often preferred over alternative approaches.

The first reason is the flexibility it offers to model the often complex interactions and relations within and between the components of an ALM problem as discussed. This point is illustrated by the following quote from one of the earliest ALM publications in which Kingsland (1982) states:

*“The dynamic behavior of a pension plan is clearly dominated by rules and methodology which are discontinuous and non-linear function of its financial condition. The task of developing a closed-form solution to evaluate the potential state of a pension plan following a series of stochastic investment and inflation experiences would be extremely difficult, if not impossible. To date, the only approach that has proven feasible is the application of Monte Carlo Simulation, wherein an investment and inflation scenario is generated by random draws based on the expected probability distribution of year to year investment and inflation behavior. In order to develop an accurate assessment of the range of potential uncertainties, it is necessary to repeat this simulation process by generating dozens or hundreds possible scenarios, consistent with statistical expectations.”*

The second reason for the popularity of scenario analysis is that it offers great possibilities for learning about the problem under consideration besides just obtaining some “optimal” solution. There certainly also exist some very elegant analytical models that are transparent enough to learn about a problem. However in order to achieve the required transparency these models often have to be rather simplified versions of the problem at hand which prevents an integral analysis of the total problem with all its interactions and interdependencies.

The third reason, which applies more to the practical than to the academic applications of ALM, is that the strong visual aspects of scenario analysis cause the models and the solutions obtained to be more easily accepted by decision makers. Decision makers actually look at scenario pictures as those in Figure 1.2 to see how relevant aspects of their organization (solvency ratio) react under different economic circumstances (interest rates and equity returns) if some strategic policy would be pursued. Acceptation of the models by decision makers is crucial for the recommendations coming from these models to be actually implemented instead of remaining some interesting theoretical experiment.

Now we know that scenarios play an important role in ALM and therefore also in the strategic policy making process of institutions such as pension funds, insurance companies and banks, an interesting question becomes to what extent the type of scenarios that are used influences the strategic policy decisions that are made. In the next section an example is given that illustrates that this sensitivity can be enormous.

## 1.2 Scenario sensitivity of strategic policy decisions

From the literature on one-period mean-variance optimization models, as started by the work of Markowitz (1952), it is well known that assumptions on risk and return can have a large impact on the optimal solutions obtained. Small changes in expected values, volatilities and correlations of future asset returns change the risk and return trade off between the various asset classes and thereby their allocations in an optimal investment portfolio. Obviously this also applies to multi-period ALM models, however with two extensions. First, because of the presence of liabilities, besides returns on asset classes, in ALM also other economic variables such as inflation rates or GDP growth are important that determine the growth of the liabilities. For example, pension fund liabilities are to a large extent driven by inflation rates while for a social security institution the liabilities may be highly dependent on the unemployment rate. Second, because of the multi-period aspect of ALM, also the “dynamics” or autocorrelation properties of the risk and return factors are important.

Because of their size and simplicity, one might get the impression that statistical figures such as (auto) correlations are not very important in a total ALM problem formulation. However, exactly the opposite is the case. They can have an enormous impact on the optimal policy outcomes of an ALM model. To support this claim, many examples can be given coming from practical ALM applications for various types of institutions. Here we illustrate the sensitivity of policy decisions for the type of scenarios by means of the following example for a Dutch pension fund. Here we only vary the strategic asset allocation between stocks and bonds while leaving all other policy instruments (pension scheme, indexation policy and contribution policy) unchanged. For each of the asset allocations, we calculated the expected investment return and the solvency risk of the pension fund for the next 25 years based on the two sets of 500 economic scenarios that are described next and which at first sight are not very different. All assets (stocks and bonds) are valued at market value while the valuation of the pension liabilities is still based on traditional actuarial and accounting principles, using a fixed discount rate of 4%.

### *Scenario set 1*

The long term averages (expected values) of the first scenario set are based on expert opinions about the future course of the relevant economic variables: inflation rates, interest rates and stock returns. The standard deviations, correlations and autocorrelations of the scenarios are exactly the same as those of representative *historical* annual Dutch benchmark time series for the 1970-2002 sample period<sup>3</sup>. Table 1.1 summarizes the statistical properties of the first scenario set. The cross correlations are the correlations with a one-year time lag between the variables involved. The averages are calculated on an arithmetic (instead of geometrical) basis. More information on the interpretation of the reported statistical properties can be found in section 3.1.

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<sup>3</sup> To achieve this exact replication of historical statistics, a VAR(1) model was estimated using the Yule-Walker approach. More information on the Yule-Walker estimation technique can be found in section 3.4.4.



Table 1.1 Statistical properties of scenario set 1.

	<i>Avg</i>	<i>Stdev</i>	<i>Correlations (t,t)</i>					<i>Cross correlations (t,t-1)</i>				
Price Inflation	2.25%	2.80%	1.0					0.9	0.9	0.4	0.5	-0.3
Wage Inflation	3.25%	3.90%	0.9	1.0				0.9	0.9	0.3	0.4	-0.3
Short Interest Rate	4.25%	2.90%	0.4	0.3	1.0			0.2	0.3	0.6	0.5	0.0
Long Interest Rate	5.25%	1.60%	0.6	0.4	0.8	1.0		0.5	0.4	0.7	0.8	-0.1
Stocks Returns	9.00%	22.10%	-0.2	-0.2	-0.2	-0.1	1.0	-0.1	-0.1	-0.1	0.1	0.0

*Scenario set 2*

The statistical properties of the second scenario set are virtually identical to those of the first scenario set. The only difference is that stock prices are assumed to follow a so called *random walk* by assuming that all auto and cross correlations in which the stock returns are involved, are zero instead of the (small) historical correlations<sup>4</sup>. Table 1.2 summarizes the statistical properties of the second scenario set in which the differences with the first scenario set are marked gray.

Table 1.2 Statistical properties of scenario set 2.

	<i>Avg</i>	<i>Stdev</i>	<i>Correlations (t,t)</i>					<i>Cross correlations (t,t-1)</i>				
Price Inflation	2.25%	2.80%	1.0					0.9	0.9	0.4	0.5	0.0
Wage Inflation	3.25%	3.90%	0.9	1.0				0.9	0.9	0.3	0.4	0.0
Short Interest Rate	4.25%	2.90%	0.4	0.3	1.0			0.2	0.3	0.6	0.5	0.0
Long Interest Rate	5.25%	1.60%	0.6	0.4	0.8	1.0		0.5	0.4	0.7	0.8	0.0
Stocks Returns	9.00%	22.10%	-0.2	-0.2	-0.2	-0.1	1.0	0.0	0.0	0.0	0.0	0.0

Following the concepts of Figure 1.2, in an ALM model 500 scenarios of the funding ratio were calculated resulting from holding different strategic asset allocations of 0%, 10%, 20%,..., 60% stocks and the remainder in bonds, in each of the two scenario sets. Based on the resulting scenarios of the funding ratio and the investment returns, we then calculated the average investment return and the solvency risk<sup>5</sup> of the pension fund for all combinations of asset allocation and scenario set. The resulting risk and return figures are plotted in Figure 1.3. Each circle represents one of the indicated asset allocations in which the darker part represents the percentage of stocks in the strategic assets allocation. The horizontal axis shows the expected return and the vertical axis shows the solvency risk. The two lines of connected asset allocations correspond to the results calculated on one of the two scenario sets. In both cases we see that as the stock allocation increases, the expected return increases while at the same time solvency risk increases, possibly with some diversification benefits between stocks and bonds at low stock allocations. In order to actually reach a decision, based on such model outcomes, policy makers need to state their maximum risk appetite. For illustrative purposes, suppose that the board of the pension fund finds a solvency risk below 2.0% acceptable. If the first

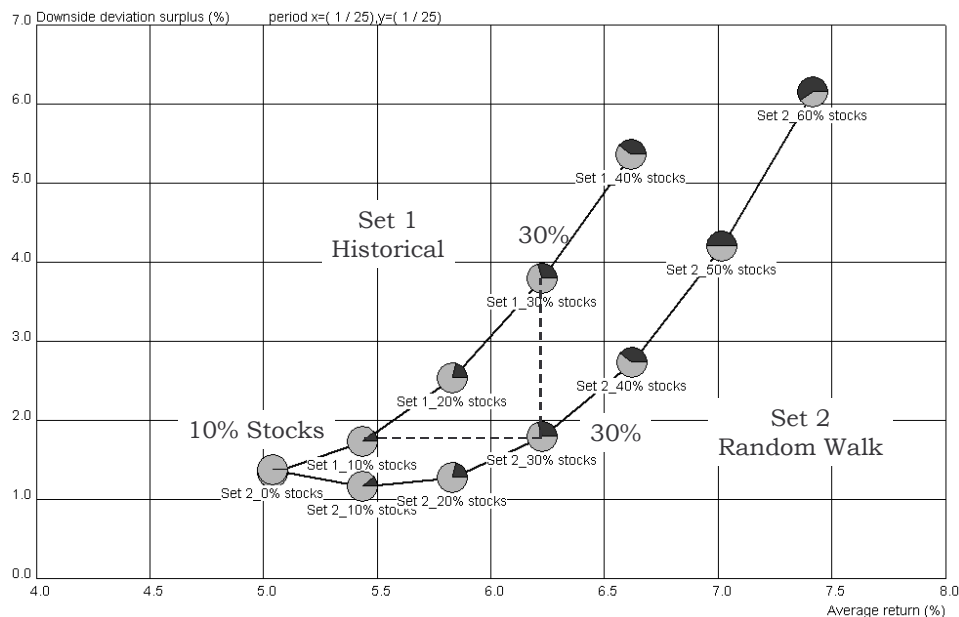
<sup>4</sup> An alternative representation of the random walk hypotheses in a multivariate context in terms of the auto and cross correlations is to assume that none of the economic variables, including stocks themselves, have any predictive power for future stock returns but that current stock returns can have predictive power for the future values of the other economic variables. In that case, only the entire bottom row of the cross correlations matrix would become zero, instead of both the bottom row and last column.

<sup>5</sup> The solvency risk is calculated as the downside deviation of the fund surplus as a percentage of its liabilities. This is the semi standard deviation of all funding ratios falling below the 100% level in the scenarios. This downside deviation gives an adequate description of both the probability of the funding ratio falling below 100% (a situation of underfunding) and the amount of underfunding that can occur.



scenario set would have been used, this maximum risk appetite would lead to an optimal asset allocation of 10% stocks and 90% bonds. After all, this allocation leads to the highest expected return with solvency risk still within the maximum amount allowed. If the second scenario set would have been used, the optimal asset allocation would have been 30% stocks and 70% bonds, a difference of 20%-points (!) stocks in the strategic asset allocation. These rather extreme effects illustrate that differences in the statistical properties of scenarios that might at first sight seem not very important, and might not even be known at all, can have a major impact on policy decisions that are based on the outcomes of models that use these scenarios.

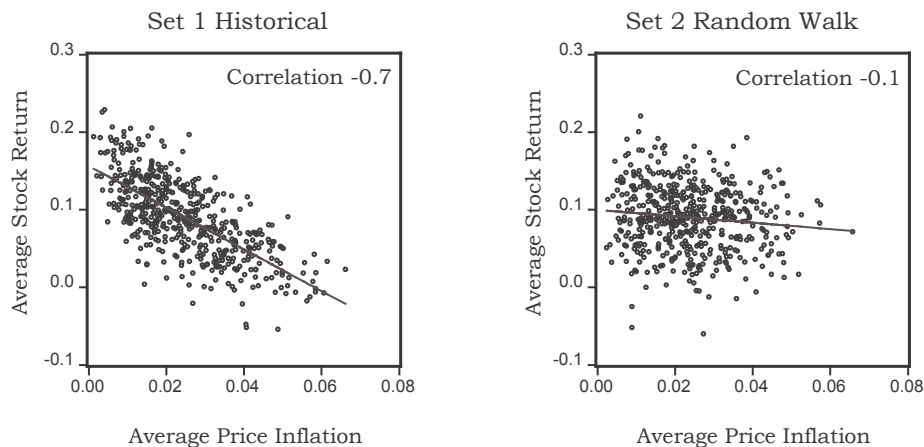
Figure 1.3 Solvency risk and average asset return for different strategic asset allocations based on scenario set 1 and 2.



This example is not meant to say what these apparently very important cross correlation numbers should be in a scenario set. After all, the historical estimates of the correlations are estimates that are uncertain themselves. And even if we would know them for sure for historical data, these correlations do not need to hold for the (scenario) future. Nor does this example intend to provide arguments in a discussion on the pros and cons of ALM models in terms of the sensitivity of the model outcomes. The example “merely” aims to illustrate the enormous effects that (not well known) scenario properties can have on ALM scenario results and how important it therefore is to proceed as clear, careful and precise as possible when constructing a set of economic scenarios. Simply generating an average value with “some volatility” around it, is clearly not enough and can even be very dangerous in terms of policy consequences.

The example would of course not be complete without a more intuitive explanation for the enormous differences between the model outcomes for the two scenario sets. For this purpose, Figure 1.4 shows the correlations between the *average stock return* in a scenario and the *average price inflation* in a scenario calculated on 500 scenarios with a length of 25 years, for both scenario sets<sup>6</sup>. Each picture plots 500 average stock returns against 500 average inflation rates and an estimated linear relation. The resulting correlation can be interpreted as the *long term* correlation between stock returns and price inflations present in the scenarios, as opposed to the *short term* (1 year) correlations shown in the correlation matrices in Tables 1.1 and 1.2 (−0.2). In the first (historical) scenario set, the long term correlation is −0.7. This means that long term stock returns and long term inflation rates are strongly negatively correlated: high average inflation rates often occur together with low average stock returns and vice versa. This evidently has very averse consequences for the funding ratio and hence the solvency risk of the pension fund. After all, its liabilities depend on the inflation rate while its assets (partially) depend on the stock returns. In the second (random walk) scenario set, the long term correlation is only −0.1 which causes a much lower solvency risk for the pension fund. It is important to note that the short term correlation is the same (−0.2) in both scenario sets. Hence, if we would not have looked beyond the conventional averages, (contemporaneous) correlations and volatilities, we would not have noticed the important differences in the long term properties of the scenarios at all and would also not have been able to explain the enormous policy consequences. Finally, note that section 16.4.3 presents an analysis of both the short and long term historical inflation hedging capacities of stocks.

Figure 1.4 Long term correlation between stock returns and price inflation rates in scenario set 1 and 2.



This example shows that economic scenarios play a crucial role in the world of ALM and that their stochastic properties can have an enormous impact on the outcomes of ALM models and thereby also on long term strategic policy decisions that are based on these outcomes. The obvious next step is then of course to discuss how these scenarios are generated. This is the topic of the next section which leads to the formulation of the two basic objectives of the research presented here.

<sup>6</sup> The averages are simple arithmetic averages. Using geometrical averages leads to virtually identical results.

### 1.3 Research objectives

The general objective of the research presented here is to contribute to a higher quality of economic scenarios as used in both academic and practical applications of ALM and thereby also to a higher quality of strategic policy making based on ALM models that use these economic scenarios. To arrive at a more specific formulation of the objectives let us start with the following two definitions from the literature on scenario analysis.

*“A scenario is a possible evolution of the future  
consistent with a clear set of assumptions.”*

Bunn and Salo (1993)

*“A scenario is a description of a possible future state of an  
organization’s environment considering possible developments of  
relevant interdependent factors in the environment.”*

Brauers and Weber (1988)

Because scenarios relate to the future, by definition assumptions need to be made about what these scenarios should look like. The first definition stresses that, above all, these assumptions need to be clear. If the assumptions are unclear in some way it is difficult, possibly even useless, to interpret the outcomes of ALM models that use these scenarios. When generating economic scenarios most would agree that the scenarios should give in some way “reasonable” descriptions of what the economic future could look like. The fact that the second definition stresses the interactions between the different variables in a scenario is basically also a requirement of this “reasonableness”. No matter which models are used to generate scenarios, whether they are theory based or more data driven, most would agree that the scenario properties should at least to some extent reflect what we know about the behavior of economic variables in the past. After all, economic history is basically our only source of information to decide on what “reasonable” scenarios should look like. Therefore one approach in ALM is to assume that the scenarios should reflect the properties, particularly the volatilities and dynamics and to a lesser extent also the expected values of the variables, of some chosen set of historic “benchmark” time series during some, also chosen, historic sample period. For example one may want to simulate scenarios for inflation and equity returns with statistical properties that are the same as observed for a Consumer Price Index and the MSCI World index during the period 1970 until present. This way, decision makers can view the outcomes of ALM models from the perspective of the chosen historic sample period without having to know the technical details of how the scenarios were generated.

Although at an abstract level this may seem a rather clear assumption, at a deeper and more detailed level it is much less clear what it actually means if one tries to extrapolate the (time series) past stochastically into the (scenario) future. The first and most important objective of the research presented here is therefore to “answer” the following question.

### Question 1

Which to scenario analysis and ALM relevant stochastic structures and relations have characterized the macroeconomic process of developed countries in the past, with respect to the real and financial sectors of the economy and their interaction, the properties at different frequencies (ranging from the very long run to shorter term business cycles) and possible changes of these structures over time?

Besides the obvious relevance of this purely empirical question to ALM and scenario analysis, it is also relevant for the gathering of *stylized facts* about macroeconomic fundamentals which can be used to develop and test theoretical models, both in a purely macroeconomic context but also in for example the ALM related context of optimal investment portfolio choices. In general, we can speak of stylized facts if some observed empirical behavior of economic variables turns out to be robust with respect to the country and the historical time period for which it is observed. More on these stylized facts will be said in the next chapter. Compared to research done by others, the distinguishing properties of the research presented here with respect to the first research question are:

#### 1. *Complete and consistent*

The research presents a combined analysis of both real and financial macroeconomic variables for different countries and different time periods and at different frequencies using one and the same set of techniques. Most variables have been studied before for some country, some time period, using some set of techniques for analyzing some aspect of the behavior of the variable. The objective here is however to present a broad and consistent set of stylized facts about all these macroeconomic variables based on one and the same methodology and covering all elements of the behavior of the variables. Chapter 17 summarizes the results in terms of more than eighty stylized facts. In general, only little theoretical explanations are given for the results. After all, the objective is the collection of empirical regularities which should precede the postulation and testing of theories as indicated in Chapter 2. However, room was left for some obvious and intuitive discussion of the results.

#### 2. *Techniques and methodology*

A very specific methodology is developed and used for analyzing the time series of macroeconomic variables. This methodology is described in Chapter 6. On the one hand the techniques constituting this methodology must be as pure and correct as possible in a methodological sense in order to prevent spurious results. On the other hand they also have to be very efficient allowing for a thorough analysis of a great number of time series on several (rolling) sample periods within a reasonable amount of time. The techniques developed and applied here satisfy both these requirements. What also distinguishes them from the techniques applied in other lines of research is that they do not belong to the classes of techniques that are commonly used in conventional econometric research. The three basic techniques applied here are

Vector AutoRegressive (VAR) models, Maximum Entropy (ME) Spectral Analysis and Filtering in the frequency domain.

### 3. *Monte Carlo experiment ME Spectral Analysis*

Section 4.7 describes the results of an extensive frequency domain based Monte Carlo experiment for finding the best combination of estimation procedure and order selection criteria to be used in the ME Spectral Analysis estimation procedure as a part of the methodology from Chapter 6.

### 4. *Statistical significance of ME spectral estimates*

Section 4.8 describes a new way of testing the statistical significance, in the sense of deviating from a white noise process, of spectral densities estimated by ME Spectral Analysis techniques as a part of the methodology from Chapter 6.

### 5. *Zero phase frequency domain filter*

Section 5.4 describes a high quality zero phase frequency domain filter based on the ideas from Bloomfield (1976) and the approach described in Schmidt (1984) which circumvents the problems of filtering finite time series in the frequency domain. The filter facilitates perfect, user specified, pass-bands, induces no phase shifts and does not lead to a loss of data at the start and end of the sample. Here, this filter is refined and extensively tested by means of the Monte Carlo experiments described in section 5.5 in order to find the optimal parameter settings for the current research purposes. The filter is a part of the methodology from Chapter 6.

### 6. *Cycles*

Without being some explicit objective, various economic cycles are prominently present in this research. This ranges from the Long Wave or Kondratieff type of cycles (period length some 50 years) until the Juglar (period length some 10 years) and Kitchin (period length some 4.5 years) business cycles. These types of cycles “simply” emerged from the broad empirical research conducted here and are not necessarily seen as self generating mechanisms.

For the second objective of this research, let us take another look at the scenario definition of Bunn and Salo (1993). Besides that the assumptions about the scenarios have to be *clear* it is of course also necessary that the scenarios generated are indeed *consistent* with these assumptions. This may seem trivial at first sight but this is certainly not always the case. In ALM, often Vector AutoRegressive (VAR) time series models are used to generate scenarios of economic variables. Examples of explicit scenario applications of VAR models are Boender (1997) and Damm (1995). Examples of VAR applications in the ALM related literature on optimal investment portfolio choices are Campbell and Viceira (2001), Kandel and Stambaugh (1987), Campbell (1991, 1996), Hodrick (1992) and Barberis (2000). Although VAR models existed long before that time, since the publication of the path breaking article by Sims (1980) titled “Macroeconomics and reality”, hence the title of this research, VAR models have been extensively used all over macroeconomics. Sims advocated the use of these models instead of the large structural econometric models that had been used until that time because the latter depended too heavily on a priori theory to allow identification and estimation (i.e. to find the “true” model). VAR models are much smaller and simpler linear models that can be seen as the reduced form of



almost any underlying (unknown) structural model. By analyzing the properties of estimated VAR models, more data based evidence on macroeconomic behavior can be obtained and no a priory theory is needed.

The simplicity of VAR models together with the fact that they can describe almost any type of stochastic (economic) process stimulated their use for generating scenarios for ALM. However, this great flexibility of VAR models may at the same time pose problems when trying to estimate them on the in general limited macroeconomic data available. One way to see this is to think of two models which are meant to model the same process and which have to be estimated on the same sample of data. Suppose the first model includes some kind of (valid) restrictions about the true properties of the process while the second model lacks such restrictions. However, both models are able to adequately describe the process under investigation. Especially when the available data sample is short compared to the number of parameters that need to be estimated, it is not hard to imagine that the first model will lead to better estimates of the process than the second model. The analogy is now that VAR models are like the second model containing no specific information whatsoever about the process to be estimated. In small samples this lack of prior model “structure”, either theory driven or based on empirical observations, can lead to poor quality estimates. Besides these short sample problems of estimating VAR models, also choices need to be made about the model order, the estimation procedure and possible transformations of the time series used for estimation. Each of these choices can have a substantial impact on the properties of the stochastic process described by an estimated VAR model. It is therefore by no means the case that estimating a VAR model automatically leads to scenarios which are consistent with the properties of the historic time series used for the estimation of the model. The second objective of this research is therefore to answer the following question.

## Question 2

Do VAR models and, more important, the way they are applied in both academic and practical ALM, indeed lead to scenarios that are consistent with the empirical knowledge obtained from the first research question and, if this is not the case, what modifications can be made to resolve the shortcomings of the current applications?

At this stage, two important remarks are in order with respect to this second research objective. The first remark is that the consistency mentioned here specifically relates to the empirical knowledge obtained with the specific methodology developed for the first research objective. Although this approach is very complete, consistent and as pure and correct as possible in a methodological sense, the empirical knowledge obtained from it is in the end still conditional on the specific methodology and data used to conduct the empirical research. By testing scenario models against this specific empirical knowledge, we implicitly raise its status to being the “true” general empirical behavior of macroeconomic variables. Although we are aware of the still specific nature of the results, we do consider the empirical knowledge obtained here as being the “benchmark” knowledge which current and new scenario models should be able to describe (or adjust) properly.

The second remark with respect to this second research objective is that both the testing of existing scenario models as well as the development of new scenario models is completely confined to the statistical (dynamic) properties of the economic scenarios themselves. By this we mean that for example scenario properties in terms of their consequences for strategic policy decisions are not considered here, despite the fact that, as we know from the example in the previous section, such effects can be enormous. The main reason for this is that any strategic ALM policy decision is in the end *conditional* on the properties of the scenarios that are used. Of course, knowing in itself the impact different scenario properties can have on strategic policy choices is very relevant. This might even give us clues about which scenario properties are the most relevant ones. However, at a fundamental level, policy outcomes cannot give us information on what good quality economic scenarios should look like, simply because the policy outcomes depend on the scenarios and there is no such thing as the “true” policy outcome on which to evaluate or develop a scenario model.

Compared to research done by others, the distinguishing properties of the research presented here with respect to the second research question are:

#### 1. *The testing itself*

VAR applications for generating scenarios have not been tested this way before on being consistent with empirical macroeconomic knowledge or stylized facts. This is by no means an easy test. As Reiter (1995) puts it in the context of more theoretical (explanatory) models “...it is easy to get a model with a good fit, much harder to get one that explains the stylized facts, and extremely difficult to find one that does both. Requiring both from a model is a severe test”<sup>7</sup>.

#### 2. *Spectral properties*

To a large extent, the testing of the VAR model applications is done by comparing the model properties in the frequency domain with the empirical spectral properties of the variables included in the models. This turns out to be a very transparent way of assessing the dynamics of a VAR model.

#### 3. *Consistency*

The models are estimated on the same time series data that was used for gathering the empirical knowledge. The same holds for the spectral and filtering techniques used. This ensures that any defects found are due to modeling issues only.

#### 4. *A frequency domain Vector AutoRegressive scenario framework*

To solve the shortcomings of conventional VAR model applications that were found, a new framework for scenario analysis is developed. Its development is described in section 20.1 and applied in sections 20.2 until 20.5. Next, the most prominent parts of this framework are described.

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<sup>7</sup> A high  $R^2$  is not enough for a dynamic model to correctly replicate historic stylized facts. This is so because the  $R^2$  is calculated by using the one-step-ahead prediction errors within the sample. The predicted value of period  $t$  is calculated by using the observed (not predicted) values up to time  $t-1$ . The path of the predictions is thereby continuously corrected by the actual data. Errors do not accumulate as when using a model to generate multi-period scenarios. Requiring that these scenarios replicate historic dynamics is therefore a much harder requirement than only having a high  $R^2$ .

### 5. *Multifactor VAR (MVAR) models*

A new class of “stacked” VAR models is presented that allow for a separate and therefore flexible, transparent and high quality modeling of the often complex behaviour of its variables in different frequency ranges. Current VAR models are included as a special case.

### 6. *Truncated VAR models*

A new class of VAR models based on truncated Normal distributions is presented that prevents (transformations of) its variables falling below some critical level.

### 7. *Eigenvalue Restricted VAR (EVR VAR) models*

A new class of VAR models is developed for which it is possible to apply restrictions on the eigenvalue and eigenvector parameters of the models that directly determine their (frequency domain) dynamics instead of on the conventional parameters. The theory and estimation of these models is described in section 3.7 while section 20.5 presents an application. The theory also allows for adjusting VAR model dynamics by hand by changing the eigenvalues and eigenvectors of a model.

Finally it is important to note that although the two research questions are derived from and placed in the specific context of ALM scenario models, the results have relevance in many other lines of economic research and applications. For example, the stylized facts from the first research question are very relevant for macroeconomic model building and testing in general while the results from testing VAR models are also relevant for applications in the field of optimal investment portfolio choices.

## 1.4 Outline

Besides the two explicit objectives of answering the two research questions, also the following implicit objectives were pursued with respect to the general presentation of the results: *completeness*, *easy access* and *consistency*. Completeness means that any graduate student with a background in a related (technical) field should be able to read and understand it without having to resort to much additional literature. Given the objective of completeness, also easy access is required. This means that readers of various levels and backgrounds should be able to easily find and focus on the parts most relevant to them while skipping parts that they are already familiar with or not that interested in, without rendering all further readings useless. Finally, consistency means that all the theory and results are presented on a unified basis of definitions and notations. This especially holds with respect to the technical aspects presented in Part II and their applications in Part III and IV.



The research presented here resulted in an extensive book of more than six hundred pages. Partially, this comes from the fact that simply a lot of work has been done. Partially, this is also the result of the general objectives of completeness, easy access and consistency. These general objectives therefore ask for a modular set up in four parts. Please note that the reader's guide presented directly after the table of contents can be used to make a choice about which sections are the most suitable to read, depending on the specific level of expertise and interests of the reader. The four sections are as follows.

*Part I: Introduction* contains, besides this chapter, a second chapter in which the methodology and techniques are discussed at a general level.

*Part II: Theoretical Background* contains three specific chapters on the basic techniques applied. These techniques are Vector Autoregressive (VAR) models, Maximum Entropy (ME) Spectral Analysis and Filtering in the frequency domain. To keep these chapters as easy to read as possible, proofs and derivations are given in the separate Appendix B as much as possible. To the extent that the reader is familiar with these techniques, some parts of these chapters can be skipped while still being able to understand the results of the other parts. Note however that Part II not only contains the relevant theory on the topics mentioned, it also presents the argumentation for the choices that were made with respect to the techniques applied. Furthermore, Part II presents the results of several rather extensive Monte Carlo experiments to test the chosen techniques and setting their parameters. Finally note that not all techniques fall within what may be called "main stream econometrics" as it prevails today. This especially holds for (parametric) spectral analysis and the developed filtering technique in the frequency domain.

*Part III: Stylized facts* contains the results and conclusions on the first research objective about the empirical behavior of macroeconomic variables. This part starts with a separate chapter on the specific methodology which makes heavy use of each of the three basic techniques from Part II. It is important to note that the techniques are used in a combined sense instead of competing with each other. Based on this methodology, the results are presented of analyzing a great number of macroeconomic variables from both the real sector (GDP, employment rates etc.) and the financial sector (price and wage indices, interest rates etc.) based on the longest possible (annual) samples for the United States, the United Kingdom and the Netherlands. The results for each variable are presented in a separate chapter. Part III closes with the main conclusions formulated in terms of almost ninety stylized facts about the empirical behavior of macroeconomic variables.

*Part IV: Scenario Consistency* contains the results and conclusions on the second research objective about the consistency of scenarios with respect to the empirical properties of the macroeconomic variables from Part III. Just as Part III, this part starts with a separate chapter on the specific methodology. Besides the testing of existing models, also a new framework for scenario analysis is developed to resolve some of the shortcomings of the current applications.

Finally, *Chapter 21* contains a separate summary for each of the four parts and directions for future research. The various *Appendices* contain, amongst others, some basic theory about sine and cosine functions and complex numbers (both very relevant for the theory in Part II), the proofs and derivations of results from Part II and the data and sources of the time series used in the empirical research of Part III.

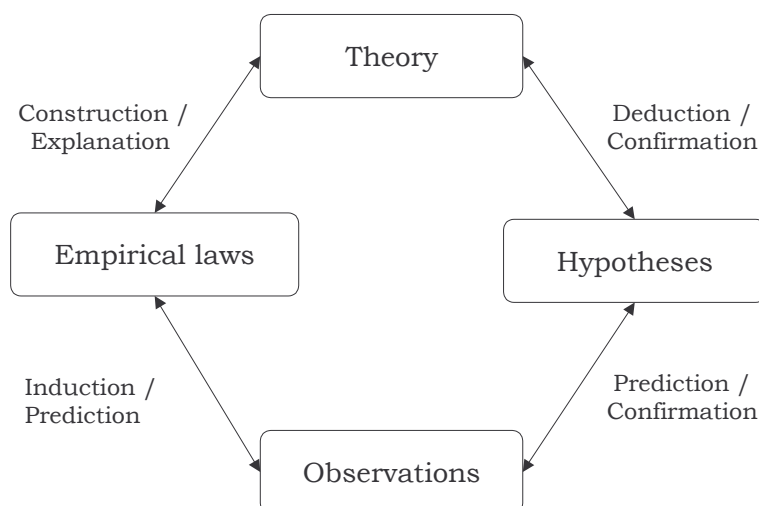
## 2 Methodology

The purpose of this second chapter on methodology is threefold. First, it positions the research in the more or less standard approach for scientific research as it prevails today. Second, a primary requirement for good quality research is to follow as good a methodological approach as possible. For this, it is of course not always necessary to develop entirely new methods and techniques. This is why the second section presents an overview of the most relevant existing paradigms and techniques. The third section motivates the choice for three specific techniques from this overview that are used for the present research purposes. Following the rather general discussion in this chapter, the three chapters of Part II on the theoretical background present the relevant theory of the three chosen techniques in more detail. At the start of both Part III and IV, the general theory of these techniques is combined to arrive at the specific methodology followed for each of the two research objectives as described in Chapter 1.

### 2.1 Positioning the research

Consider the more or less standard methodological cycle shown in Figure 2.1. Such a cycle can for example be found in the Dutch textbooks by Koningsveld (1987) and Bouma (1968).

Figure 2.1 “Standard” cycle of scientific research



The ultimate objective of any science should be to arrive at as complete and valid a set of theories on some object of study as possible. To arrive at such theories, science should be an ongoing process of the steps in Figure 2.1. It should start at the bottom with the collection, description and ordering of observations on the object of study. Moving clockwise, these observations hopefully contain some empirical regularities for which a theory that explains them can be proposed. In order to test a theory's validity, by logical reasoning hypotheses can be deduced from the theory which should subsequently be tested against (other) empirical observations. If the

hypotheses are consistent with reality, this can be seen as a confirmation of the proposed theory. If they are not consistent, the theory has to be modified and tested again. Once more and more hypotheses from a theory are found to be correct when empirically tested, confidence in the theory will grow. Although a theory by definition will never reach the status of “truth”, in practice we can then start using it as an explanation for the empirical laws observed which in turn can be used to predict individual observations.

If one would have to position the research presented here in the standard approach, for the first research objective of analyzing the historical time series to reveal their most important properties, it would certainly be at the phases of collecting observations and, even more, inducing as much empirical laws from them as possible. That is, at the two phases at the bottom and left hand side of the cycle in Figure 2.1. The second research objective of testing whether VAR models, as used in ALM for generating scenarios, adequately describe the empirical regularities found, does not neatly fit in any of the phases of the standard scientific cycle. One could say that while both the research objectives serve a direct practical purpose in the world of ALM and scenario analysis, the first research objective also serves the extra purpose of collecting stylized facts as a benchmark for the draft and testing of macroeconomic theories. Thereby the work on the first research objective is an obvious part of the scientific process while this is not the case for the second objective because it “solely” has a practical relevance.

The next section presents an overview of the most relevant paradigms and techniques available today. For obtaining such an overview one needs to read through several decades of macroeconomic research. While doing this, it is hard not to get the impression that macroeconomic science has been less progressive than for example the natural sciences and that part of the reason for this may be that the standard methodological approach as described here, has not always been applied very strict. In the natural sciences there is a clear frontier to which knowledge (theory) has progressed thus far and which one tries to further expand along the lines of the standard approach. Of course there are huge basic differences between macroeconomics and the natural sciences in terms of the possibilities for setting up experiments and testing hypotheses. Still, the paradigms and techniques used in macroeconomics in the past all together feel too much like a “patchwork” of different ideas and approaches to be consistent with one clear and unified approach to science from which one can “easily” grasp the current frontier of macroeconomic knowledge. The overview of paradigms and techniques in the next section may serve as an illustration of this point.

Striking is that there appears to have been a lot of attention for both theoretical and technical aspects compared to the relatively little attention there has been for first obtaining empirical facts which theories should try to explain and against which they should be tested. At the start of the 1990's, the macroeconomic profession became aware of this potential (dogmatic) problem and initiated a change. Belongia and Garfinkel (1992) contains a number of papers on this subject. In one of these papers, Laidler favors to “...reinstate empirical evidence as a factor more important than a priori principles...” while Blanchard argues that “... progress will require more systematic confrontation with the data and more systematic integration. This task is far from impossible. But it will require a return to more pragmatic, data-

*oriented research...*". In another article from that year, Summers (1991) argues that *"...formal econometric work...virtually always fails. The only empirical research that has contributed to thinking about substantive issues and the development of economics is pragmatic empirical work, based on methodological principles directly opposed to those that have become fashionable in recent years"*. Although some of these statements may seem a little rough or possibly even out of date, they did to some extent inspire the strong empirical orientation of the research presented here.

## 2.2 Overview of paradigms and techniques

As in any other line of research, in macroeconomics it is extremely important to use techniques that are as pure and correct as possible in a methodological sense. This minimizes the probability of finding spurious results which have no real meaning but are the result of technical side effects or methodological errors instead. Since developing or even finding these kind of techniques is by no means an easy task, this section starts off by reviewing the paradigms and techniques that have been used in past research. Without claiming to be complete, the most dominant paradigms and techniques are described while the next section motivates the choice for three specific techniques from this list as being the most suitable for the current research objectives. The review in the next section also serves as a common background to which is referred in later parts. Besides the good review on business cycle research given by Jacobs (1998), further readings can be found in the references stated with each topic.

### 2.2.1 Classical and Keynesian macroeconomics

Macroeconomic theory as found in many standard text books, mostly consists of two components. The first is what is generally called *Classical macroeconomics* in which the clearing of markets based on (inter-temporal) preferences and possibilities play a central role. On the commodity, labor and bond markets, equilibrium values for respectively commodity prices, wages and interest rates are formed by the laws of supply and demand. A crucial assumption is that there are no barriers which prevent these quantities from perfectly adjusting to their equilibrium values. Hereby the classical theory also assumes that money can have no real economic effects.

The second component is called *Keynesian macroeconomics* and started with the famous work of Keynes (1936). Keynes abandons the idea of perfect adjustment of prices and wages to their equilibrium values. Instead these are assumed to be "sticky" which causes markets not always to clear and for example unemployment to occur. Also because of this, the effects of money are no longer neutral and monetary shocks can have real economic effects.

### 2.2.2 Decomposition

At a conceptual level, it has long been recognized that when studying macroeconomics one has to make a clear distinction as to which aspect of macroeconomics one is interested in. It seems likely that we will be dealing with very different "forces" when we are studying for example the long term growth of economies, comprising many decades, compared to the intra day trading effects on a stock exchange. If different forces are at work also different models or approaches

may be needed to adequately analyze and describe the relevant economic behavior. Tinbergen (1946) formalized this idea by proposing a *decomposition* of a time series of an economic variable as

$$\text{Time Series} = \text{Trend} + \text{Cycle} + \text{Seasonal} + \text{Random} \quad (2.2.1)$$

One way or another, this type of decomposition plays a role in most of the paradigms and techniques discussed in this chapter.

A specific decomposition that has drawn much attention is between the trend on the one hand and the other components on the other. At an informal level, trends are often associated with the very slow and long term movements of economic variables while the other components comprise the fluctuations around these trends. The desire to first eliminate trends from economic variables before analyzing or modeling them, to a large extent stems from the prerequisite of a time series to be *stationary*<sup>8</sup>. Simply stated, stationarity stands for the assumption that the properties of a time series under consideration do not change over time, hence a kind of “time invariance”. Note that this does not mean that the variable itself does not change over time but merely that the stochastic properties of the process that is driving the variable remains the same. At a technical level, stationarity is required to be able to estimate one model on a complete sample instead of a different model for each observation in the sample, which would clearly be impossible. At a conceptual level, stationarity also corresponds to a kind of economic stability which is intuitively appealing. It is now clear why so much attention has been paid to the elimination of trends. Because of their slow moving behavior trends typically describe expected values which are not constant during the sample period, thereby clearly violating the stationarity assumption. By removing the trends, only the fluctuations around the trends remain. These no longer suffer from this kind of non-stationarity and thereby become more suitable for analysis<sup>9</sup>.

Since the optimal method for separating the trend from the other components of a time series does not exist, many types of detrending methods are available today. Some relatively old examples are moving average detrending, exponential smoothing and the estimation of linear or polynomial trends. More recent examples are the model based detrending techniques which are applied in Structural Time Series Analysis and the popular Hodrick Prescott (HP) filter which is widely used, for example in the Real Business Cycle literature and for the construction of Business Cycle Indicators. Last but not least, there is the differencing operator which is closely related to the Classical Time Series Analysis following the work of Box and Jenkins (1976). It is well known by now that one has to take caution when using detrending techniques, especially when the “true” model contains a so called stochastic trend. Nelson and Kang (1981) show that detrending a random walk by estimating a linear trend leads to pseudo-cyclical and purely artificial autocorrelation structures for the detrended series. King and Rebelo (1993) and Harvey and Jaeger (1993) show that spurious cycles can also occur when applying the HP filter on integrated processes (i.e. processes containing a stochastic trend). Osborn (1995) shows that the same holds for moving average detrending. At a broader level, Canova (1998) compares a

<sup>8</sup> Another reason could be the presence of non-stochastic trends. Such deterministic components are often unwanted in a time series framework and they are therefore eliminated from the time series under investigation.

<sup>9</sup> Note that this does not prevent other kinds of non-stationarity to occur in the fluctuations around the trends. For example volatilities may still change over time.



large number of detrending techniques and concludes that both the quantitative and qualitative properties of the detrended series vary widely across detrending methods and that the different methods extract different types of information from the data.

### 2.2.3 Cycles

This section draws on the overviews given in Tylecote (1992) and Van Duijn (1999). The evolution of economies all over the world is known to be far from smooth. There are two ways of looking at these economic fluctuations. The first is the often adopted view that in the absence of exogenous shocks, economies will have a stable evolution towards the equilibrium values of prices, wages etc. In this view, recessions can occur only because of incidents or shocks such as wars, political mistakes or natural disasters. In the second view, economic fluctuations are not (only) caused by exogenous shocks, but instead the fluctuations are (also) endogenously generated from within the economy itself. In such self-generating *cycles* each next phase is caused by the course of events that took place during the previous phase. In the literature several types of cycles can be found which are now summerized.

#### *Kitchin, Juglar and Kuznets cycles*

A first type of endogenous cycle is (or was) the so called *Kitchin “inventory” cycle*. This cycle is named after Kitchin (1923) who found a pattern of fluctuation in economic growth rates with a length of three to five years. This cycle seems to be caused by constant over- and undershoots in business inventories compared to the required level. The cycle can be generated by a simple mathematical inventory investment model as shown by Van Duijn (1983).

Long before that time, Juglar (1862) found the *Juglar “investment” cycle* with a length of seven to eleven years. This cycle appears to be driven by the same kind of over- and undershoots as the Kitchin cycle but instead of the inventories, in this case coming from investments in fixed assets such as plants and machinery<sup>10</sup>. The longer period reflects the slower adjustment process involved. Postwar (Dutch) Juglar-recessions occurred around 1958, 1967, 1974, 1982 and 1993.

A third “short term” cycle is the *Kuznets “building” cycle* as found by Kuznets (1930) with a length of fifteen to twenty-five years. While the Kitchin and Juglar cycles have been found for different periods and different countries, the Kuznets cycle has been found mainly in the United States before 1914. Also a self-generating mechanism such as with the Kitchin and Juglar cycles seems unlikely for this cycle. Some exogenous factors such as the migration waves during the nineteenth century from Europe to the United States or even the weather have been suggested as having caused these types of cycles which no longer seem to exist today.

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<sup>10</sup> Note that at the moment of writing, at the end of 2001, both in the United States and Europe severe reductions in economic growth were observed and it even had to be seen if a world recession can be avoided. A possible explanation are the huge investments companies had done in Information and Communication Technology (ICT) during the preceding decade. These have led to large over-investments which seems supported by the fact that the ICT companies were the first to give out warnings for lower future profits and to reduce employment on a large scale.

### *Business cycles*

If one speaks of the *business cycles*, one in general refers to economic fluctuations in the range of the Kitchin and Juglar cycles. During a business cycle, phases of prosperity, recession, depression and recovery appear in an alternating fashion. The term “recession” most of the times refers to a Juglar-recession while the intermediate Kitchin-recessions manifest themselves as a kind of mid-cycle growth reductions which can cause an M-shaped pattern in the business cycle. M-shaped business cycles have been noted for Germany by Helmstädter (1989) and for the Netherlands by Hogenboom and Scholten (1997). Van Duijn (1999) explains such a pattern as a superposition of a Juglar and a Kitchin cycle. More on business cycles follows in the next section on Business Cycle Indicators.

Although the existence of business cycles had by then already been recognized for a long time, during the decades of high growth following World War II, Maddison (1989) and Hobsawm (1994) indicate this period as the “golden age of fast growth”, doubt emerged in the literature as to whether the business cycle had not become “obsolete” as Bronfenbrenner (1969) puts it. Also Burns (1961) talks of a possible stabilization of economic fluctuations. Instead of business cycles, during that time one spoke of *growth cycles* which do not contain recessions with an actual reduction in output but merely periods with lower, though still positive, growth. Mintz (1972) is an example of the literature on this subject. It is not hard to imagine that business cycles and growth cycles are actually the same phenomena only with different levels of long term growth underlying them. As an illustration, during the “slowdown” with lower average growth rates which started around 1975, the usual terminology of the business cycle returned.

### *Kondratieff cycles or long waves*

Although the existence and causes of Kitchin, Juglar and Kuznets cycles are already heavily debated, there is even less agreement among economists on the existence of the *Kondratieff cycle* or the *long wave* with a length of some forty-five to sixty years. Just as business cycles this cycle is believed to describe alternating periods of prosperity, recession, depression and recovery only with much longer durations. The little agreement on the existence of long waves of course to a large extent stems from the length itself. A cycle can hardly be seen to exist unless it repeats itself a considerable number of times. The problem with the long wave is that we have had only time for about four cycles since the start of the Industrial Revolution in the nineteenth century. However, depending on the economic climate (good times or bad times) and driven by different motivations (Marxist versus Capitalist), the Kondratieff cycle has received considerable attention in the past, for some reason especially in the Netherlands. Although Kondratieff (1926, 1928, 1935) is often seen as the “father” of the long wave, it was actually the Dutch Marxist Van Gelderen (1913) who was the first to notice long term fluctuations in commodity prices and growth rates and believed that these were not confined to a specific country or time period but were more or less synchronized present in all capitalist economies around the world. Van Gelderen however got little credit for his work because he only published in Dutch. After these pioneers many others such as De Wolff (1929), Van Duijn (1977, 1983), Mensch (1979), Van Ewijk (1981), Kleinknecht (1984), Morsink (1986) and Solomou (1987) have studied the long wave with different conclusions. The most famous of them is however without a doubt Schumpeter (1939) who, besides proposing a theory for the long wave, also theoretically integrated the different cycles

that had separately been found before and named them after what he thought to be their “discoverers”. From his theory on what causes the long wave and the extensions and modifications made to it by others, the explanation that appeals the most is that of the “basic” innovations. In this view, during a downswing, striving for prosperity and profits, people are stimulated to come up with new important innovations which then form the basis for the next upswing. During the upswing, these basic innovations are exploited and primarily less radical “process” innovations take place. As the benefits from the basic innovations start to diminish and the next downswing sets in, the incentive to come up with new basic innovations arrives and we are full circle. In fact, this can be seen as a kind of *product life cycle* only on a much larger scale. If this theory is true, it should be possible to associate some basic innovation(s) associated with each of the four Kondratieff cycles which allegedly have occurred in the past. This indeed appears to be possible. Table 2.1 gives an approximate dating of the cycles that emerges from the literature. The basic innovations or “technological styles” as given by Tylecote (1992) that may have caused the relevant upswing are stated with each of the cycles. The table also suggests that at the present moment we may find ourselves in the upswing of the fifth Kondratieff cycle induced by the new possibilities offered by the Information and Communication Technology (ICT).

Table 2.1: Approximate dating of Kondratieff cycles and possible basic innovations.

	<i>Trough – Peak – Trough</i>	<i>Basic Innovation</i>
1	1780 – 1810 – 1850	Steam transport
2	1850 – 1875 – 1890	Steel and electricity
3	1890 – 1915 – 1950	Cars and assembly lines
4	1950 – 1973 – 1990	Microelectronics and biotechnology
5	1990 – ? – ?	ICT ?

#### 2.2.4 Business Cycle Indicators

In Mitchell (1913), for the first time economic fluctuations are thought to consist of sequences of cycles which can be divided into the four distinct phases of prosperity, recession, depression and recovery. This work was continued by Mitchell (1927) and the famous work of Burns and Mitchell (1946) who defined business cycles as

*“... a type of fluctuations found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle; this sequence of changes is recurrent but not periodic; in duration business cycles vary from more than one year to ten or twelve years; they are not divisible into shorter cycles of similar character and amplitudes approximating their own.”*

Stimulated by the great depression of the 1930's, at the National Bureau of Economic Research (NBER) in the United States, Burns and Mitchell set up an extensive methodology to collect a broad spectrum of economic data and to construct so called



*Business Cycle Indicators* from these data. These indicators can be divided into leading, coincidental and lagging indicators. The leading indicators can be used to predict what will be the future state of the economy while the coincidental indicators show the current state of the economy. Besides their predicting purposes, Burns and Mitchell also intended the empirical business cycle facts to be used for constructing and testing economic theories. Unfortunately, after Koopmans (1947) criticized the work of Burns and Mitchell as being “measurement without theory”, reporting business cycle facts has not been very popular in macroeconomics for a long time. Koopmans argued that economists should first hypothesize that the series under consideration are generated by some probability model which should then be estimated and tested and that doing otherwise is unscientific. As should be clear from the previous section, I agree with Kydland and Prescott (1990) who state that “... the reporting of facts – without assuming the data are generated by some probability model – is an important scientific activity.” and “... see no reason for economics to be an exception”. The standard cycle of scientific research from Figure 2.1 even prescribes that measurement should take place before any form of theory can be postulated at all. In that sense, one can also criticize strongly theory driven research as being “theory without measurement”. Also see the remarks at the end of section 2.1.

Until World War II, the NBER collected and reported business cycle facts for the United States, the United Kingdom, France and Germany. Because of the presumed absence of business cycles in the decades following World War II and the growing attention for growth cycles instead, it was not until the 1970's that the business cycle again received considerable attention, stimulated by the work of Lucas (1977). Lucas defined business cycles a little different than Burns and Mitchell did. He viewed business cycle regularities as “*co-movements of the deviations from trend in different aggregate time series*” and viewed the business cycle itself as the “*movements about trend in gross national product.*” Until now, this revived attention for business cycles has not diminished and the existence of business cycles now seems universally accepted, although there is still no agreement on what drives them. For most countries both official and commercial institutions construct and administrate leading indicators of the state of the business cycle. Most of the time, gross national product or industrial production is seen as the coincidental indicator. More information on business cycle indicators can be found in for example Oppenländer (1995), Klein (1990), Lahiri and Moore (1991) and Stock and Watson (1993). Furthermore in the Real Business Cycle literature (section 2.2.10) models are tested against empirical business cycle *stylized facts* which partly is a return to the standard approach to scientific research as described in section 2.1. In general, we can speak of stylized facts if some observed empirical behavior of economic variables turns out to be robust with respect to the country and the historical time period for which it is observed. For sure it would be too much to sum up all that is currently known about the behavior of business cycles. However, some of the best known or most discussed business cycle properties are the following. Other examples of stylized facts are given in Romer (2001).

### 1. Volatility

First, it is well known that business cycle fluctuations show up more volatile and pronounced in variables directly linked to the state of business enterprises. For example industrial production shows much clearer cyclical fluctuations than do

private consumption or national product. Given that the over- and undershooting of inventories and investments in fixed assets, which take place in the same business enterprises, are a possible explanation for the Kitchin and Juglar cycles, this observation seems intuitively logical.

## 2. *Term spread*

Second, it has long been known that the slope of the term structure of interest rates, the term spread, contains information on current and future economic growth. Articles and books such as Kessel (1965), Hardouvelis (1988), Fama (1990), Harvey (1991) and Estrella and Hardouvelis (1991) show that during periods of high growth term structures are steep while at the top of a cycle they are flat or even inverted indicating a recession or at least a reduction of growth to be at hand. Strongin and Petsch (1997) and Hardouvelis (1988) state that the predictive power of the term spread in the United States has increased since the Federal Reserve (FED) around 1979 changed their monetary policy into explicitly adjusting short term interest rates to control the rate of inflation. Because of its reliability in predicting economic growth, the term spread is often included in leading business cycle indicators.

## 3. *Stock prices*

Third, there has been a lot of discussion on whether stock prices also have some relation to the business cycle or whether they behave as a non informative random walk. Intuitively it seems logical that stock prices should at least have something to do with what happens in the real economy. After all, just as the level of industrial production they are closely linked to what is at the very hart of all business cycle fluctuations: *“work organized in business enterprises”* as Burns and Mitchell put it. Indeed, evidence for a link between equity returns and the business cycle is found by for example Fama and French (1988, 1989), Poterba and Summers (1988) and Campbell and Shiller (1988a and 1988b). Some of these articles also find negative autocorrelations which seem to get stronger as the return interval is lengthened. Because they find this in equity portfolios of different sizes and of different compositions, they conclude there must be some kind of “common factor” at work, presumably the business cycle. Earlier, Chen, Roll and Ross (1986) had already tried to identify the economic “state variables” that according to asset-pricing theories should be behind the observed co-movements in asset prices. They found equity returns being connected to for example industrial production, interest rates, the term spread and credit spreads. The fact that they found more than one important variable merely illustrates that the business cycle has more to do with the co-movement of many economic variables than with the value of one specific variable. Besides the reported negative autocorrelations on yearly or longer returns, Lo and MacKinlay (1988) find positive autocorrelations for weekly and monthly returns. These findings may very well be consistent with business cycles fluctuations having a length of several years. Lo and Wang (1995) provide more references to recent research on autocorrelations in equity returns. Balvers, Cosimano and McDonald (1990) and Cecchetti, Lam and Mark (1990) show that mean reversion in stock returns is consistent with the hypothesis of market efficiency if the underlying fundamentals or state variables are mean reverting. If the business cycle in national product or industrial production is mean reverting around the underlying trends and all this information is contained in stock prices, stock prices themselves should also be at least to some extent mean reverting.

If equity returns are shown to contain business cycle type fluctuations they are often found to do so with a lead of about six months up to one year over for example national product. Although it is recognized that it may not be a very reliable leading indicator<sup>11</sup>, Fisher and Merton (1984) and Barro (1989) for example identify the stock market as being a good predictor of economic growth. These leading capacities of stock markets are intuitively logical since stock markets are discounting mechanisms of both long and short term expectations about future economic growth, company profits, inflation etc. However because of their unreliable predictive qualities they are not often included in leading business cycle indicators anymore. Because also the slope of the term structure has been shown to be a leading indicator it is logical that Keim and Stambaugh (1986) and Asprem (1989) find a positive relation between equity returns and the slope of the term structure<sup>12</sup>.

#### 4. *Changing business cycle properties*

Fourth and last, it has become clear that business cycle properties need not be constant over time because the structure of economies themselves can change as they develop. A first example of such changes is the alleged reduction of the business cycle volatility in output and employment over the course of the previous century. Zarnowitz (1992), Weir (1986, 1992) and Balke and Gordan (1989) all claim such reductions have occurred in the United States. Romer (1986a, 1986b) warns that such observations may be caused by the poor quality of early estimates of for example gross national product. Based on a multi-county study, Backus and Kehoe (1992) confirm a higher volatility during the interwar than during the postwar period. A comparison of the prewar and postwar period however gave no consistent pattern. Calomiris and Hanes (1994) suggest that the reduced volatility may have been caused by the increased economic importance of the service sector which behaves less cyclical than the manufacturing industry. Furthermore they suggest that the increase in business cycle volatility during the nineteenth century as observed by James (1993) may have been caused by the introduction of the manufacturing industry into a previously largely agricultural economy. Finally Shapiro (1988) questions the reduced volatility of real economic variables by noting that the volatility of financial variables such as equity returns has remained relatively unchanged.

A second example of changing business cycle properties is given by Diebold and Rudebusch (1992), Moore and Zarnowitz (1986), Watson (1994) and Romer (1994) who all claim that after World War II the business cycle has known longer expansion phases and shorter contraction phases while the total length has approximately stayed the same. If one looks at growth cycles however, the length seems to have increased, which is strange if business cycles and growth cycles are indeed the same phenomenon. Stiglitz (1998) goes one step further and states that although before the Great Depression business cycles in terms of a self generating mechanism may have existed, after World War II there has not been such a thing as a business cycle. He suspects that it is improved macroeconomic policy that accounts for much of the change. This rather remarkable claim is based on running logit regressions on US recession dates which show that after World War II the

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<sup>11</sup> Samuelson (1966) stated that "*The stock market has predicted nine out of the last five recessions*" to illustrate that a drop in equity prices is certainly not always followed by a recession.

<sup>12</sup> Note that at the moment of writing, at the end of 2001, the prospects for economic growth were sober while at the same time equities had performed badly and term structures were rather flat or even inverted. The predictive power of stock prices and the term spread with respect to economic growth therefore also seems to hold for this set back in economic growth.

probability of *entering* a recession phase has been independent of the length of the expansion phase while the probability of *leaving* a recession phase positively depended on the length of the recession. Before the Great Depression the opposite appeared to be the case. Note that the situation after World War II very naturally combines with an asymmetrical economic policy which tries to stay in an expansion phase as long as possible and tries to leave a recession phase as soon as possible.

A third and last example of changing business cycle properties is given by Kydland and Prescott (1990), Backus and Kahoe (1992) and Cooley and Ohanian (1991) who all claim that the United States price level was pro-cyclical before World War II but that after World War II the price level has become counter-cyclical. This is consistent with the observation that after World War II prices and wages have become more “sticky” and that they show lagged and less flexible reactions to what happens to other economic variables.

### 2.2.5 Structural Econometric Models

Just as the dating of business cycles by the NBER, the use of large scale *Structural Econometric Models* was also partly initiated by the great depression of the 1930's. The Cowles Commission asked Tinbergen to test several business cycle theories. The approach that he proposed was new in the sense that he translated theories into parameterized econometric models which he subsequently estimated using statistical techniques. The idea was that the estimated models, amongst other things, could be used to support economic and political decision making. These often non-linear models generally were very complex and consisted of possibly more than a hundred equations which obviously led to identification (finding the actual form of the model) and estimation problems. One solution was simply to set some of the parameters which had counterparts in the real world at the values observed in practice. In the light of the limited economic data available for estimation, it should come as no surprise that the quality of the predictions these models generated was not very good. Similar to current ALM scenario analysis practice, these (multi-period) predictions were partly obtained by first performing stochastic simulations and then using the average values from these simulations as the predictive values. The standard deviations around these averages served as an indication of the uncertainty associated with the predictions. For example Biachi, Calzolari and Corsi (1976) performed such simulations only with respect to the error terms while Fair (1980) describes an example of simulations including both the error terms and the uncertainty in the parameter estimates. Although the quality of the predictions was not very good, setting up such extensive models by itself was already a useful exercise to learn more about macroeconomic behavior. Models in this tradition are still being used by for example public institutions who have an advisory function towards central governments<sup>13</sup>.

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<sup>13</sup> For example the “Centraal Plan Bureau (CPB)” in the Netherlands.



### 2.2.6 Classical Time Series Analysis

At the start of the previous century Yule (1927), Slutsky (1927) and Frisch (1933) laid the foundations for what could be described as *Classical Time Series Analysis*. They proposed a new class of purely statistical models which were able to generate the observed cyclical behavior in time series. Basically these models consisted of simple linear difference equations which are constantly moved away from their equilibrium values by random shocks. Slutsky showed how low order stable difference equations are able to generate cycles as the infinite sums of past random disturbances, the so called “Yule-Slutsky effect”. The important underlying theorem was given by Wold (1938). It states that any (second order) stationary stochastic process can be described as a Moving Average (MA) of past random disturbances with the appropriate weights. By inverting these MA models to equivalent though more parsimonious AutoRegressive (AR) models they could easily be estimated on real world data. On these foundations, Box and Jenkins (1976) built their famous work on the modeling of time series processes using the wide class of *AutoRegressive Integrated Moving Average* (ARIMA) models. This led to a widely used methodology on how to identify the most appropriate model within the class of ARIMA models using information from empirical (partial) autocorrelation functions and statistical testing on the order of integration (i.e. testing for so called *unit roots* which are used to identify the number of differencing operations needed in order to obtain a stationary time series). What followed was an enormous amount of literature on the right way to test for unit roots, as started by Dickey and Fuller (1979), and the associated discussions on whether time series contained stochastic trends or not, for example in the influential article of Nelson and Plosser (1982)<sup>14</sup>. Also research was done on finding appropriate *order selection criteria* which could help identify the best model by balancing the fit of the models and the number of parameters that have to be estimated. Well known examples are Akaike (1971) and Schwarz (1978) in which the Akaike Information Criterion (AIC) and Schwarz Criterion (SC) are proposed. Furthermore, in a multivariate setting the discussion on stochastic trends also led to the class of so called *co-integrated models* in which the separate variables follow stochastic trends but at the same time certain linear (co-integrating) combinations of these variables exist that are stationary. Engle and Granger (1987) and Johansen (1988) started the statistical testing for the existence of such stable relations. These co-integrating or *error-correction models* can be shown to be special cases of the popular class of Vector AutoRegressive (VAR) models to which we turn next.

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<sup>14</sup> Note that, based on the Box and Jenkins methodology and statistical testing for unit roots, Nelson and Plosser found a so called ARIMA(0,1,1) model to fit most macroeconomic time series well. Such a model looks like  $x_t = x_{t-1} + \varepsilon_t + \alpha \varepsilon_{t-1}$  where  $\varepsilon_t \sim N(0, \sigma^2)$  is a white noise process and  $\alpha$  is a parameter. In the light of the immense literature on business cycles, it is surprising to say the least that such a model is not able to describe any real cyclical characteristics at all. A simple AR(2) model, which is equivalent to an ARIMA(2,0,0) model, may also be consistent with the observed autocorrelations but is also capable of describing cyclical processes. This at least warns us not solely to look at the outcome of statistical testing but to use other, external, information as well when looking for an appropriate model.

### 2.2.7 Vector AutoRegressive Models

*Vector AutoRegressive* (VAR) *models* are the multivariate versions of the univariate AR models which can even be extended to the class of VARIMA models. As already explained in section 1.1, VAR models were introduced in macroeconomics by Sims (1980) as an alternative to the large Structural Econometric Models that had been used before. Since they can be seen as the reduced form of many unknown structural models they suffer less from the difficult process of identification. However, because of the large number of parameters that have to be estimated in unrestricted VAR models, their great flexibility is at the same time the biggest drawback of these models. For example a simple VAR(1) model for ten variables already requires the estimation of more than a hundred parameters. The adverse effects are for example illustrated by the large confidence intervals reported by Runkle (1987) when VAR models are estimated on the limited economic data available. He concludes that “VAR models may let the data speak for themselves but they are not speaking very loudly.” Den Butter and Jansen (2004) give a recent example of how unrestricted VAR models can lead to implausible model behavior from an economic point of view. These difficulties in estimating VAR models, though less than for the much larger Structural Econometric Models, and the fact that they start from the data instead of starting from theoretical foundations, has always divided economists into those in favor of and those opposed to using VAR models. Clements and Mizon (1991) compromise by stating that Structural Econometric Models and VAR models should be used simultaneously. A parsimonious VAR model can serve as a kind of benchmark for large structural models in a sense that a Structural Econometric Model should at least be able to do what a VAR model can do. They call this “*encompassing the VAR*”.

Different ways have been proposed to reduce the problem of over-parameterization when estimating unrestricted VAR models. The first is to restrict statistically insignificant parameters to zero and delete them from the estimation by various stepwise procedures. Example of such procedures are given in Lütkepohl (1991). The second approach is to use *Bayesian VAR models* in which prior information is added before estimating the model. See for example Litterman (1986). Third, there are the *index models* as suggested by Sargent and Sims (1977), Sims (1981) and Quah and Sargent (1993). These try to describe an  $n$ -dimensional process by an  $r$ -dimensional index where  $r$  is (much) smaller than  $n$ . This approach is motivated by the idea that in some way the “state of the economy” (business cycle?) is responsible for most of the dynamics of economic variables. Fourth, there are the *Reduced Rank VAR* (RR VAR) *models* as described in for example Lütkepohl (1991, section 5.3). These models show resemblance to the index models in the sense that also they reduce the number of linear equations to be estimated in a VAR model by assuming there are a few linear combinations that contain all the relevant information about the variables described by the model. The fifth and last approach to increase the estimation efficiency is to incorporate structural information into a VAR model. This can but need not necessarily be done in a *Structural VAR* (SVAR) *model* in which one applies restrictions to arrive at “behavioral disturbances” that give a meaningful interpretation to the stochastic error components of the models. For example Blanchard and Quah (1989) were the first to identify permanent (supply) and transitory (demand) disturbances in the output process by applying the appropriate restrictions on a VAR model. In general the approach of incorporating

structural information into a VAR model is based on the fact that many theoretical models have an equivalent (restricted) VAR representation. This both holds for models from the literature on Real Business Cycles (section 2.2.10) and on Structural Time Series Analysis (section 2.2.9). Examples of such links between VAR and other models are given in Kydland and Prescott (1982), King, Plosser and Rebelo (1988) and Hendry and Mizon (1990). In general one could say that in this approach guidance from economic theory is used to impose some structure onto a VAR model that may increase the estimation efficiency. Also such economic theory may prevent the models from describing relations that may seem statistically valid but which are inconsistent with economic theory. The formulation “may” is used because a prerequisite is of course that the restrictions applied are indeed *valid* restrictions. Note that the further one goes in applying theory based restrictions, the more the VAR approach loses its data oriented and reduced form character and the more it becomes exposed to the subjectivity of the underlying theories which was the reason of their introduction in macroeconomics by Sims (1980) in the first place.

Applications of VAR models can roughly be divided into two parts. First, there are the macroeconomic applications in which models are used to learn about the dynamics of economic processes from historical data. The estimated models are also used to predict important economic variables such as GDP growth. The second type of applications of VAR models is in the ALM literature and in the related literature on optimal investment portfolio choices. Here VAR models are used to represent the uncertainty in interest rates, inflation rates and returns on various asset classes. In most cases a VAR model is estimated on historical data to capture and model the historic uncertainty in these types of economic variables. Next, the VAR model can either be used in an analytical format or it can be used to generate scenarios for the relevant economic variables. The latter is easy because of the recursive format of VAR models. Examples of scenario applications of VAR models in the context of ALM are given by Dert (1995), Boender (1997) and Damm (1995). Examples of VAR modeling of asset returns are given by Campbell and Viceira (2001), Kandel and Stambaugh (1987), Campbell (1991, 1996), Hodrick (1992) and Barberis (2000).

### 2.2.8 London School of Economics

The approach of the *London School of Economics* as advocated in for example Hendry (1995), states that models should be consistent with (low-level) theory, sample information and previously constructed models. In this view progress can only be made if new models can at least do what previous models could. A “general to specific” approach is followed in which one starts from very general models which subsequently are made more specific by rigorous statistical testing of model assumptions and restrictions. A whole battery of statistical tests for Normality, heteroskedasticity (i.e. non-stable variances in residuals), unit roots, residual autocorrelations, etc. has come from this line of research to find the “true” model within the class of models that is assumed to contain the Data Generating Process.

One may wonder whether by relying so heavily on statistical testing one indeed arrives at the “best” possible model. Economic relations are extremely complex and “*all model economies are abstractions and are by definition false*” as Kydland and Prescott (1996) put it. Therefore models can never be more than a crude approximation of economic reality and cannot be true in a literal sense, thereby rendering the validity of formal statistical testing doubtful. Furthermore, most of the

times in macroeconomics the amount of data will be limited while testing procedures are often based on asymptotic results assuming an infinite amount of data. Also data will often be subject to measurement errors. This is not to say that statistical testing is without value but merely that too much relying on such procedures may cause missing out on some interesting aspects of economics or arriving at models that are not consistent with other economic knowledge. Of course this also holds for other approaches such as for example the Classical Time Series Analysis as discussed in section 2.2.6. Indeed the remark in the footnote from that section illustrates the point made here.

### 2.2.9 Structural Time Series Analysis

Thus far, two extreme approaches for the modeling of economic relations were discussed. One is to build a Structural Econometric Model, guided by economic theory whenever possible. The second is to build a model without any a priori structure at all, by using a data driven VAR model. *Structural Time Series Analysis* as initiated by Harvey (1981, 1989) can be seen as a kind of in between approach. It suggests to build a separate model for each of the components from Tinbergen's (1946) decomposition into trend, cycle, seasonal and random components. By applying such a decomposition and specifying functional forms for the separate components, a structure is imposed onto the model, though not necessarily based on theoretical relations. In these models one often finds *unobserved components* such as "cycles" which are not directly observable but do have a clear economic interpretation. Note that just as the VAR models and the Real Business Cycle models from the next sub-section, also the structural time series models are used nowadays for the identification of different types of shocks with an economic interpretation. Den Butter and Koopman (2001) for example use a structural time series model for the identification of (permanent) technology shocks and (temporary) demand shocks.

These models are estimated by using the technique of the *Kalman filter*. Although the Kalman filter is more complicated than the regular Ordinary Least Squares (OLS) technique, estimation is simplified by the fact that the question whether or not series are driven by stochastic trends and the related testing for unit roots are less of an issue here. This is so because in this approach trend components can be modeled explicitly. Until now, mostly univariate Structural Time Series Models have been applied. Because in macroeconomics the relations between different variables are extremely important, having only univariate structural models will clearly not be enough. Recently however the first multivariate versions of such models have been developed. At the moment the numerical optimizations involved in estimating these models seems to be the limiting factor. For an overview of these types of models see for example Harvey and Koopman (1997). Finally, as noted before, Structural Time Series models often can be shown to be equivalent to special restricted versions of the more general VAR(IMA) models.



### 2.2.10 Real Business Cycles

*Real Business Cycle models* are currently very popular in macroeconomics. Contrary to the Business Cycle Indicators, these models do not intend to predict the business cycle. Instead, they aim to describe the general structure of economic fluctuations starting from the optimization behavior of individuals as described by Classical Macroeconomics while continuously being disturbed by technological shocks. Because of the enhanced computational power it has only recently become possible to study the aggregate business cycle consequences of Classical Macroeconomics. One of the first Real Business Cycle models is given by recent Nobel Prize winners in Kydland and Prescott (1982) while Cooley (1995) and Danthine and Donaldson (1993) give overviews of other literature on this subject.

The approach is best illustrated by a typical Real Business Cycle experiment. First, a theoretical model is constructed which describes the economic behavior of individuals based on equations describing output, income and capital and on a process driving technology, for example an AR(1) model. Random shocks in the technology process and the constant reactions of the individuals towards the equilibrium solutions of the various equations determine the course of the economy as a whole. In the second step the parameters of the model are not estimated but calibrated to be consistent with the real world values for observable variables which are present in the model. In the third step the model is used as a kind of “artificial economy” by simulating scenarios of for example gross national product, mostly on a quarterly basis. In the fourth and final step, properties of historical business cycles are compared to the properties of the simulated business cycles. Often the Hodrick Prescott (HP) filter is used to separate trends and business cycle fluctuations. The business cycle properties are compared with respect to statistics such as standard deviations and correlations but also with respect to auto (cross) correlations to determine the lead and lag relations between the different variables. The Real Business Cycle models have also renewed the interest for collecting *stylized facts* about economic fluctuations. The more the simulation statistics are consistent with these historic stylized facts, the more confidence one has in the theoretical model proposed in the first step. If they are not consistent, the type of the inconsistency can give clues on how to adjust the model and try again. To avoid the dangers of “data snooping” it is of course necessary that the stylized facts indeed reflect robust properties of macroeconomic fluctuations as observed for various countries and time periods instead of being the properties of just a single historical time series. Note that in general this approach matches very well with the standard approach to science as described in section 2.1. Comments heard on the Real Business Cycle approach therefore most of the time do not concern the methodology but rather the specification of the theoretical models used. Calomiris and Hanes (1995) and Hillinger (1992) for example state that the technological shocks are a very unlikely cause of cyclical fluctuations since they are hard to identify in the real world and are not believed to be consistent with economic history.

In the original Real Business Cycle literature, the optimization behavior of the (identical) individual agents is only disturbed by one type of technological shocks. Caballero (1992) however argues that direct application of microeconomic explanations to aggregate data can be seriously misleading, a “fallacy of composition” in the words of Samuelson (1955). Important recent extensions of the Real Business Cycle models are therefore models that allow for shocks both at the aggregate

(macroeconomic) level and at the (microeconomic) agent level. The last type of shocks are also called idiosyncratic (agent specific) shocks. Recent examples of such applications in the labor market are Gautier (1997) and Davis and Haltiwanger (1999) while Cooper et al. (1999) is an application in the field of capital replacements.

Canova (1993) illustrates the possibility to see Real Business Cycle models as special, restricted, versions of the more general VAR models while Watson (1993) proposes different ways to determine a measure of fit for calibrated models such as these. He concludes that the most and the best information on the differences between the dynamics of historical and simulated data comes from comparing so called *spectral densities*. In the next and last sub-section of this review of paradigms and techniques more will be said on the spectral density of stochastic processes and spectral analysis in general.

### 2.2.11 Spectral Analysis

While in main stream macroeconomics Classical Time Series Analysis based on the ARIMA class of models has become very popular, in the natural sciences another technique called *Spectral Analysis* has become widely used. Because Spectral Analysis is not that well known and often applied in economics<sup>15</sup> (in the Netherlands) as for example the Box and Jenkins ARIMA methodology and because Spectral Analysis requires a rather different view on empirical data and stochastic processes, this section elaborates more on the historical background of spectral analysis than was the case for the previous paradigms and techniques. The following is based on the overviews given in Bloomfield (1976), Marple (1987) and Koopmans (1983).

Just as fluctuations are inherent to macroeconomics, also nature itself is known to contain many cyclical or recursive processes. For example the length of a day, the seasonal changes and the phases of the moon enabled mankind long ago to measure time and devise calendars. Another example is Pythagoras who around 600 b.c. noted a relation between the sine wave vibrations of musical notes as produced by a string with a fixed tension on the one hand and the length of the string on the other. The mathematical foundations of spectral analysis were laid in the seventeenth century by Sir Isaac Newton who gave the first mathematical description of the periodicity of the sine wave vibrations as noted by Pythagoras. He also observed that sunlight guided through a glass made prism is dissected into many colors of light, each with its own wavelength. White light as produced by the sun contains all wavelengths and thereby also all colors. He introduced the word *spectrum* as the scientific name for this decomposition of light. It is derived from the Latin word “specter” which means “image” or “ghostly appearance”.

In the eighteenth century the general solution for the mathematical description of the vibrating string was given by both Bernouilli and Euler. At the start of the nineteenth century, Fourier extended their results by showing that any mathematical function  $f(x)$ , even with a limited number of discontinuities, could be written as an infinite sum of sine and cosine waves.

$$f(x) = \sum_{k=1}^{\infty} (A_k \cos(kax) + B_k \sin(kax)) \quad (2.2.2)$$

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<sup>15</sup> Economic courses on time series analysis can for example contain only half a lecture on spectral analysis while most of the other lectures are on the Box and Jenkins ARIMA methodology.

The mathematical exercise of calculating the so called *Fourier series*  $A_k$  and  $B_k$  for a function  $f(x)$ , which may just as well be a sample of empirical observations, was called *harmonic analysis*. Just as a prism dissects white light into many colors of different wavelengths, the Fourier transform decomposes a function or a time series into a set of sine and cosine waves of different wavelengths or frequencies. The frequencies are the reciprocal of the wavelengths and are indicated by  $k\alpha$  in (2.2.2). The most appropriate frequency interval for analysis in discrete time lies between the lowest frequency of zero, with fluctuations of infinite wavelengths (i.e. a constant term), and the highest frequency of  $1/2$  with the shortest wavelength of only two time periods.

During the second half of the nineteenth century, harmonic analysis was extensively applied in the search for cyclical behavior in various natural phenomena such as sound, weather and sunspots. To simplify the search for unknown periodicity in empirical data, Schuster (1898, 1900, 1906) introduced the concept of the *periodogram* as the sum of the squared Fourier coefficients ( $A_k^2 + B_k^2$ ) calculated on an interval of frequencies. The periodogram gives the squared amplitude for each of the sinusoid functions in the Fourier transform of a series, thereby indicating to what extent the relevant frequencies are “present” in the fluctuations of the time series. For this also see (A.1.9) and (A.1.10) in Appendix A. Instead of the conventional way of analyzing a time series in the *time domain* the periodogram thereby enables analyzing it in the *frequency domain*. An important property of the periodogram is that by summing its values across “all” frequencies one obtains the conventional variance of the time series. Just like a probability distribution distributes a “probability mass” of one over the possible outcomes of an experiment, the periodogram therefore distributes the “variance mass” of a time series over an interval of frequencies. Schuster applied periodogram analysis for example to investigate the cycle in the intensity of sunspots which he estimated to have a period length of around eleven years. He was already very aware of many of the drawbacks and peculiarities of the periodogram as we know them today. Depending on the sample being analyzed, he noted that the periodogram randomly indicated periodic behavior at frequencies that were not really present in the data analyzed. He knew that calculating several periodograms on different samples and then averaging them could eliminate these what he called *accidental periodicities*. However the calculation of such a *mean periodogram* was too complex at that time. Schuster was also aware of the side lobes in the periodogram and the *spurious periodicities* occurring around dominant frequencies in the periodogram which are inherent in the Fourier analysis of any finite time series.

The initial enthusiasm for periodogram analysis was tempered on the one hand by the large computational efforts required to perform the Fourier transforms. For example the periodogram analysis of a single wheat price index given by Beveridge (1921, 1922) was considered a major computational achievement. On the other hand, researchers at the start of the twentieth century became aware that, such as Schuster had noted before, applying periodogram analysis on empirical data resulted in wildly fluctuating periodograms, which did not stabilize as the sample size was increased. Later, it was shown that this was because the empirical periodogram is an inconsistent estimator (i.e. with a non-decreasing variance as the sample size increases) of its theoretical counterpart: the *spectral density* or *spectrum* of a

stochastic process. Being the “expected value” of the periodogram, the spectrum very much looks like the mean periodogram as suggested by Schuster. If more people at that time would have been aware of this, possibly the interest for periodogram or spectral analysis would not have decreased as much as it did. However, the disappointment did encourage scientists to develop other techniques for analyzing time series. As a matter of fact Yule (1927) introduced the autoregressive time series models (section 2.2.6) as an alternative to periodogram analysis for analyzing the disturbed periodic behavior in the intensity of sunspots. He estimated these models using the statistical technique of linear regression which was developed in those days. Besides laying the foundations for the Classical Time Series Analysis, by using the estimated models to obtain information on the periodic behavior of sunspots, he also laid the foundations for a technique which would later be called *parametric spectral analysis*: characterizing the periodic behavior of stochastic processes by first estimating a time series model and subsequently analyzing the spectral density of that model.

The important work of Wiener (1930) put spectral analysis on a solid statistical basis and thereby induced a period of renewed interest. He showed that the autocorrelation function of a stationary stochastic process and the spectral density can be calculated from one another by the relatively simple Fourier transformation and therefore basically contain the same information, only presented differently. This important relation is known as the Wiener-Khintchine theorem since it was Khintchine (1934) who introduced stationary processes and developed the related correlation theory. While Wold (1938) showed that any stationary stochastic process can be described as a Moving Average (MA) of past random disturbances with the appropriate weights, Cramér (1942) discovered its spectral counterpart by showing that any stationary process can also be described as an infinite sum of sinusoids of different (fixed) frequencies with a stochastic amplitude and phase.

Once the theoretical basis was there, a great interest in the empirical question on how to estimate spectral densities emerged during the 1940's and 1950's. Daniell (1946) pointed out that a smoothed form of the periodogram is a suitable estimator. This was followed by Barlett (1948) and Kendall (1948). Tukey (1949) can be seen as an important pioneer in the field of empirical spectral analysis. He laid the foundations for the estimation of spectral densities on finite samples. Many of the terminology and techniques from the conventional (non-parametric) spectral analysis such as “aliasing”, “windowing”, “prewhitening”, “tapering” and “smoothing” come from this line of work. Tukey also made large contributions to the development of fast algorithms to calculate Fourier transforms. The most prominent is the *Fast Fourier Transform* (FFT) as described in Cooley and Tukey (1965). Together with advances in computer technology, this algorithm enabled routine Fourier analysis of extensive sets of data. However, these applications of spectral analysis again mostly were in areas of the natural sciences. Some early works on the application of spectral analysis in econometrics are Granger and Hatanak (1964), Fishman (1969) and Koenig and Wolters (1972). The lack of interest for applying spectral analysis in economics probably had two reasons. First, spectral analysis can only be sensibly applied to stationary time series while many economic variables are known to be non-stationary because of their trending behavior. If these trends are not correctly removed from the series this leads to the “typical spectral shape” that Granger (1966) finds for time series of economic variables. In such spectra most of the “spectral



mass” lies so dominantly at the low frequencies that information on other interesting (business cycle) frequencies is hardly visible. Second, samples of economic time series are in general too short for conventional (non-parametric) spectral estimation techniques to be applied successfully. While the first problem can be solved by applying appropriate detrending or filtering techniques, a solution to the second problem can be found by returning to the parametric approach to spectral analysis as first applied by Yule (1927). Although *parametric spectral analysis* as initiated by Burg (1967, 1975) is not well known, Akaike (1969), Barlett (1948) and Parzen (1957a, 1957b) had already suggested using AutoRegressive (AR) models to estimate spectral densities. Based on the theoretical information criteria of *Maximum Entropy*, Burg proved that the “optimal” way to estimate spectral densities is by first estimating an AR model and then calculating the spectral density from this model. This is why Maximum Entropy spectral analysis is strongly related to *AutoRegressive Spectral Estimation* of which Parzen (1983) gives a good overview. Perhaps because the work of Burg had its background in the natural sciences, his approach to spectral analysis has hardly been applied in macroeconomics. An exception is Sebold-Bender (1990) who successfully applied Maximum Entropy Spectral Analysis to gather business cycle stylized facts. Theory on Maximum Entropy and AutoRegressive Spectral Analysis can be found in Koopmans (1974) and Priestley (1981).

Because macroeconomics by definition concerns the analysis of all kinds of variables and the way they interact, *multivariate Spectral Analysis* is of crucial importance. Analogous to the frequency decomposition of the variance of a stochastic process in the univariate case, in multivariate Spectral Analysis also the covariance is decomposed into frequencies. Although the resulting *co-spectra* are difficult to interpret, fortunately they can be split into a *coherence spectrum* and a *phase spectrum* which have a clear and intuitively appealing interpretation. A coherence spectrum gives a separate (absolute value) correlation between the fluctuations at each frequency in two different variables. This correlation is corrected for possible lead and lag relations (phase differences) that exist between the variables at these frequencies. The information on the expected lead and lag relations is separately given for each frequency by the phase spectrum. Multivariate spectra therefore in a sense decompose a single correlation number between two variables into three components. First, the univariate spectral densities show which specific frequencies play a role in the correlation. Second, the coherence spectrum shows what is the, phase corrected, absolute correlation between the phase and amplitude shocks at these frequencies. Third and finally, the phase spectrum shows how the overall correlation number is determined by phase differences between the fluctuations at the relevant frequencies. One might note that all this information about the dynamics of a multivariate process is also contained in its (cross) autocorrelation function in the time domain. This is certainly true. In fact it can be shown that the information in the time and frequency domain is exactly the same. However, the advantage of the spectral analysis approach is that it presents the information in a way that is much more transparent and easier to understand.

With respect to the estimation of multivariate spectral densities, the Maximum Entropy concept extends to the multivariate case showing that the “best” way to estimate multivariate spectra is by first estimating a VAR model and second calculating the multivariate spectra from the estimated model. Reiter (1995) for

example contains a multivariate follow up on the line of research set out in Sebold-Bender (1990).

## 2.3 Three central techniques

As I confronted the research objectives from section 1.3 with the techniques and paradigms available to answer them as best as possible, I arrived at three central techniques which in a way constitute all of the methodology applied for the research presented in Part III and IV. These are

1. Filtering in the frequency domain
2. Maximum Entropy Spectral Analysis
3. Vector AutoRegressive (VAR) time series models

First of all it should be stressed that these three techniques are *not* used *separately* in a competitive sense. Instead, the first chapter of Part III, Chapter 6, describes how these three techniques work *together* in a specific methodology developed for the research presented in that part. Chapter 18 does the same for the research presented in Part IV. Let us now motivate the choice for these three central techniques.

The requirement of appropriate filtering techniques first stems from recognizing the useful decomposition of economic processes and time series into trends, cycles, seasonals and random fluctuations. Second, a fruitful analysis of empirical economic time series, either using techniques from the Classical Time Series Analysis or from Spectral Analysis, can only be done when trends have been adequately removed. Noting the variety of detrending techniques available and the potential dangers of obtaining spurious results, the first requirement of such filtering techniques is that they are as “correct” as possible. Given the specific objective to analyze the evolution of economic variables at different frequencies, ranging from the very long run to short term business cycles, the second requirement for the filtering techniques is that they offer enough flexibility to “zoom in” on the various components of a time series. The only way to meet these requirements is to define and arrive at filtering techniques in the frequency domain instead of in the time domain. Since most of the conventional detrending techniques applied in macroeconomics are defined in the time domain and offer little or no flexibility to arrive at or even define the interesting components of a time series, Chapter 5 describes the construction and testing of such a filter along the lines of filters in Schmidt (1984), Baxter and King (1995, 1999) and Christiano and Fitzgerald (1999). In the absence of knowing the “true” model of a time series, filtering should “merely” be seen as applying a set of adequate statistical techniques to extract the components of interest from the time series. Designing filters directly based on their properties in the frequency domain comes closest to doing this in an “optimal” manner.

The choice for the second technique, Spectral Analysis, on the one hand stems from the chosen approach to designing filters. On the other hand, Spectral Analysis also matches very naturally with the different types of alleged economic cycles of which the existence of business cycles seems the least disputed. When trying to collect stylized facts of economic fluctuations for the purpose of the first research objective, estimating (multivariate) spectral densities seems the most efficient way to report information on the dynamics of economic variables. The obvious problem

when applying Spectral Analysis to (filtered) economic time series is the shortness of the available time series which renders classical non-parametric spectral estimation techniques of little use. However, parametric Maximum Entropy spectral estimation techniques are able to efficiently extract information from short sample time series. Chapter 4 builds up the relevant spectral theory starting from the very definition of simple sine and cosine functions and ends with rather extensive experiments to determine the optimal (estimation procedure and order selection criteria) settings for the Maximum Entropy spectral estimation and a new way to test the statistical significance of spectral densities estimated this way.

Because the Maximum Entropy spectral techniques require estimating autoregressive models, the class of Vector AutoRegressive (VAR) time series models constitute the third technique. These models are also important with respect to the second research objective in which some of the scenario models applied in ALM are tested on their consistency with the stylized facts found in the empirical analysis of historical time series. Since a lot of these models are of the VAR type, the workings of VAR models becomes extra important here. Chapter 3 therefore describes the relevant aspects of autoregressive models in general and introduces new classes of restricted VAR models that will be used for the construction of a new scenario framework in Part IV.





## *Part II*

# Theoretical Background

*“The purpose of computing is insight, not numbers.”*

Richard W. Hamming (1962)

Part II consists of three chapters. Each of these chapters provide the relevant theory and experiments of one of the three central techniques chosen in section 2.3 to carry out the research presented in Part III and IV. It should be stressed that these three techniques are *not* used *separately* in a competitive sense. Instead, the first chapter of Part III, Chapter 6, describes how these three techniques work *together* in a specific methodology developed for the research presented in that part. Chapter 18 describes the specific methodology for the research presented in Part IV. The three techniques are respectively (Vector) AutoRegressive models (Chapter 3), Spectral Analysis (Chapter 4) and Linear Filters (Chapter 5). Those readers already familiar with some or all of these techniques may skip the relevant parts and continue with Part III of Part IV. Note however that each chapter results in specific new applications of the relevant techniques that are explicitly used for the analysis in Part III and IV. For example, Chapter 3 presents the new class of EigenValue Restricted (EVR) VAR models that is used in a new scenario framework that is described and applied in Part IV. Chapter 4 presents an extensive Monte Carlo experiment for an optimal use of Maximum Entropy (ME) spectral analysis in the methodology of Part III. Finally, Chapter 5 presents an optimal zero phase frequency filter that is used in the methodology of both Part III and IV.



### 3 AutoRegressive Models

This first chapter of Part II on the theoretical background discusses both univariate and multivariate autoregressive models. The text does not contain a full scale theory of such models but merely discusses them with respect to the issues that are relevant for the research presented here. The books by for example Hamilton (1994) and Lütkepohl (1991) give more complete overviews of the theory and practice of autoregressive models. Despite their simple linear format, autoregressive models, especially the multivariate versions, are sometimes thought of as rather complex models of which the workings are hard to understand. In order to simplify their interpretation, autoregressive models can be seen to have simple counterparts in the world of (deterministic) differential equations which enables an easy assessment of their dynamics. Furthermore, an example model is given which is used at various places in the text. The same example is used in the next chapter on spectral analysis to illustrate how the shape of a spectral density corresponds to the dynamic properties of an autoregressive model. Furthermore, this chapter contains two separate sections on how to determine the appropriate order of an autoregressive model and on how to estimate its parameters once the order is known. Finally it is shown how confidence intervals of autoregressive models can be calculated and two new types of models are introduced. These new EigenValue Restricted Vector AutoRegressive (EVR VAR) models and Truncated Vector AutoRegressive (TVAR) models are used in a new scenario framework that is described and applied in Part IV. However, let us start with some basic properties of stochastic processes.

#### 3.1 Stochastic processes

The usual way to model the value of a variable that displays some kind of “random” behavior through time is by a *stochastic process*. Basically, such a (univariate) stochastic process describes a probability distribution for the value of the variable at each point in time, possibly related to previous and future values of the variable. Stated otherwise, it describes a joint probability distribution for the value of the variable at all points in time. The stochastic process of some variable  $x_t$  can be characterized by its moments such as the *mean* or *expected value* (first moment)

$$\mu = E(x_t) \quad (3.1.1)$$

and the *k-th order autocovariances* (second moments)

$$\gamma_k = E(x_t - \mu)(x_{t-k} - \mu) \quad (3.1.2)$$

The mean is the value around which the variable tends to evolve while the autocovariances describe the inter-temporal relations between values of the variable at different points in time.

The autocovariance of order  $k=0$  is the *variance* which describes the volatility of the variable. Autocovariances can have any value. Dividing them by the variance however gives the *autocorrelations*

$$\rho_k = \frac{\gamma_k}{\gamma_0} \quad (3.1.3)$$

which always lie between a value of minus one and plus one, indicating respectively a perfect negative and positive (linear) relation between values of the variable,  $k$  time periods apart. Defining these autocorrelations as a function of the time lag  $k$  results in what is called the *AutoCorrelation Function* (ACF). The ACF is crucial in describing the dynamic properties of a stochastic process. Besides these first and second order moments, also higher order moments such as the *skewness* and the *kurtosis* (i.e. the thickness of the tails of the distribution) can be used to characterize a stochastic process. If one assumes a *Normal probability distribution* however the skewness and the kurtosis will always have a value of respectively zero and three. That is why assuming a Normal probability distribution implies that the stochastic process can be completely characterized by the first and second moments.

A crucial assumption made here is that the process is *stationary* which means that the properties of the process such as the mean and the autocovariances are fixed and do not depend on the time  $t$ <sup>16</sup>. Previously, stationarity was already assumed by leaving out a time index with the mean and autocovariances. Note that such *time invariance* of the process does not mean that the variable itself does not change over time but merely that the properties of the process driving the variable are fixed. Stationarity is a crucial assumption for being able to describe the stochastic behavior of some variable by a single model and to be able to estimate the parameters of such a model on one sample of data. Otherwise each point in time would require another model and only one observation would be available to estimate each of these models. At a more intuitive level assuming stationarity for example for a set of macroeconomic variables, comes down to assuming that the economic forces driving these variables remain the same throughout the sample.

The expressions (3.1.1) and (3.1.2) for the first and second moments of a univariate stochastic process easily generalize to the multivariate case. Suppose that  $\bar{x}_t = (x_{1,t}, \dots, x_{n,t})'$  is a vector of  $n$  variables. The mean now simply becomes a vector consisting of the means of each of the  $n$  variables.

$$\bar{\mu} = E(\bar{x}_t) = (\mu_1, \dots, \mu_n)' \quad (3.1.4)$$

The autocovariances become the  $n \times n$  matrices

$$\Gamma_k = E(\bar{x}_t - \bar{\mu})(\bar{x}_{t-k} - \bar{\mu})' \quad (3.1.5)$$

For  $k=0$  this gives the well known *covariance matrix* of the variables in  $\bar{x}_t$  which describe the (contemporaneous) correlations between the variables at the same moment  $t$ . Although the univariate autocovariances are symmetrical around  $k=0$

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<sup>16</sup> Strictly speaking, under this condition the process is called *second-order* or *covariance stationary* because it only requires that the first and second moment of the process are independent of time.

( $\gamma_k = \gamma_{-k}$ ) this does not exactly holds for the multivariate  $\Gamma_k$  matrices. Here the relation is

$$\Gamma_k = \Gamma_{-k}' \quad (3.1.6)$$

The sequence of the  $i$ -th diagonal element in each of the  $\Gamma_k$  matrices form the autocovariances of the  $i$ -th variable  $x_{i,t}$ . The off diagonal elements in the matrices for  $k \neq 0$  describe the *cross correlations* between the variables. The *autocorrelation matrices*  $P_k$  containing both the individual autocorrelations and the cross correlations is

$$P_k = D^{-1} \Gamma_k D^{-1} \quad (3.1.7)$$

where

$$D^{-1} = \begin{bmatrix} 1/\sqrt{\gamma_0^1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\sqrt{\gamma_0^n} \end{bmatrix}$$

in which  $\gamma_0^i$  is the variance of variable  $x_{i,t}$  from the  $i$ -th diagonal element of  $\Gamma_0$ .  $D^{-1}$  is therefore a diagonal matrix of the inverted standard deviations of the variables in  $\bar{x}_t$ . Finally, the stationarity assumption in the multivariate case generalizes to the vector  $\bar{\mu}$  and matrices  $\Gamma_k$  being independent of time  $t$ . Note that in the remainder of the text the explicit  $\bar{x}_t$  notation will no longer be used to denote a vector variable. The interpretation of  $x_t$  will be clear from the context in which it is used.

### 3.2 Difference equations

A very flexible class of models that can describe all kinds of (stationary) stochastic processes is the class of *AutoRegressive* (AR) models. The simplest of such models is the univariate AR(1) model

$$x_t = \nu + \beta_1 x_{t-1} + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(0, \sigma^2) \quad (3.2.1)$$

The *error process*  $\varepsilon_t$  is a *white noise* process that is assumed to have a Normal distribution. A white noise process has a zero mean, constant variance and all inter-temporal correlations equal to zero. That is

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ E(\varepsilon_t^2) &= \sigma^2 \\ E(\varepsilon_t \varepsilon_{t-k}) &= 0 \quad \text{for } k \neq 0 \end{aligned} \quad (3.2.2)$$

Besides the variance  $\sigma^2$  of these error terms, the other parameters of the AR(1) model are the constant term  $\nu$  and the autoregressive parameter  $\beta_1$ . This type of model is called autoregressive because the value of the variable  $x$  at time  $t$  is “explained” by

the previous value of the same variable at time  $t-1$ . An equation such as (3.2.1) is also called a *linear first-order difference equation*. A difference equation is an expression relating a variable  $x_t$  to its previous (or coming) values. This is a first-order difference equation because only the first lag of the variable appears in the equation. Such an AR(1) model describes a stochastic process for the variable  $x_t$  with some mean value, variance and autocorrelation structure. In the following sections two extensions of the simple AR(1) model are discussed. For both the class of higher order AR( $p$ ) models and for the class of multivariate *Vector Autoregressive* (VAR) models we will see how the properties of the stochastic processes they describe can be calculated from the model parameters and vice versa.

### 3.2.1 Univariate (AR) models

A general AR( $p$ ) model is defined by a  $p$ -th order difference equation combined with a white noise error process.

$$x_t = \nu + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_p x_{t-p} + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(0, \sigma^2) \quad (3.2.3)$$

To obtain the properties (i.e. mean, variance and autocorrelations) of the stochastic process described by such a model it is instructive to first look at the simplest case for  $p=1$  and to write the AR(1) model (3.2.1) in a *Moving Average* (MA) representation. By repeated substitution for  $x_{t-1}$ ,  $x_{t-2}$ ,... one obtains

$$x_t = \nu \sum_{l=0}^{\infty} \beta_1^l + \sum_{l=0}^{\infty} \beta_1^l \varepsilon_{t-l} \quad (3.2.4)$$

If  $|\beta_1| < 1$  the first infinite sum converges which gives

$$x_t = \frac{\nu}{1 - \beta_1} + \sum_{l=0}^{\infty} \beta_1^l \varepsilon_{t-l} \quad (3.2.5)$$

Because the statistical properties of the error process  $\varepsilon_t$  are known, from this representation the mean value  $\mu$  and variance  $\gamma_0$  of the  $x_t$  process can be calculated.

$$\mu = E(x_t) = \frac{\nu}{1 - \beta_1} \quad (3.2.6)$$

$$\gamma_0 = E(x_t - \mu)^2 = E\left(\sum_{l=0}^{\infty} \beta_1^l \varepsilon_{t-l}\right)^2 = \sigma^2 \sum_{l=0}^{\infty} \beta_1^{2l} = \frac{\sigma^2}{1 - \beta_1^2} \quad (3.2.7)$$

where the last equality again assumes  $|\beta_1| < 1$  for the infinite sum to converge. If this crucial condition is satisfied both the mean and the variance of the process are independent of  $t$ . Because this corresponds to stationarity of the process as discussed in the previous section, the condition  $|\beta_1| < 1$  is also called the *stationarity condition* of the AR(1) process. If it is not satisfied, given that the process starts at a

certain time  $t$ , the variance will increase over time causing it clearly to be dependent of time. Besides the time invariance of the properties of the stochastic process, another interpretation of stationarity can now be seen from the  $MA(\infty)$  representation in equation (3.2.5). If  $|\beta_1| < 1$ , the influence of past error terms  $\varepsilon_{t-k}$  on the value of  $x_t$  will decay towards zero as the time lag  $k$  increases. That is, past errors do not cumulate into the value of  $x_t$ . If they would, for example if  $\beta_1=1$ ,  $x_t$  displays what is called a *stochastic trend* and the  $AR(1)$  model becomes a *random walk*.

Besides the stationarity condition it is also important to note that the mean value of the process is not equal to the constant term  $\nu$  as one may think and that the variance of the process is not equal to the variance of the error process  $\sigma^2$ . Instead, both the mean value and the variance are functions of all model parameters, including the autoregressive parameter  $\beta_1$ .

Arriving at the autocovariances of an  $AR(1)$  model is a little more difficult. It is easiest to first rewrite the model in the following mean corrected format.

$$x_t - \mu = \beta_1(x_{t-1} - \mu) + \varepsilon_t \quad (3.2.8)$$

where  $\mu$  is as given in (3.2.6). Multiplying by  $x_{t-k} - \mu$  and then taking expectations on both sides leads to

$$\gamma_k - \beta_1\gamma_{k-1} = E(\varepsilon_t x_{t-k}) \quad (3.2.9)$$

in terms of the autocovariances  $\gamma_k$ . Because

$$E(\varepsilon_t x_{t-k}) = \begin{cases} \sigma^2 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.2.10)$$

this reduces to the so called *Yule-Walker equations* named after Yule (1927) and Walker (1931).

$$\begin{aligned} \gamma_0 - \beta_1\gamma_1 &= \sigma^2 \\ \gamma_k &= \beta_1\gamma_{k-1} \quad \text{for } k > 0 \end{aligned} \quad (3.2.11)$$

These equations provide the link between the parameters  $\beta_1$  and  $\sigma^2$  of the  $AR(1)$  model on the one hand and the autocovariances  $\gamma_0, \gamma_1, \gamma_2, \dots$  of the stochastic process it describes on the other. If one knows the parameters, the equations enable the calculation of the autocovariances. Also, the other way around, if one knows the autocovariances, they enable the calculation of the corresponding model parameters. Substituting for example from the second equation  $\gamma_1 = \beta_1\gamma_0$  ( $k=1$ ) into the first equations yields

$$\begin{aligned} \gamma_0 &= \frac{\sigma^2}{1 - \beta_1^2} \\ \gamma_k &= \beta_1\gamma_{k-1} \quad \text{for } k > 0 \end{aligned} \quad (3.2.12)$$

as the solution for the autocovariances of the process. Dividing these by the process variance  $\gamma_0$  leads to the autocorrelations

$$\begin{aligned}\rho_0 &= 1 \\ \rho_k &= \beta_1^k \text{ for } k > 0\end{aligned}\tag{3.2.13}$$

From this yet another interpretation of stationarity can be seen. If the AR(1) process is stationary (i.e.  $|\beta_1| < 1$ ) the autocorrelations decay at an exponential rate towards zero as the order  $k$  increases. That is, the correlation between values of the process decreases as the time between these observations increases.

For the general AR( $p$ ) model (3.2.3), repeated substitution for  $x_{t-1}, x_{t-2}, \dots$  just as for the AR(1) model leads to the MA( $\infty$ ) representation

$$x_t = v \sum_{l=0}^{\infty} (\beta_1 + \beta_2 + \dots + \beta_p)^l + \sum_{l=0}^{\infty} \psi_l \varepsilon_{t-l}\tag{3.2.14}$$

The weights  $\psi_l$  are functions of the autoregressive parameters  $\beta_1, \dots, \beta_p$ . From this we see that for the first infinite sum to converge it is sufficient that  $|\beta_1 + \beta_2 + \dots + \beta_p| < 1$ . If this condition is satisfied the mean of the process is

$$\mu = E(x_t) = \frac{v}{1 - \beta_1 - \beta_2 - \dots - \beta_p}\tag{3.2.15}$$

However in order for also the autocovariances to be independent of time, and thereby causing stationarity in both first and second moments, this condition is not enough. For this to be true it is required that for the roots  $\lambda=r_1, \dots, \lambda=r_p$  of the so called *characteristic equation*<sup>17</sup>

$$1 - \beta_1 \lambda - \beta_2 \lambda^2 - \dots - \beta_p \lambda^p = (1 - \frac{1}{r_1} \lambda)(1 - \frac{1}{r_2} \lambda) \dots (1 - \frac{1}{r_p} \lambda) = 0\tag{3.2.16}$$

it must hold that  $|r_i| > 1$  for  $i=1, \dots, p$ . Note that for an AR(1) model this condition is equivalent to  $|\beta_1| < 1$  as noted before because the single root of the characteristic equation is  $1/\beta_1$ . Since stationarity requires the roots to have a modulus greater than one, it is also said that the roots must lie outside the unit circle in the complex plain<sup>18</sup>.

<sup>17</sup> Leaving out the constant term, a model as the AR( $p$ ) model (3.2.3) is often written in terms of a *lag polynomial* as  $\phi(L)x_t = \varepsilon_t$  where  $\phi(L) = 1 - \beta_1 L - \beta_2 L^2 - \dots - \beta_p L^p$  and  $L$  is the lag operator defined as  $L^k x_t = x_{t-k}$ . In this representation because of the stationarity condition,  $\phi(L)$  can be inverted such that  $x_t = \phi(L)^{-1} \varepsilon_t$  is the MA( $\infty$ ) representation of the process. The weights in the polynomial  $\phi(L)^{-1}$  correspond to the weights  $\psi_l$  in (3.2.14). Note that the characteristic equation (3.2.16) is simply the lag polynomial with the lag operator replaced by the variable  $\lambda$ .

<sup>18</sup> Some basic theory of complex numbers is given in Appendix A.2.



The variance and autocovariances of the general AR( $p$ ) process can again be obtained from the Yule-Walker equations which follow from writing the model in a mean corrected format, multiplying by  $x_t - \mu$  and taking expectations on both sides. In terms of the autocovariances this yields

$$\gamma_k - \beta_1 \gamma_{k-1} - \beta_2 \gamma_{k-2} - \dots - \beta_p \gamma_{k-p} = E(\varepsilon_t x_{t-k}) \quad (3.2.17)$$

Because of (3.2.10), the first  $p+1$  of these equations are equivalent to

$$\gamma_0 - \gamma_1 \beta_1 - \dots - \gamma_p \beta_p = \sigma^2 \quad (3.2.18)$$

and the  $p \times p$  system

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix} \quad (3.2.19)$$

Given the first  $p+1$  autocovariances  $\gamma_0, \dots, \gamma_p$  of a stochastic process, the parameters  $\beta_1, \dots, \beta_p$  and  $\sigma^2$  of the corresponding AR( $p$ ) model can be solved from (3.2.18) and (3.2.19) as

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix} \quad (3.2.20)$$

and

$$\sigma^2 = \gamma_0 - \gamma_1 \beta_1 - \dots - \gamma_p \beta_p \quad (3.2.21)$$

In order to calculate, the other way around, the autocovariances from the model parameters, the equations (3.2.18) and (3.2.19) have to be rewritten as a  $(p+1) \times (p+1)$  system having  $\beta_1, \dots, \beta_p$  and  $\sigma^2$  as the parameters, instead of  $\gamma_0, \dots, \gamma_p$ .

This yields the system

$$B\bar{\gamma} = \bar{\sigma} \quad (3.2.22)$$

where

$$B = \begin{bmatrix} 1 & -\beta_1 & -\beta_2 & -\beta_3 & \cdots & -\beta_{p-2} & -\beta_{p-1} & -\beta_p \\ -\beta_1 & 1-\beta_2 & -\beta_3 & -\beta_4 & \cdots & -\beta_{p-1} & -\beta_p & 0 \\ -\beta_2 & -\beta_1-\beta_3 & 1-\beta_4 & -\beta_5 & \cdots & -\beta_p & 0 & 0 \\ -\beta_3 & -\beta_2-\beta_4 & -\beta_1-\beta_5 & 1-\beta_6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\beta_{p-2} & -\beta_{p-3}-\beta_{p-1} & -\beta_{p-4}-\beta_p & -\beta_{p-5} & \cdots & 1 & 0 & 0 \\ -\beta_{p-1} & -\beta_{p-2}-\beta_p & -\beta_{p-3} & -\beta_{p-4} & \cdots & -\beta_1 & 1 & 0 \\ -\beta_p & -\beta_{p-1} & -\beta_{p-2} & -\beta_{p-3} & \cdots & -\beta_2 & -\beta_1 & 1 \end{bmatrix}$$

$$\bar{\gamma} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_p \end{bmatrix} \text{ and } \bar{\sigma} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from which the first  $p+1$  autocovariances (including the variance) can be solved as

$$\bar{\gamma} = B^{-1}\bar{\sigma} \quad (3.2.23)$$

Higher order autocovariances can be obtained through the recursion

$$\gamma_k = \beta_1\gamma_{k-1} + \beta_2\gamma_{k-2} + \dots + \beta_p\gamma_{k-p} \text{ for } k > p \quad (3.2.24)$$

### 3.2.2 Multivariate (VAR) models

The multivariate extension of the AR( $p$ ) model is the VAR( $p$ ) model

$$x_t = v + A_1x_{t-1} + A_2x_{t-2} + \dots + A_px_{t-p} + \varepsilon_t \quad (3.2.25)$$

where

$$x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad A_i = \begin{bmatrix} a_{i,1,1} & a_{i,1,2} & \cdots & a_{i,1,n} \\ a_{i,2,1} & a_{i,2,2} & \cdots & a_{i,2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,n,1} & a_{i,n,2} & \cdots & a_{i,n,n} \end{bmatrix} \text{ for } i = 1, \dots, p$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix}, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t \varepsilon_s') = \begin{cases} \Sigma & \text{for } s = t \\ 0 & \text{for } s \neq t \end{cases}$$

Instead of a single variable, the model describes a multivariate stochastic process for the  $n$  variables  $x_{1,t}, \dots, x_{n,t}$ . The parameters that determine the properties of this stochastic process are the constant vector  $v$ , the  $n \times n$  matrices  $A_1, \dots, A_p$  and the  $n \times n$  covariance matrix  $\Sigma$  of the now multivariate error process  $\varepsilon_t$ . Model (3.2.25) can also be seen in terms of a multivariate  $p$ -th order difference equation which links the value of each variable at time  $t$  to the  $p$  previous values of both the variable itself and of all the other  $n-1$  variables.

To arrive at the properties such as the mean, variance and autocovariances of the stochastic process described by a VAR( $p$ ) model, we start with the following VAR(1) model.

$$x_t = v + A_1 x_{t-1} + \varepsilon_t \quad (3.2.26)$$

where the error process is defined as in (3.2.25). The stationarity condition for a VAR( $p$ ) model requires all roots  $r_j$ ,  $j=1, \dots, n \times p$  of the characteristic (determinant) equation

$$\left| I_n - \lambda A_1 - \lambda^2 A_2 - \dots - \lambda^p A_p \right| = 0 \quad (3.2.27)$$

to lie outside the unit circle and therefore to have a modulus greater than one. If this stationarity condition is satisfied, recursive substitution yields

$$\mu = E(x_t) = (I - A_1)^{-1} v \quad (3.2.28)$$

for the mean vector of the VAR(1) process. Writing the model in a mean corrected format, post-multiplying by the row vector  $(x_{t-k} - \mu)'$  and taking expectations on both sides leads to

$$E(x_t - \mu)(x_{t-k} - \mu)' = A_1 E(x_{t-1} - \mu)(x_{t-k} - \mu)' + E\varepsilon_t(x_{t-k} - \mu)' \quad (3.2.29)$$

which in terms of the autocovariance matrices gives the Yule-Walker equations for  $k=0$  and  $k>0$ .

$$\begin{aligned} \Gamma_0 &= A_1 \Gamma_{-1} + \Sigma = A_1 \Gamma_1' + \Sigma \\ \Gamma_k &= A_1 \Gamma_{k-1} \quad \text{for } k > 0 \end{aligned} \quad (3.2.30)$$

Again these equations provide the link between the model parameters  $A_1$  and  $\Sigma$  on the one hand and the autocovariances of the stochastic process described by the model on the other. Given the first two autocovariance matrices  $\Gamma_0$  and  $\Gamma_1$ , the corresponding parameter matrices are

$$\begin{aligned} A_1 &= \Gamma_1 \Gamma_0^{-1} \\ \Sigma &= \Gamma_0 - A_1 \Gamma_1' = \Gamma_0 - \Gamma_1 \Gamma_0^{-1} \Gamma_1' \end{aligned} \quad (3.2.31)$$

The other way around, calculating the autocovariances from the parameter matrices  $A_1$  and  $\Sigma$ , starts by substituting  $\Gamma_1=A_1\Gamma_0$  into the equation for  $\Gamma_0$  from (3.2.30). This yields

$$\Gamma_0 = A_1\Gamma_0A_1' + \Sigma \quad (3.2.32)$$

Because  $x_t$  has dimension  $n$ , this basically is a system of  $n \times n$  equations in an equal number of  $n \times n$  unknowns, being the elements of the  $\Gamma_0$  matrix. As Lütkepohl (1991, p. 22) shows, this system can be solved as

$$\text{vec}(\Gamma_0) = (I_{n \times n} - A_1 \otimes A_1)^{-1} \text{vec}(\Sigma) \quad (3.2.33)$$

The  $\text{vec}$  and  $\otimes$  (Kronecker product) operator are defined as (A.3.1) and (A.3.2). Once  $\Gamma_0$  is known, the higher order autocovariance matrices  $\Gamma_k$  can be recursively calculated from the second equation in (3.2.30). The autocorrelation matrices  $P_k$  can then be calculated as indicated by (3.1.7). The mean value of the general VAR( $p$ ) model (3.2.25) is

$$\mu = E(x_t) = (I - A_1 - A_2 \dots - A_p)^{-1}v \quad (3.2.34)$$

The Yule-Walker equations for a  $n$  dimensional VAR( $p$ ) model easily follow from writing (3.2.25) as the following equivalent  $p \times n$  dimensional VAR(1) model.

$$y_t = w + By_{t-1} + \xi_t \quad \text{where } \xi_t \sim N(0, \Psi) \quad (3.2.35)$$

and

$$y_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ \vdots \\ x_{t-p+1} \end{bmatrix}, \quad w = \begin{bmatrix} v \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & A_2 & \dots & \dots & A_p \\ I_n & 0 & \dots & \dots & 0 \\ 0 & I_n & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} \Sigma & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}$$

The autocovariance matrices of this derived process are

$$\begin{aligned} \Phi_k &= E \begin{bmatrix} x_t - \mu \\ \vdots \\ x_{t-p+1} - \mu \end{bmatrix} \begin{bmatrix} (x_{t-k} - \mu)' & \dots & (x_{t-p+1-k} - \mu)' \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_k & \Gamma_{k+1} & \dots & \Gamma_{p-1+k} \\ \Gamma_{k-1} & \Gamma_k & \dots & \Gamma_{p-2+k} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{k-p+1} & \Gamma_{k-p} & \dots & \Gamma_k \end{bmatrix} \end{aligned} \quad (3.2.36)$$

Because this derived process is a VAR(1) model, its autocovariance matrices  $\Phi_k$  on the one hand and its model parameter matrices  $B$  and  $\Psi$  on the other can be calculated from one another by solving the Yule-Walker equations (3.2.30) for a VAR(1) model. The parameters of the original process,  $A_1, \dots, A_p$  and  $\Sigma$ , and its autocovariances  $\Gamma_k$  can then easily be read from their counterparts in the derived process. Finally note that a derived process such as (3.2.35) can equally well be applied to transform a univariate AR( $p$ ) model into a multivariate VAR(1) model. Take for example the AR(2) model

$$x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(0, \sigma^2) \quad (3.2.37)$$

which can be written as the equivalent  $n=2$  dimensional VAR(1) model

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} + \xi_t \quad \text{where } \xi_t \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}\right) \quad (3.2.38)$$

For high order VAR( $p$ ) models describing a large number of variables, solving the Yule-Walker equations for the derived VAR(1) process using the direct solution (3.2.33) may lead to numerical difficulties when calculating the autocovariance matrices. For example a VAR(2) model for  $n=10$  variables already requires inverting a  $400 \times 400$  matrix because of the Kronecker product involved. Note that such large matrices are not involved when solving for the model parameters as given by (3.2.31). To prevent numerical problems in such (rare) cases, the following alternative method for calculating the autocovariances can be used. This solution is based on the MA( $\infty$ ) representation of a VAR(1) model with zero mean that can be obtained by recursive substitution for  $x_{t-1}, x_{t-2}, \dots$  which gives

$$x_t = \varepsilon_t + A_1 \varepsilon_{t-1} + A_1^2 \varepsilon_{t-2} + A_1^3 \varepsilon_{t-3} + \dots = \sum_{i=0}^{\infty} A_1^i \varepsilon_{t-i} \quad (3.2.39)$$

If the model is stationary this infinite sum converges. Based on this MA( $\infty$ ) representation now follows

$$\Gamma_0 = E(x_t x_t') = \Sigma + A_1 \Sigma A_1' + A_1 (A_1 \Sigma A_1') A_1' + \dots \quad (3.2.40)$$

Starting from  $\Sigma$ , constant pre-multiplying by  $A_1$  and post-multiplying by  $A_1'$  therefore gives a recursive format to calculate each next (matrix) element to be added to the infinite sum. In case of stationarity, the elements added will decrease towards zero. A criterion to stop adding elements can be that all eigenvalues of the additional (matrix) element are below some critical value. Once  $\Gamma_0$  is obtained this way, higher order autocovariance matrices  $\Gamma_k$  can be recursively calculated from the second equation of (3.2.30) as usual. For VAR models of orders  $p > 1$  this procedure can equally well be applied to the derived VAR(1) model (3.2.35).

### 3.3 Dynamics

There are several ways to explore the dynamics of a VAR model. One way is through the autocorrelations and cross correlations from the autocovariance matrices  $\Gamma_k$  as calculated from the model parameters. Another way is to look at the *impulse response functions*. The purpose of such a *multiplier analysis* is to investigate the response of all variables to a one-time shock in just one of the variables. It uses the deterministic part of the VAR model (3.2.25) by leaving out the error process  $\varepsilon_t$ . Since we are not interested in the mean of the process in such an exercise, the constant vector  $v$  is left out of the model which gives

$$x_t = A_1 x_{t-1} + A_2 x_{t-2} + \dots + A_p x_{t-p} \quad (3.3.1)$$

Starting with a vector  $x_t$  of all zeros and just a single 1 at the  $i$ -th position can then be seen as a one-time shock in the  $i$ -th variable. Recursive substitution for  $x_{t+k}$  starting from this initial vector then leads to  $n$  impulse response functions showing how each of the  $n$  variables reacts to such a one-time shock in the  $i$ -th variable. Repeating this for each of the variables leads to  $n \times n$  impulse response functions. Here a slightly different approach is used to analyze the dynamics of a VAR model which will be called a *deterministic simulation* of a VAR model.

#### 3.3.1 Deterministic simulations

Just as the impulse response analysis, a deterministic simulation also uses the deterministic version (3.3.1) of a VAR model but instead of starting with a vector of all zeros and a single 1 for the  $i$ -th variable, it starts with some arbitrary vector of initial values. By recursive substitution one can then study in one picture (instead of  $n \times n$  pictures as with the impulse response analysis) the dynamic behavior of the VAR model. As an example let's look at the following two dimensional VAR(2) model with  $v=0$  and therefore  $E(x_t)=0$ .

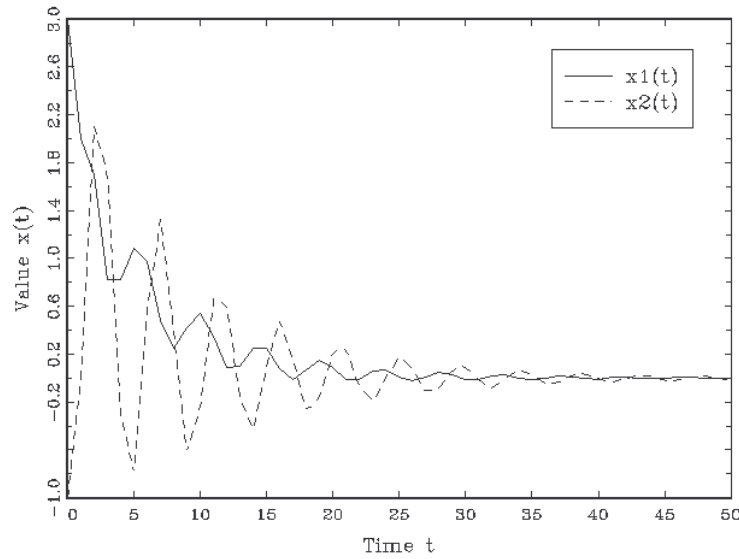
$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.3 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & -0.7 \end{bmatrix} \begin{bmatrix} x_{1,t-2} \\ x_{2,t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \quad (3.3.2)$$

where

$$\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Figure 3.1 shows a deterministic simulation of this model for some chosen initial values  $x_0=(3,-1)'$  and  $x_1=(2,0)'$ .

Figure 3.1: A deterministic simulation of VAR(2) model (3.3.2).

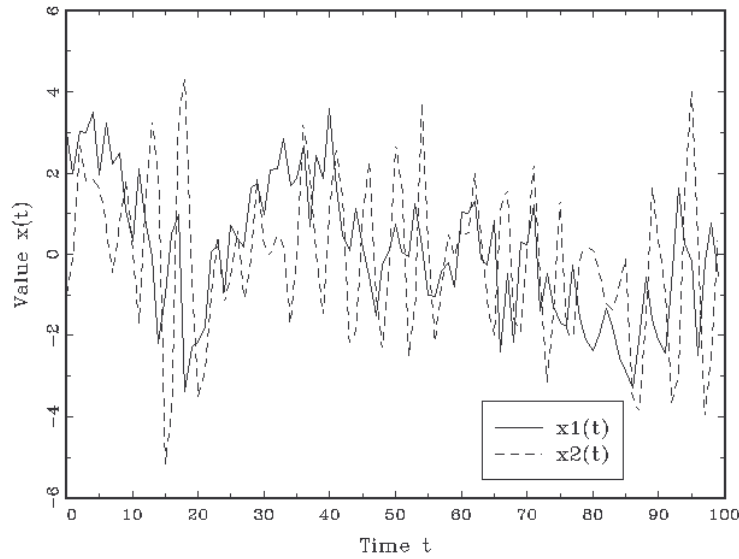


From this deterministic simulation we can learn a lot about the dynamic behavior of the model. It can clearly be seen that in the absence of the error terms the value of both variables decays towards the mean  $E(x_t) = (0, 0)'$ . Once the variables reach this equilibrium value they remain unchanged. The path towards the equilibrium value however is different for both variables. The first variable  $x_{1,t}$  shows a slow exponential decay combined with damped cyclical fluctuations with a period length of around five time periods. In the second variable  $x_{2,t}$  the slow decay is hardly visible while the cyclical fluctuations are more clearly present. Furthermore the peaks and troughs in the first variable lead the peaks and troughs in the second variable by approximately one or two time periods. Note that although the choice of the initial values of course determines the specific path of the two variables, it has no effect on the general conclusions about the dynamic behavior of the model.

To see how the aforementioned properties of a deterministic simulation turn up in true stochastic simulations of the model, Figure 3.2 plots one such a simulation of the two time series starting from the same initial values. By adding stochastic error terms the variables will never permanently reach their equilibrium values as with the deterministic simulation. Instead, the random shocks will constantly move the variables away from the paths towards their equilibrium values. The variables however do have the tendency to decay towards their equilibrium value in the fashion displayed by the deterministic simulation. Although the variables will never show truly periodic behavior because of the constant interruptions by the error terms, it can be seen that both variables do show a kind of “pseudo periodic” behavior with a period length of around five time periods. In case of the first variable, these pseudo periodic fluctuations are shaped around a slowly moving pattern consistent with the slow decay observed in the deterministic simulation. Also the lead of the first variable on the second variable is visible in a stochastic sense, meaning it is not exactly the same in all time periods but it is one or two time periods on average.



Figure 3.2: Stochastic simulation of VAR(2) model (3.3.2).



While a deterministic simulation is by construction closely linked to an impulse response analysis, it is also has an equivalence to the univariate *autocorrelation function* (autocorrelations  $\rho_k$  as a function of the time lag  $k$ ) of the variables described by the model. The link can best be seen by comparing the deterministic difference equation of a general AR( $p$ ) model

$$x_t = \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_p x_{t-p} \quad (3.3.3)$$

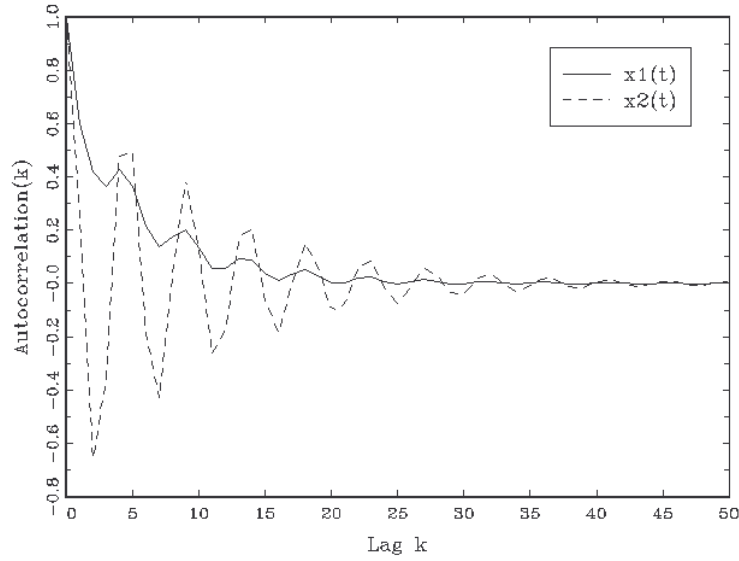
and the univariate Yule-Walker equation (3.2.24) divided by the variance of the process

$$\rho_k = \beta_1 \rho_{k-1} + \beta_2 \rho_{k-2} + \dots + \beta_p \rho_{k-p} \quad (3.3.4)$$

The equations (3.3.3) and (3.3.4) only differ with respect to the variables. A deterministic simulation is the solution to (3.3.3) while the autocorrelation function is the solution to (3.3.4). Therefore a deterministic simulation and the autocorrelation function will show a similar pattern. Indeed, the autocorrelation functions of the two variables from the model (3.3.2) as shown in Figure 3.3 look very much like the deterministic simulation from Figure 3.1. This equivalence implies that most of the conclusions that can be drawn from a deterministic simulation can also be based on the autocorrelation functions. There are however also two differences to be noticed. The first is that a deterministic simulation starts from a random initial vector  $x_t$  while the autocorrelation functions by definition all start at a value of one. Since a vector of ones may just as well be seen as some initial vector this difference doesn't matter that much. The second difference however is that the autocorrelation functions do not show the lead / lag relations between the two variables as they are visible in a deterministic simulation. These lead / lag relations are in a way taken out of the autocorrelation functions. Of course these relations are still present in the sequences of cross correlations. However, these cross correlations are less easy to interpret than a simple deterministic simulation. The fact that a deterministic simulation shows in one picture both the univariate and the multivariate properties

of a VAR model is what renders it more suitable to assess the dynamic properties of the model than an impulse response or correlation analysis.

Figure 3.3 Autocorrelation functions of VAR(2) model (3.3.2).



### 3.3.2 Eigenvalues and eigenvectors

The easiest way to understand the shape of a deterministic simulation of a VAR model and the corresponding behavior of the stochastic process is based on the following general result for (deterministic) linear difference equations. This result can for example be found in Barnett (1990, p. 155). It says that linear difference equations of the form

$$X_{k+1} = AX_k \text{ for } k = 0, 1, 2, \dots \quad (3.3.5)$$

where  $X_k = (x_{1,k}, \dots, x_{n,k})'$  and  $A$  is an  $n \times n$  matrix have the general solution

$$\begin{bmatrix} x_{1,k} \\ \vdots \\ x_{n,k} \end{bmatrix} = \begin{bmatrix} v_{1,1} \\ \vdots \\ v_{1,n} \end{bmatrix} \dots \begin{bmatrix} v_{n,1} \\ \vdots \\ v_{n,n} \end{bmatrix} \begin{bmatrix} \beta_1 \lambda_1^k \\ \vdots \\ \beta_n \lambda_n^k \end{bmatrix} \quad (3.3.6)$$

Here  $\lambda_1, \dots, \lambda_n$  and  $v_1, \dots, v_n$  are respectively the  $n$  eigenvalues and the  $n$  corresponding eigenvectors of the matrix  $A$ . For any square  $n \times n$  matrix  $A$ , the eigenvalues  $\lambda_i$  and eigenvectors  $v_i$  for  $i=1, \dots, n$  are defined as the pairs of numbers and vectors which satisfy

$$Av_i = \lambda_i v_i \quad (3.3.7)$$

The eigenvalues can be found by solving the determinant equation

$$|A - \lambda I| = 0 \quad (3.3.8)$$

while the corresponding eigenvectors can be found along lines as described in almost any standard textbook on linear algebra. The  $\beta_1, \dots, \beta_n$  in (3.3.6) are arbitrary constants which are determined by the initial values of the vector  $X_k$ . These constants can also be understood by noting from (3.3.7) that eigenvectors are only unique up to a constant term. If  $v_i$  is an eigenvector,  $cv_i$  is also an eigenvector for any constant  $c$ . Depending on the matrix  $A$ , the eigenvalues and eigenvectors can both be real or complex (vector)numbers. If  $X_k$  is real valued, any complex eigenvalue will have a corresponding complex eigenvector. Real eigenvalues will correspond to real eigenvectors. Furthermore complex eigenvalues and eigenvectors will appear as complex conjugate pairs. That is, if  $\lambda_i$  is a complex eigenvalue with complex eigenvector  $v_i$ , one of the other eigenvalues and eigenvectors will be the complex conjugates  $\lambda_i^*$  and  $v_i^*$ . Also the corresponding  $\beta_i$  coefficients in (3.3.6) have to be complex conjugates for the vector  $X_k$  to be real valued.

To illustrate the link between (3.3.6) and a deterministic simulation of a VAR model consider a simple VAR(1) model for  $n=4$ . The deterministic simulation of such a model can be written as

$$x_{t+k+1} = Ax_{t+k} \quad (3.3.9)$$

which is a first order differential equation just as (3.3.5). The general solution of this deterministic simulation is given by (3.3.6) with the only difference that the constant  $\beta$  terms will depend on  $t$ . Suppose the  $4 \times 4$  matrix  $A$  has two complex conjugated eigenvalues and two (different) real eigenvalues. Let's write these eigenvalues in polar format as

$$\begin{aligned} \lambda_1 &= R_1 \exp(i\omega_1) = R_1(\cos \omega_1 + i \sin \omega_1) \\ \lambda_2 &= \lambda_1^* = R_1 \exp(-i\omega_1) = R_1(\cos \omega_1 - i \sin \omega_1) \\ \lambda_3 &= R_3 \\ \lambda_4 &= R_4 \end{aligned}$$

and the corresponding eigenvectors as

$$v_1 = \begin{bmatrix} \Delta_{1,1} \exp(i\phi_{1,1}) \\ \vdots \\ \Delta_{4,1} \exp(i\phi_{4,1}) \end{bmatrix}, \quad v_2 = v_1^* = \begin{bmatrix} \Delta_{1,1} \exp(-i\phi_{1,1}) \\ \vdots \\ \Delta_{4,1} \exp(-i\phi_{4,1}) \end{bmatrix}, \quad v_3 = \begin{bmatrix} \Delta_{1,3} \\ \vdots \\ \Delta_{4,3} \end{bmatrix} \quad \text{and} \quad v_4 = \begin{bmatrix} \Delta_{1,4} \\ \vdots \\ \Delta_{4,4} \end{bmatrix}$$

Furthermore write the constant terms in (3.3.6) as

$$\begin{aligned}\beta_{1,t} &= a_{1,t} \exp(ib_{1,t}) \\ \beta_{2,t} &= \beta_{1,t}^* = a_{1,t} \exp(-ib_{1,t}) \\ \beta_{3,t} &= a_{3,t} \\ \beta_{4,t} &= a_{4,t}\end{aligned}$$

The use of the index  $t$  in the constant  $\beta$  terms is to show that they depend on the initial values of  $x_{t+k}$  for  $k=0$  in (3.3.9). If we substitute the eigenvalues, eigenvectors and constant terms into the general solution (3.3.6) we get

$$\begin{aligned}\begin{bmatrix} x_{1,t+k} \\ \vdots \\ x_{n,t+k} \end{bmatrix} &= \begin{bmatrix} \Delta_{1,1} \exp(i\phi_{1,1}) & \Delta_{1,1} \exp(-i\phi_{1,1}) & \Delta_{1,3} & \Delta_{1,4} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{4,1} \exp(i\phi_{4,1}) & \Delta_{4,1} \exp(-i\phi_{4,1}) & \Delta_{4,3} & \Delta_{4,4} \end{bmatrix} \begin{bmatrix} a_{1,t} \exp(ib_{1,t}) R_1^k \exp(i\omega_1 k) \\ a_{1,t} \exp(-ib_{1,t}) R_1^k \exp(-i\omega_1 k) \\ a_{3,t} R_3^k \\ a_{4,t} R_4^k \end{bmatrix} \\ &= \begin{bmatrix} \Delta_{1,1} a_{1,t} R_1^k \{ \exp(i(\phi_{1,1} + b_{1,t} + \omega_1 k)) + \exp(-i(\phi_{1,1} + b_{1,t} + \omega_1 k)) \} + \Delta_{1,3} a_{3,t} R_3^k + \Delta_{1,4} a_{4,t} R_4^k \\ \vdots \\ \Delta_{4,1} a_{1,t} R_1^k \{ \exp(i(\phi_{4,1} + b_{1,t} + \omega_1 k)) + \exp(-i(\phi_{4,1} + b_{1,t} + \omega_1 k)) \} + \Delta_{4,3} a_{3,t} R_3^k + \Delta_{4,4} a_{4,t} R_4^k \end{bmatrix}\end{aligned}$$

Using the Euler equation

$$\cos \omega = \frac{\exp(i\omega) + \exp(-i\omega)}{2} \quad (3.3.10)$$

it now follows that the deterministic simulation of a four dimensional VAR(1) model with two complex and two real eigenvalues can be written as

$$\begin{bmatrix} x_{1,t+k} \\ \vdots \\ x_{n,t+k} \end{bmatrix} = \begin{bmatrix} \Delta_{1,1} a_{1,t} R_1^k 2 \cos(\phi_{1,1} + b_{1,t} + \omega_1 k) + \Delta_{1,3} a_{3,t} R_3^k + \Delta_{1,4} a_{4,t} R_4^k \\ \vdots \\ \Delta_{4,1} a_{1,t} R_1^k 2 \cos(\phi_{4,1} + b_{1,t} + \omega_1 k) + \Delta_{4,3} a_{3,t} R_3^k + \Delta_{4,4} a_{4,t} R_4^k \end{bmatrix} \quad (3.3.11)$$

for  $k=0,1,2,\dots$ . From this representation we see that the deterministic simulation of each of the four variables is a linear combination of exponentially dampened smooth and cyclical patterns. The dampened smooth patterns come from the real eigenvalues of the matrix  $A$  while the dampened cyclical patterns come from the complex conjugate eigenvalues. The speed at which the dampening takes place depends on the modulus  $R_i$  of the relevant eigenvalue. The closer this is to zero, the faster the dampening. The faster this dampening the more the error process will dominate the properties of the stochastic process and the less clear the pattern caused by the relevant eigenvalue will be present. The other way around, the closer the modulus is to one the clearer the eigenvalue patterns will be, in the end almost causing non-stationarity. The frequency of the cycles is determined by the argument  $\omega_1$  of the complex eigenvalue in terms of its polar coordinates. Positive real eigenvalues will not show any cyclical

behavior. Instead, the effects of past random shocks on the value of  $x_t$  will smoothly decay at an exponential rate. The closer real eigenvalues are to zero, the faster the effects of past shocks will fade away and the more erratic the process will behave. The other way around, the closer real roots are to one, the longer past shocks are accumulated into the value of the process and the smoother it will behave. Note that a negative real eigenvalue ( $R_i < 0$ ) describes a dampened alternating pattern which may also be seen as a complex eigenvalue with  $\omega_1 = \pi$ . The extent to which the pattern of eigenvalue  $j$  appears in the simulation of variable  $i$  is determined by  $\Delta_{ij}$  of the corresponding eigenvector. Since the eigenvectors are only unique up to a constant term, these  $\Delta$  terms only show the *relative* importance of an eigenvalue in a variable. The phase differences between the cycles of a complex root in two variables is determined by the difference between the relevant  $\phi_{ij}$  from the eigenvectors. Again, because of the non-uniqueness of eigenvectors, only the difference between such values is informative. The absolute value has no meaning because of the non-uniqueness of the eigenvectors. The constant terms  $\beta_{i,t}$ , represented by the  $a$  and  $b$  terms, are  $t$  dependent deviations from the  $\Delta$  and  $\phi$  terms which cause the consistency with the initial values  $x_t$ . Note that since the number of such constants is equal to the number of variables, the constant terms can be solved for any vector of initial values.

The representation (3.3.11) of a VAR model is a very important one. Although it is an equivalent formulation of the model it reveals much more information about the stochastic process described by the model. From the standard parameter matrices  $A_1, A_2, \dots, A_p$  one can say nothing about how the model behaves while the representation in terms of eigenvalues and eigenvectors directly gives important information about the dynamics of the model. In a sense one could say that the  $R, \omega, \Delta$  and  $\phi$  terms from the eigenvalues and eigenvectors are more *structural type parameters* than the standard parameters matrices. The disadvantage of this representation is of course its complex non-linear format compared to the simple linear format of the original VAR model. Three more important things have to be said about the representation (3.3.11).

### 1. Univariate AR and higher order VAR( $p$ ) models

The first thing is that the representation (3.3.11) may equally well be applied to both univariate AR models and higher order VAR( $p$ ) models. For this one only has to transform these models into a derived VAR(1) model such as given by (3.2.35) and (3.2.40). Once the eigenvalue and eigenvector representation for this derived model is known one simply selects the equations from this model that correspond to the original process. For an AR model this will always be the first equation while for a VAR( $p$ ) model this will be the first  $n$  equations. With respect to such derived VAR(1) models it is important to note the following property of the eigenvalues and eigenvectors of the parameter matrix of a model such as (3.2.35). For example, for  $p=3$  such a matrix looks like

$$B = \begin{bmatrix} A_1 & A_2 & A_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}$$

Suppose  $\lambda$  and  $v$  are an eigenvalue and the corresponding  $3n \times 1$  dimensional eigenvector of this special type of matrix. By definition then holds

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{3n} \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_{3n} \end{bmatrix}$$

which implies

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_{n+1} \\ \vdots \\ v_{2n} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_{n+1} \\ \vdots \\ v_{2n} \end{bmatrix} = \lambda \begin{bmatrix} v_{2n+1} \\ \vdots \\ v_{3n} \end{bmatrix}$$

From this we see that the special structure of the  $B$  matrix implies

$$\begin{bmatrix} v_{n+1} \\ \vdots \\ v_{2n} \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_{2n+1} \\ \vdots \\ v_{3n} \end{bmatrix} = \frac{1}{\lambda^2} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad (3.3.12)$$

Therefore the first  $n$  elements of the  $3n$  eigenvectors combined with the  $3n$  eigenvalues are the only true parameters of the derived VAR model. The other  $2n$  elements of the eigenvectors are uniquely determined by these parameters.

## 2. Equivalence between eigenvalues and roots of characteristic equation

The second thing worth mentioning is the equivalence between the eigenvalues of the, possibly derived, parameter matrix and the roots of the characteristic equations (3.2.16) and (3.2.27) which determine whether an autoregressive model describes a stationary process or not. For the most general case of a VAR( $p$ ) model the (derived) parameter matrix is as we know

$$B = \begin{bmatrix} A_1 & A_2 & \cdots & \cdots & A_p \\ I_n & 0 & \cdots & \cdots & 0 \\ 0 & I_n & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}$$

The  $np$  eigenvalues of this matrix are by definition the solutions of

$$|B - \lambda I_{np}| = 0 \quad (3.3.13)$$

It turns out that these eigenvalues are the same as the inverted roots of the characteristic equation. To see this note that

$$\begin{aligned} |B - \lambda I_{np}| = 0 &\Leftrightarrow \left| I_{np} - \left( \frac{1}{\lambda} \right) B \right| = 0 \Leftrightarrow \\ \left| I_n - \left( \frac{1}{\lambda} \right) A_1 - \left( \frac{1}{\lambda} \right)^2 A_2 - \dots - \left( \frac{1}{\lambda} \right)^p A_p \right| &= 0 \end{aligned} \quad (3.3.14)$$

and compare the last equation to (3.2.27). The stationarity condition which requires that the roots of the characteristic equation of a VAR model are *outside* the unit circle, i.e. have a modulus greater than one, is therefore equivalent to the requirement that the eigenvalues of the parameter matrix are *inside* the unit circle, i.e. have a modulus smaller than one. From the representation (3.3.11) it can now be easily seen why such a condition leads to a stationary model. If all eigenvalues have a modulus  $R_i < 1$  the deterministic simulation of each variable will decay towards its equilibrium value. In case of some modulus  $R_i > 1$ , the deterministic simulations will show an exploding behavior because of the  $k$ -th powers of the modulus in (3.3.11). Such behavior clearly implies non-stationarity. Furthermore, in section 2.2.7 it was noted before that VAR models offer great flexibility in modeling all kinds of stochastic processes while at the same time this great flexibility may cause problems when estimating VAR models on limited samples of data. Note that this great amount of flexibility may also be seen from the large number of  $np$  roots / eigenvalues of the model. For example a simple VAR(1) model on ten variables already has ten roots which can all be present in each of the variables. For comparison, a univariate AR(1) model only has one root with the corresponding much simpler stochastic behavior. Zellner and Palm (1974) show that a VAR( $p$ ) model for  $n$  variables implies univariate ARMA( $np$ ,  $(n-1)p$ ) models for each of the individual variables. To describe some stochastic process for one variable, the required order of an encompassing high dimensional VAR model will be much lower than the order of a univariate AR model.

### 3. Calculating model parameters from eigenvalues and eigenvectors

The third and last important thing to note is the fact that one cannot only calculate the eigenvalues and eigenvectors from the parameter matrices, one can also, the other way around, calculate the parameter matrices from some eigenvalues and eigenvectors specified<sup>19</sup>. If  $A_e$  is an  $n \times n$  matrix in terms of the standard basis and  $A_v$  is the same matrix with respect to the basis consisting of the eigenvectors  $v_1, \dots, v_n$  then these can be calculated from one another through the relation

$$A_e = P A_v P^{-1} \quad (3.3.15)$$

$$\text{where } A_v = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \text{ and } P = \begin{bmatrix} \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{1,n} \end{pmatrix} & \dots & \begin{pmatrix} v_{n,1} \\ \vdots \\ v_{n,n} \end{pmatrix} \end{bmatrix}.$$

<sup>19</sup> Note that this may prove a very useful property when applying VAR models to generate scenarios for the purpose of ALM as describes in Chapter 1. By defining different eigenvalues and eigenvectors one can directly influence the crucial properties of the scenarios, for example to perform a sensitivity analysis.



So  $A_v$  is the zero matrix with the  $n$  eigenvalues  $\lambda_i$  on the main diagonal and  $P$  is the matrix having the  $n$  eigenvectors as its columns. Calculating the parameters of a VAR model from the  $R$ ,  $\omega$ ,  $\Delta$  and  $\phi$  terms of the eigenvalues and eigenvectors now simply comes down to first constructing the eigenvalues and eigenvectors from these parameters and then applying (3.3.15). For univariate AR( $p$ ) or higher order VAR( $p$ ) models this approach can be combined with the eigenvalue and eigenvector relations (3.3.12) and the formulations of the derived VAR(1) models as given by (3.2.35) and (3.2.40).

Let us now return to the example VAR(2) model (3.3.2) for  $n=2$  and see what the eigenvalues and eigenvectors of its parameter matrices tell us about the dynamic behavior of the model. The four eigenvalues of the derived matrix are

$$\begin{aligned}\lambda_1 &= 0.167 + 0.890i = 0.905 \exp(i \cdot 0.220 \cdot 2\pi) \\ \lambda_2 &= 0.167 - 0.890i = 0.905 \exp(-i \cdot 0.220 \cdot 2\pi) \\ \lambda_3 &= 0.833 \\ \lambda_4 &= -0.366\end{aligned}$$

The first two rows of the corresponding eigenvectors have been normalized with respect to the first row because of the non-uniqueness of the eigenvectors. This way it is easy to assess the relative amplitudes and phases of the eigenvalues in the two variables. The result is

$$\begin{aligned}v_1 &= \begin{bmatrix} 1 \\ -1.414 - 3.504i \end{bmatrix} = \begin{bmatrix} 1 \\ 3.778 \exp(-i \cdot 0.311 \cdot 2\pi) \end{bmatrix} \\ v_2 &= \begin{bmatrix} 1 \\ -1.414 + 3.504i \end{bmatrix} = \begin{bmatrix} 1 \\ 3.778 \exp(i \cdot 0.311 \cdot 2\pi) \end{bmatrix} \\ v_3 &= \begin{bmatrix} 1 \\ 0.466 \end{bmatrix} \\ v_4 &= \begin{bmatrix} 1 \\ 0.057 \end{bmatrix}\end{aligned}$$

From these eigenvalues and normalized eigenvectors we can see several things. First of all the amplitudes of all eigenvalues are smaller than one indicating a dampened deterministic simulation which satisfies the stationary condition. Second we see that two of the eigenvalues are complex conjugates indicating (pseudo) periodic behavior with a period length of about  $1/0.22 \approx 4.5$  time periods. With a modulus of 0.905 this periodic behavior is slowly dampened. The other two eigenvalues are real valued, one positive and one negative. The negative real eigenvalue implies an alternating pattern in the deterministic simulation. However because in absolute value it is rather small it is strongly dampened which explains why it is hard to see in the deterministic simulation in Figure 3.1. The positive real eigenvalue causes slowly dampened non-cyclical patterns in the deterministic simulation. The normalized eigenvectors give us information on how these four eigenvalues turn up in the two variables. The modulus in the second row of  $v_1$  and  $v_2$  is 3.778. This indicates that the pseudo period behavior of the complex eigenvalue is almost four times greater in the second

variable than it is in the first variable. The phase shift between the first and second variable with respect to this complex eigenvalue can be seen from the argument of the second row element of  $v_1$  and  $v_2$ . Dividing this by the argument of the corresponding eigenvalue tells us that the second variable leads the first variable by approximately  $-0.311/0.220 \approx -1.4$  time periods. The minus sign indicates that in fact this is a lag of about 1.4 time periods. The eigenvector  $v_4$  is not very relevant because of the low modulus of the real eigenvalue  $\lambda_4$ . However the eigenvector  $v_3$  shows that the slow decay of the positive real eigenvalue should be more than twice as clear in the first variable than it is in the second variable. All these properties indicated by the eigenvalues and eigenvectors are consistent with what we already knew from the deterministic simulation in Figure 3.1.

### 3.4 Estimation

The previous sections discussed autoregressive models assuming that the parameters of the models are known. When using models to investigate the properties of empirical time series this will of course not be the case. The models will have to be estimated on real data. With respect to the estimation of autoregressive models there are two important questions. The first is how to decide on the order  $p$  of the model? Several criteria designed for this purpose are described in section 3.5. Once the order is “known” the second question is what technique to use to actually estimate the parameters of the model? This section describes several estimation procedures, some of which are well known while others are rarely encountered in conventional econometric research. The focus is here on the aspects most relevant for the research presented in Part III and Part IV. Further details on the more familiar techniques can be found in econometric textbooks such as Stewart (1991) and Judge et al. (1980) while Lütkepohl (1991) discusses the estimation of multivariate VAR models.

#### 3.4.1 Ordinary Least Squares and Maximum Likelihood

The well known estimation method of *Ordinary Least Squares* (OLS) is based on the *Classical Regression Model* which was developed at the start of the twentieth century. This model consists of the following linear equation

$$y_i = \beta_1 + \beta_2 x_{2,i} + \dots + \beta_p x_{p,i} + \varepsilon_i \quad \text{for } i = 1, \dots, T \quad (3.4.1)$$

where

$$\varepsilon_i \sim N(0, \sigma^2) \quad \text{and} \quad E(\varepsilon_i \varepsilon_s) = 0 \quad \text{for } i \neq s$$

This equation explains  $T$  observations of the variable  $y$  from a constant term ( $x_1$ ) and a linear combination of the variables  $x_2, \dots, x_p$  which are also called *regressors* or *explanatory variables*. The independent *error terms*  $\varepsilon_i$  fill in the gap between the true values of  $y$  on the left hand side and the predicted or explained values of  $y$  on the right hand side. Strictly speaking, in this model the variable  $y$  is endogenous while  $x_2, \dots, x_p$  are fixed exogenous variables. Note that, contrary to the time series models discussed in the previous sections, originally this model was designed as a *cross section* model explaining for example a person’s income from variables such as education, age, place of birth, etc. In matrix notation the model becomes

$$y = X\beta + \varepsilon \quad (3.4.2)$$

where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix}, X = \begin{bmatrix} 1 & x_{2,1} & \cdots & x_{p,1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2,T} & \cdots & x_{p,T} \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \text{ and } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_T \end{bmatrix} \sim N(0, \sigma^2 I)$$

The *Least Squares* (LS) criterion means that given observed values for  $y$  and  $X$ , the parameters  $\beta$  and  $\sigma^2$  are estimated such that the sum of squared residuals  $\hat{\varepsilon}_i$ , the empirical counterparts of the error terms  $\varepsilon_i$ ,

$$V(\hat{\beta}) = \sum_{i=1}^T \hat{\varepsilon}_i^2 = \hat{\varepsilon}' \hat{\varepsilon} = (y - X\hat{\beta})'(y - X\hat{\beta}) \quad (3.4.3)$$

is minimized as a function of the estimated parameters  $\hat{\beta}$ . The solution to this minimization problem is the well known OLS estimator

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'y \quad (3.4.4)$$

Inverting  $X'X$  is only possible if the columns of  $X$  are linear independent, that is, if none of the explanatory variables  $x_2, \dots, x_p$  is a linear combination of the other variables. Assuming the model is correctly specified (i.e.  $y = X\beta + \varepsilon$  indeed is the “true” model) it can be shown that

$$\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1}) \quad (3.4.5)$$

which indicates that  $\hat{\beta}$  is an unbiased estimator of the true  $\beta$  parameters. The unbiased estimator for the variance of the error terms is

$$\hat{\sigma}_{OLS}^2 = \frac{1}{T-p} \sum_{i=1}^T \hat{\varepsilon}_i^2 = \frac{\hat{\varepsilon}' \hat{\varepsilon}}{T-p} \quad (3.4.6)$$

for which can be shown to hold

$$\frac{\hat{\varepsilon}' \hat{\varepsilon}}{\sigma_{OLS}^2} \sim \chi_{T-p}^2 \quad (3.4.7)$$

In order to assess the quality of a model estimated by OLS, one can use the familiar measure for the goodness of fit given by

$$R^2 = 1 - \frac{\sum_{i=1}^T \hat{\varepsilon}_i^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \quad (3.4.8)$$

which indicates to what extent the variance of the endogenous variable  $y$  can be explained by the model. Assuming the model contains a constant term, it holds that  $0 \leq R^2 \leq 1$ . By simply including just as many explanatory variables as there are observations it is always possible to achieve a perfect fit, i.e.  $R^2=1$ . Because obviously this does not have to be an adequate model<sup>20</sup> also the adjusted  $R^2$

$$\bar{R}^2 = 1 - \frac{(1 - R^2)(T - 1)}{T - p} \quad (3.4.9)$$

can be used which puts a penalty on increasing the number of explanatory variables  $p$ . A second way to measure the quality of a model estimated by OLS is by statistical testing whether the model parameters are different from zero. For this purpose the  $t$ -values of the estimated parameters can be used. The statistical properties of the estimators (3.4.5) and (3.4.7) imply

$$t_j = \frac{\hat{\beta}_j - \beta_j}{\hat{\sigma} \sqrt{a_{j,j}}} \sim t_{T-p} \quad (3.4.10)$$

where  $a_{j,j}$  is the  $j$ -th diagonal element of  $(X'X)^{-1}$  and  $\hat{\sigma} \sqrt{a_{j,j}}$  is the estimated standard deviation or *standard error* of  $\hat{\beta}_j$ . The relation (3.4.10) can be used to test the null-hypothesis  $H_0: \beta_j=0$  versus the alternative  $H_1: \beta_j \neq 0$  following standard statistical testing procedures.

In practice not all assumptions of the classical regression model, needed for example to derive the statistical properties of the OLS estimators, will always be satisfied. Fortunately, there exist asymptotic results that indicate that in such cases the results still hold approximately where the quality of the approximation increases with the sample size. One important violation of the classical assumptions, which is particularly relevant for the research presented here, occurs when applying the OLS technique to estimate AR models. In an AR model the regressors on the right hand side of (3.4.1) become lagged values of the same endogenous variable on the left hand side and can therefore no longer be seen as fixed and exogenous but become stochastic variables themselves. In that case the OLS estimators are no longer unbiased though the bias becomes smaller as the sample size increases. Provided the error terms are independent and identically distributed and the process is stationary, the OLS estimators of an AR model can be shown to be consistent (i.e. to have a variance that decreases with the sample size) and asymptotically Normally distributed. In that case the OLS technique can still be applied to estimate AR

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<sup>20</sup> Also see the remark at the end of section 1.3 and the corresponding footnote for a more general comment on using  $R^2$  for measuring the quality of a model.

models although one has to be aware of the limitations of the asymptotic justification when estimating AR models on small samples. Also the underlying assumptions of zero (auto)correlation in the residuals and stationarity of the process have to be tested.

Statistical testing for *residual serial correlation* in AR models can be done by using tests such as the Ljung-Box (also called modified Box-Pierce) or de Breusch-Godfrey test. Note that the occurrence of such correlations between the error terms is more likely in time series models than it is in cross section models. For testing *stationarity* of a process an extensive literature exists on so called *unit root tests* and their statistical properties. Theory and practice of these tests can be found in for example Fuller (1985) and Dickey, Bell and Miller (1986). Related to this topic is the well known danger of so called “spurious regressions”. This means that when two independent non-stationary variables are regressed on each other using OLS techniques, statistical testing using *t*-values may lead to the conclusion that there exists a statistical significant relation between these variables. However, because of the independence of the variables this relation obviously has to be spurious. This phenomenon is an example of the false conclusions which may be drawn from statistical testing procedures if the assumptions underlying the tests are not satisfied. Final examples of statistical testing procedures to test the validity of the assumptions of the classical regression model are the Jarque-Bera test for *Normality* of the residuals and various tests for *heteroskedasticity*, that is a non-constant volatility, of the residuals.

Besides the LS criterion, another justification of the OLS estimators comes from the more general principle of *Maximum Likelihood* (ML). From (3.4.1) it follows

$$\varepsilon_i = y_i - x_i\beta \sim N(0, \sigma^2) \Leftrightarrow y_i \sim N(x_i\beta, \sigma^2) \quad (3.4.11)$$

Because of the assumed independence, the joint *probability density function* of the observations  $y_1, \dots, y_T$ , given the parameters  $\beta$  and  $\sigma^2$ , is therefore

$$\begin{aligned} p(y_1, \dots, y_T \mid \beta, \sigma^2) &= \prod_{i=1}^T (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(y_i - x_i\beta)^2}{2\sigma^2}\right\} \\ &= (2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}\right\} \end{aligned} \quad (3.4.12)$$

The *likelihood function* has the same shape as (3.4.12) but is interpreted the other way around, as a function of the parameters  $\beta$  and  $\sigma^2$ , given the observations  $y_1, \dots, y_T$ .

$$L(\beta, \sigma^2 \mid y_1, \dots, y_T) = (2\pi\sigma^2)^{-T/2} \exp\left\{-\frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}\right\} \quad (3.4.13)$$

Note that this notation does not mean to imply that the fundamental interpretation of the arguments has changed.  $\beta$  and  $\sigma^2$  are still the fixed parameters of the probability density function and  $y_1, \dots, y_T$  is still a sample from the underlying stochastic process. The ML principle is now to estimate the parameter values as the values that maximize this likelihood function and thereby identify the model that “most likely”

generated the observed samples of  $y$  and  $X$ . To find these parameters the first order derivatives of the (natural) logarithm of (3.4.13)<sup>21</sup> with respect to  $\beta$  and  $\sigma^2$  are set to zero and can be solved to result in

$$\hat{\beta}_{ML} = (X'X)^{-1}X'y \quad (3.4.14)$$

and

$$\hat{\sigma}_{ML}^2 = \frac{1}{T} \sum_{i=1}^T \hat{\varepsilon}_i^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{T} \quad (3.4.15)$$

From this it can be seen that the ML estimator of  $\beta$  is exactly the same as the OLS estimator (3.4.4). and that the only difference between the ML and OLS estimator of  $\sigma^2$  is that the ML estimator divides the sum of squared residuals by  $T$  and the OLS estimator by  $T-p$ , i.e. the sample size corrected for the number of parameters estimated. The ML estimator for  $\sigma^2$  is therefore biased but the bias decreases with an increasing sample size.

For testing statistical significance of ML estimators, asymptotic results exist which indicate that ML estimators are consistent and asymptotically Normally distributed. Also it can be shown that, within the class of all such estimators, the ML estimators have the smallest variance making them asymptotically efficient estimators within the class of consistent and Normally distributed estimators.

### 3.4.2 Seemingly Unrelated Regressions

Before going into the OLS estimation of multivariate VAR models, we first look at the situation in which the assumption  $\varepsilon \sim N(0, \sigma^2 I_T)$  of the classical regression model (3.4.2) is dropped and replaced by the more general specification  $\varepsilon \sim N(0, \sigma^2 V)$  where  $V$  is a positive definite  $T \times T$  matrix<sup>22</sup>. Because now the error terms no longer to be independent, the standard OLS theory no longer applies. For example it can be shown that for the OLS estimator now holds

$$\hat{\beta} \sim N\left(\beta, \sigma^2 (X'X)^{-1} X' V X (X'X)^{-1}\right) \quad (3.4.16)$$

instead of (3.4.5). Although the estimator  $\hat{\beta}$  is still unbiased, its variance is no longer  $\sigma^2 (X'X)^{-1}$  which was used for example to derive the distribution of the standard  $t$ -values. In other words, applying standard OLS estimates and theory may lead to false conclusions about statistical relations between variables if  $V \neq I_T$ . Furthermore the OLS estimator is no longer the unbiased estimator with the smallest variance. The estimator that does have this property is the known as the *Generalized Least Squares* (GLS) estimator.

<sup>21</sup> The  $\ln$  operator is a monotonic transformation of (3.4.12) and therefore implies the same optimal values of  $\beta$  and  $\sigma^2$  as the original likelihood function. It however simplifies the solving of the first order conditions because of the resulting linear format.

<sup>22</sup> A square matrix  $A$  is positive definite if  $x'Ax > 0$  for any vector  $x$ . If  $x'Ax \geq 0$ ,  $A$  is called positive semi definite. Statistically the requirement that  $V$  is positive definite implies that any linear combination of the error terms in  $\varepsilon$  has a variance greater than zero. Stochastic variables with a negative variance cannot exist. So this basically comes down to requiring that  $\varepsilon$  follows a "valid" stochastic process. Also see the requirements for the YW estimation technique in section 3.4.4.

To derive this estimator, first assumed that the matrix  $V$  is known. Because it is positive definite, an invertible matrix  $L$  exists such that

$$L' L = V^{-1} \text{ and } L V L' = I_T \quad (3.4.17)$$

Using the matrix  $L$ , the model (3.4.2) can be transformed as

$$L y = L X \beta + L \varepsilon \quad (3.4.18)$$

where

$$E(L \varepsilon)^2 = L(\sigma^2 V)L' = \sigma^2 L V L' = \sigma^2 I_T \quad (3.4.19)$$

Because the error terms  $L \varepsilon$  now are uncorrelated again, the transformed model satisfies the assumptions of the classical regression model and standard OLS techniques can be applied. In terms of the original model this however gives the GLS estimator

$$\hat{\beta}_{GLS} = (X' V^{-1} X)^{-1} X' V^{-1} y \quad (3.4.20)$$

for which holds

$$\hat{\beta}_{GLS} \sim N\left(\beta, \sigma^2 (X' V^{-1} X)^{-1}\right) \quad (3.4.21)$$

The unbiased GLS estimator for  $\sigma^2$  is

$$\hat{\sigma}_{GLS}^2 = \frac{1}{T-p} (L y - L X \hat{\beta}_{GLS})' (L y - L X \hat{\beta}_{GLS}) \quad (3.4.22)$$

When applying this estimator in practice, the matrix  $V$  is not known and a two-step procedure has to be applied. In the first step the estimate  $\hat{V}$  and its decomposing matrix  $\hat{L}$  are calculated from the standard OLS residuals. In the second step the model is transformed as

$$\hat{L} y = \hat{L} X \beta + \hat{L} \varepsilon \quad (3.4.23)$$



Because of its dependence on  $y$ ,  $\hat{L}$  is as stochastic matrix and the regressors of this transformed model are no longer fixed and exogenous. Just as with the AR models however the results of the classical regression model remain valid as an approximation which results in the so called “feasible” GLS estimator

$$\hat{\beta}_{GLS} \sim N\left(\beta, \sigma^2 (X' \hat{V}^{-1} X)^{-1}\right) \quad (3.4.24)$$

An obvious application of such GLS estimators is when estimating multivariate VAR( $p$ ) models. It seems logical to estimate such a model, as given by (3.3.1), by applying OLS for each separate equation and subsequently estimating the covariance matrix  $\Sigma$  of the error process from the residuals of each of the equations. Although this does result in unbiased estimators for the parameter matrices  $A_i$ ,  $i=1, \dots, p$ , these estimators need not be the most efficient (i.e. with the lowest variance) if the error terms of the various equations are correlated. Estimating such a model in which

$$\sigma^2 V = \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,T} \\ \vdots & \ddots & \vdots \\ \sigma_{1,T} & \cdots & \sigma_{T,T} \end{bmatrix} \quad (3.4.25)$$

with  $\sigma_{i,j} \neq 0$  for some  $i \neq j$ , is also called a *Seemingly Unrelated Regression* (SUR) because the equations of the model are in a way “invisibly” connected by the correlations in the error process. To see the possible deficiencies of estimating each equation of such a model separately by OLS, write a general VAR( $p$ ) model with its  $n$  equations as a model with one equation in the notation of the classical regression model. This can be done by stacking all equations at all times  $t$  in the sample on top of each other. This yields the equivalent model

$$\bar{y} = \bar{X}\bar{\beta} + \bar{\varepsilon} \quad (3.4.26)$$

where

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \bar{X} = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}, \bar{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \text{ and } \bar{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

with

$$y_j = \begin{bmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,T} \end{bmatrix}, X_j = \begin{bmatrix} 1, x_{1,0} \cdots x_{n,0} & x_{1,-1} \cdots x_{n,-1} & \cdots & x_{1,1-p} \cdots x_{n,1-p} \\ 1, x_{1,1} \cdots x_{n,1} & x_{1,0} \cdots x_{n,0} & \cdots & x_{1,2-p} \cdots x_{n,2-p} \\ \vdots & \vdots & \ddots & \vdots \\ 1, x_{1,T-1} \cdots x_{n,T-1} & x_{1,T-2} \cdots x_{n,T-2} & \cdots & x_{1,T-p} \cdots x_{n,T-p} \end{bmatrix},$$

$$\beta_j = \begin{bmatrix} (v_j, a_{1,j,1} \cdots a_{1,j,n})' \\ (v_j, a_{2,j,1} \cdots a_{2,j,n})' \\ \vdots \\ (v_j, a_{p,j,1} \cdots a_{p,j,n})' \end{bmatrix} \text{ and } \varepsilon_j = \begin{bmatrix} \varepsilon_{j,1} \\ \varepsilon_{j,2} \\ \vdots \\ \varepsilon_{j,T} \end{bmatrix} \text{ for } j = 1, \dots, n$$

and

$$E(\bar{\varepsilon}\bar{\varepsilon}') = \bar{\Sigma} = \Sigma \otimes I_T = \begin{bmatrix} \sigma_{1,1}I_T & \cdots & \sigma_{1,n}I_T \\ \vdots & \ddots & \vdots \\ \sigma_{n,1}I_T & \cdots & \sigma_{n,n}I_T \end{bmatrix}$$

For the size of the various matrices holds that  $y_j$  and  $\varepsilon_j$ :  $(T \times 1)$ ,  $X_j$ :  $(T \times (p+1)n)$ ,  $\beta_j$ :  $((p+1)n \times 1)$ ,  $\Sigma$ :  $(n \times n)$ ,  $\bar{y}$  and  $\bar{\varepsilon}$ :  $(Tn \times 1)$ ,  $\bar{X}$ :  $(Tn \times (pn^2 + n))$ ,  $\bar{\beta}$ :  $((pn^2 + n) \times 1)$  and  $\bar{\Sigma}$ :  $(Tn \times Tn)$ . If  $\sigma_{i,j} \neq 0$  for some  $i \neq j$ ,  $\bar{\Sigma}$  cannot be written as  $\sigma^2 I_{Tn}$  and therefore the GLS instead of the OLS estimator has to be used. Because the derived (3.4.26) model is an ordinary one equation model, the feasible GLS estimator (3.4.24) can be applied to estimate  $\bar{\beta}$ , and thereby also the parameters  $v$  and  $A_i$ ,  $i=1, \dots, p$  of the original VAR( $p$ ) model, as

$$\bar{\beta}_{GLS} = (\bar{X}' \hat{\bar{\Sigma}}^{-1} \bar{X})^{-1} \bar{X}' \hat{\bar{\Sigma}}^{-1} \bar{y} \quad (3.4.27)$$

For this, first  $\hat{\bar{\Sigma}}$  must be estimated by applying OLS to each equation separately and estimating the covariance between the error terms from the residual vectors  $\hat{\varepsilon}_j$  for  $j=1, \dots, n$  by the unbiased estimators

$$\hat{\sigma}_{i,j} = \frac{\hat{\varepsilon}_i' \hat{\varepsilon}_j}{T - np - 1} \text{ for } i, j = 1, \dots, n \quad (3.4.28)$$

An extension to this two-step approach is an iterative GLS procedure in which each set of GLS estimates is used to recalculate the residuals and obtain a new estimate of  $\bar{\Sigma}$  based on which the GLS estimator of the next iteration can be calculated. This procedure is repeated until it converges. The asymptotic statistical properties of the iterative GLS estimate and of the two step estimator are the same. However, only iterating until convergence can be shown to lead to the ML estimators.

Theoretically, in general, GLS estimators will be more efficient (i.e. have a lower variance) than the direct OLS estimates. In the specific case of an unrestricted VAR model however the resulting estimates turn out to be identical and there are no efficiency gains from estimating by GLS. This is so because there are two cases in which the GLS estimator (3.4.27) can be shown to reduce to the OLS estimator

$$\bar{\beta} = (\bar{X}' \bar{X})^{-1} \bar{X}' \bar{y} \quad (3.4.29)$$

which is equivalent to applying OLS for each of the  $n$  equations separately. The first case is the (trivial) situation in which all error terms are uncorrelated so that  $\sigma_{ij}=0$  for all  $i \neq j$ . The second case in which OLS and GLS estimates are the same is the situation where the regressor matrices  $X_j$  are identical for all equations  $j=1, \dots, n$ . Because this is clearly the case for an unrestricted VAR model, it is equally efficient to estimate an unrestricted VAR model equation by equation using OLS and subsequently estimating the error covariance matrix from the residual series than it is to estimate the model by GLS.

Just as for the univariate AR models, the OLS results, such as the  $t$ -tests, still apply approximately despite the stochastic character of the regressors in the case of VAR models. Most of the testing procedures on aspects such as Normally distributed residuals or residual autocorrelation also have a multivariate version. Furthermore, the OLS estimators are equal to the ML estimators in the case of unrestricted VAR models. A problem is however, as noted before, the large number of parameters of VAR models when estimating them on small samples of data. The only way to increase the efficiency of the estimates in such cases is to apply restrictions to the model, for example by deleting some of the parameters from the estimation process and give them a zero value. Because the omission of some of the relevant regressors in one or more of the  $X_j$  matrices, these are no longer identical for all  $n$  equations and the GLS procedure has to be applied to obtain the most efficient estimates. Furthermore the number of estimated parameters in each equation is different in that case, which is why it is practical to use the biased estimators of the residual covariances

$$\hat{\sigma}_{i,j} = \frac{\hat{\varepsilon}_i \hat{\varepsilon}_j}{T} \text{ for } i, j = 1, \dots, n \quad (3.4.30)$$

### 3.4.3 Top-Down and Bottom-Up strategies

If, as suggested at the end of the previous section, one decides to delete some of the parameters from a VAR model to increase the efficiency of the estimates, one of course needs some kind of procedure or criterion that indicates which parameters best to delete. Here one may for example be guided by the statistical significance of the parameters, step by step deleting the, judged by their  $t$ -values, most insignificant parameters from the model until all parameters fall within some desired significance level. Once the restrictions are known, the resulting so called *subset* VAR model can be estimated by a GLS procedure. Another type of strategy to find the “optimal” restrictions, is to estimate various models with different parameter combinations and choosing the model that is optimal according to some chosen criterion. In practice it is not unusual to use so called *order selection criteria* such as the *Akaike Information*

*Criterion* (AIC) or the *Schwarz Criterion* (SC) for this purpose. These and other order selection criteria will be discussed in the next section. For now it is only necessary to know that these criteria all make some kind of trade off between the variance of the residuals, i.e. a measure of the fit of the model to the sample data, on the one hand and the number of parameters to be estimated needed to obtain this fit, on the other. For each estimated model, the value of such a criterion can be calculated and the “optimal” order is the order of the model with the lowest criterion value.

Suppose some maximum order  $p$  is known. One strategy can then be to simply estimate all possible subset VAR models and choose the one with the lowest value of the chosen order selection criterion. Because the required computational effort rapidly increases with the dimension and order of the model, the practical usefulness of such a strategy is limited. For an  $n$ -dimensional VAR( $p$ ) model there exist

$$\sum_{j=0}^{n^2 p} \binom{n^2 p}{j} = 2^{n^2 p} \quad (3.4.31)$$

possible subset VAR models, each with a different combination of parameters to be estimated. For a bivariate VAR(4) model this already amounts to  $2^{16}=65536$  subset models, while in practice especially the dimension will be (much) greater than two. Two more practical strategies to search within all possible subset models for the optimal one are a *Top-Down strategy* and a more elaborate *Bottom-Up strategy* which are both explained next.

#### *Top-Down strategy*

Given some maximum order  $p$ , a Top-Down strategy starts from an unconstrained  $n$ -dimensional VAR( $p$ ) model and subsequently deletes parameters in an iterative manner from each of the  $n$  equations separately. The strategy consists of the following steps.

1. Determine a maximum order  $p$ . For this, one or more of the various order selection criteria as discussed in section 3.5 can be used.
2. Choose an order selection criterion to be used in the optimization of the parameter restrictions by finding its minimum value. It is known that in general the use of the Akaike Information Criterion (AIC) leads to a model with the most parameters while use of the Schwarz Criterion (SC) leads to a more parsimonious model. These and other criteria are described in the section 3.5.
3. For each of the  $n$  equations

$$\begin{aligned} x_{j,t} = & v_j + a_{1,j,1}x_{1,t-1} + \dots + a_{1,j,n}x_{n,t-1} & \text{for } j = 1, \dots, n \\ & + a_{2,j,1}x_{1,t-2} + \dots + a_{2,j,n}x_{n,t-2} \\ & \vdots \\ & + a_{p,j,1}x_{1,t-p} + \dots + a_{p,j,n}x_{n,t-p} + \varepsilon_{j,t} \end{aligned} \quad (3.4.32)$$

go through the following steps.

- a) Estimate the parameters of the complete equation using OLS and calculate the value of the chosen criterion.

- b) Delete the last parameter  $a_{p,j,n}$  by excluding the regressor  $x_{n,t-p}$  from the estimation. Recalculate the value of the criterion. If it is greater than the previous value, permanently maintain the parameter  $a_{p,j,n}$  in the equation. If the criterion value has decreased, thereby indicating a “better” parameter combination is found, permanently delete the parameter from the equation<sup>23</sup>.
- c) Repeat steps a) and b) only now with the one-but-last parameter  $a_{p,j,n-1}$  and all other parameters until the constant  $v_j$ . In each step one of the parameters in the equations is set to zero if this decreases the value of the order selection criterion.
4. Given the parameter restrictions found in step 3, estimate the complete model using GLS.

### *Bottom-Up strategy*

A drawback from such a Top-Down strategy may be that for each of the equations it starts by estimating all parameters at once. Especially in the case of a high order model of considerable dimensions, these initial estimates will not be very accurate causing a bad start of the strategy. A Bottom-Up strategy tries to overcome this problem. Such a strategy also tries to find the optimal parameter restrictions for each of the  $n$  equations separately. It consists of the following steps.

1. Determine a maximum order  $p$ .
2. Choose an order selection criterion to be used in the optimization of the parameter restrictions.
3. For each of the  $n$  equations (3.4.32) go through the following steps.
  - a) Estimate the equation by respectively including the first  $0, 1, \dots, p$  lags of the first variable  $x_{1,t}$  and determine the order  $p_1$  which minimizes the chosen criterion. That is, select the equation

$$x_{j,t} = v_j + a_{1,j,1}x_{1,t-1} + \dots + a_{p_1,j,1}x_{1,t-p_1} + \varepsilon_{j,t} \quad (3.4.33)$$

by estimating

$$x_{j,t} = v_j + a_{1,j,1}x_{1,t-1} + \dots + a_{i,j,1}x_{1,t-i} \quad \text{for } i = 0, \dots, p \quad (3.4.34)$$

and calculating and comparing the values of the order selection criterion.

- b) Keep  $p_1$  fixed and now add respectively the first  $0, 1, \dots, p$  lags of the second variable  $x_{2,t}$  to the equation already containing the  $p_1$  lags of the first variable. Determine the order  $p_2$  which minimizes the chosen criterion. This yields the equation

$$\begin{aligned} x_{j,t} = v_j + a_{1,j,1}x_{1,t-1} + \dots + a_{p_1,j,1}x_{1,t-p_1} \\ + a_{1,j,2}x_{2,t-1} + \dots + a_{p_2,j,2}x_{2,t-p_2} + \varepsilon_{j,t} \end{aligned} \quad (3.4.35)$$

Repeat this procedure until for all variables  $x_{1,t}, \dots, x_{n,t}$  the orders  $p_1, \dots, p_n$  are found.

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<sup>23</sup> Note that following such a procedure means that different orderings of the variables will actually lead to different models.

4. Starting from the equation defined by the orders  $p_1, \dots, p_n$ , apply a Top-Down strategy to possibly set intermediate parameters (i.e. with a lag less than the  $p$ ) to zero.
5. Given the parameter restrictions found in step 3 and 4, estimate the complete model using GLS.

#### 3.4.4 Yule-Walker

Section 3.3 illustrated the relations between the dynamics of a VAR model on the one hand and the modulus and argument of its roots on the other. From this it became clear that the dynamic properties of a VAR model typically extend over more than one time period. However, the OLS procedure described in the previous sections is based on the assumption that the best estimator of a VAR model is the one that minimizes the variance of only the one-step ahead prediction errors. In the OLS approach therefore implicitly each false prediction is corrected before the next one-step prediction error is calculated. The accumulation of prediction errors over several time periods does not occur and the OLS criterion does not directly judge the multi-period fit of the model. Therefore it is not clear at first sight why minimizing the one-step ahead prediction errors would also lead to the best estimate of the multi-period dynamics of a VAR model.

One alternative to meet this doubt may be to minimize (also) for example the two or three period prediction errors instead of (only) the one-step prediction errors. Another alternative is to apply the following *Yule-Walker* (YW) estimation technique which more directly relates to the multi-period dynamics of VAR models. As the name suggests it is based on the Yule-Walker equations (3.2.17) and (3.2.30) for respectively the univariate and multivariate case. Here only the univariate case is discussed which has a straightforward multivariate version. As shown before, the YW equations can be used both to calculate autocovariances from model parameters and, the other way around, to calculate model parameters from autocovariances. The simple YW estimation procedure therefore consists of the following steps.

1. Use the *unbiased* estimators

$$\hat{\mu} = \frac{1}{T} \sum_{t=0}^{T-1} x_t \quad (3.4.36)$$

and

$$\hat{\gamma}_k = \frac{1}{T-k-1} \sum_{t=|k|}^{T-1} (x_t - \hat{\mu})(x_{t-|k|} - \hat{\mu}) \quad (3.4.37)$$

to estimate the average  $\mu$  and the  $p+1$  autocovariances  $\gamma_0, \dots, \gamma_p$  from the sample data. An alternative for (3.4.37) is to use the *biased* estimator

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=|k|}^{T-1} (x_t - \hat{\mu})(x_{t-|k|} - \hat{\mu}) \quad (3.4.38)$$

2. Estimate the parameters of the corresponding AR( $p$ ) model as

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_p \end{bmatrix}_{YW} = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{p-1} \\ \hat{\gamma}_1 & \hat{\gamma}_0 & \cdots & \hat{\gamma}_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}_{p-1} & \hat{\gamma}_{p-2} & \cdots & \hat{\gamma}_0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \vdots \\ \hat{\gamma}_p \end{bmatrix} \quad (3.4.39)$$

$$\hat{\sigma}_{YW}^2 = \hat{\gamma}_0 - \hat{\gamma}_1 \hat{\beta}_1 - \cdots - \hat{\gamma}_p \hat{\beta}_p \quad (3.4.40)$$

$$\hat{\nu}_{YW} = \hat{\mu}(1 - \hat{\beta}_1 - \hat{\beta}_2 - \cdots - \hat{\beta}_p) \quad (3.4.41)$$

Although in large samples the difference from using the unbiased estimator (3.4.37) or the biased estimator (3.4.38) for the autocovariances is small, in small samples this can result in very different estimates. Besides the obvious numerical differences also the fundamental properties of these estimators turn out to be very different. The fact is that using the *biased* estimator (3.4.38) always leads to a stationary AR model while for the unbiased estimator this need not be the case. To see this, simply note that if for the empirical autocovariances of any order  $k$  the sum of squares is divided by the original sample length  $T$ , the autocovariances are guaranteed to converge to zero as the order  $k$  increases. In section 3.3 it was shown that such decreasing autocovariances or correlations correspond to a stationary AR model. Because of the direct one-to-one relations in the YW equations these properties of the empirical autocorrelations directly apply to the corresponding AR model. If, instead,  $T-k-1$  is used as in the unbiased estimator, convergence of the autocovariances need not occur because each smaller sum of squares is also divided by a smaller number. Therefore stationarity is *not* guaranteed when using the *unbiased* estimators (3.4.37). So by using the biased estimators of the autocovariances in the YW method, one implicitly adds a stationary restriction to the model to be estimated. Of course applying such a restriction is only valid if the underlying stochastic process is indeed stationary<sup>24</sup>. If this is the case however, the extra information from such a restriction can increase the efficiency of the estimates compared to for example OLS which does not apply such a restriction. Besides the sample data, there is nothing from preventing an AR model estimated by OLS from being close to or even over the non-stationarity boundary. Note that this observation can become especially important when estimating large unrestricted VAR models on small samples of macroeconomic data.

A requirement for the parameters to be estimated by YW is of course that the matrix in (3.4.39) can indeed be inverted. For this it is sufficient that the matrix is *positive definite*. A symmetrical  $n \times n$  matrix  $B$  is positive definite if

$$a' B a > 0 \quad \forall \quad a = (a_1, \dots, a_n)' \quad (3.4.42)$$

<sup>24</sup> Note from the discussions in sections 3.1 and 3.4.1 that stationarity is often needed to estimate or even construct a sensible model. From this perspective the stationarity restriction in the (biased) YW estimation technique will rarely be considered an invalid one or be a limiting factor in some sense.



Furthermore it holds

$$\begin{aligned}
 & B \text{ positive definite} \\
 & \Leftrightarrow \\
 & \text{all eigenvalues } \lambda_i > 0 \text{ for } i = 1, \dots, n \\
 & \Leftrightarrow \\
 & |B| = \prod_{i=1}^n \lambda_i > 0
 \end{aligned} \tag{3.4.43}$$

To give some statistical intuition for this mathematical requirement look at all the variables from the vectors  $x_t = (x_{1,t}, \dots, x_{n,t})'$  of a general  $n$ -dimensional VAR( $p$ ) process for the periods  $t-q, \dots, t$  as individual continuous random variables. Assuming the VAR model has a zero mean, the covariancematrix for all these individual random variables is

$$\Omega_q = E \begin{bmatrix} x_t \\ \vdots \\ x_{t-q} \end{bmatrix} \begin{bmatrix} x_t' & \dots & x_{t-q}' \end{bmatrix} = E \begin{bmatrix} x_t x_t' & x_t x_{t-1}' & \dots & x_t x_{t-q}' \\ x_{t-1} x_t' & x_{t-1} x_{t-1}' & \dots & x_{t-1} x_{t-q}' \\ \vdots & \vdots & \ddots & \vdots \\ x_{t-q} x_t' & x_{t-q} x_{t-1}' & \dots & x_{t-q} x_{t-q}' \end{bmatrix} \tag{3.4.44}$$

If the process is stationary this is equivalent to

$$\Omega_q = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \dots & \Gamma_q \\ \Gamma_1' & \Gamma_0 & \dots & \Gamma_{q-1}' \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_q' & \Gamma_{q-1}' & \dots & \Gamma_0 \end{bmatrix} \tag{3.4.45}$$

A basic requirement of any valid univariate random variable is that it has a positive variance. This means that also any linear combination of the individual random variables in the vectors  $x_t = (x_{1,t}, \dots, x_{n,t})'$  for  $t-q, \dots, t$  must have a positive variance. If  $y_q = (x_t', \dots, x_{n,t-q}')'$  then this requires that

$$\text{Var}(a' y_q) = a' \text{Cov}(y_q) a = a' \Omega_q a > 0 \quad \forall \quad a = (a_1, \dots, a_{n(q+1)})' \tag{3.4.46}$$

which is equivalent to requiring that  $\Omega_q$  is a positive definite matrix. For a valid stochastic process  $\Omega_q$  has to be positive definite for any value of  $q$  which is equivalent to

$$|\Omega_q| > 0 \text{ for } q \geq 0 \tag{3.4.47}$$

If this is satisfied for  $q=0, \dots, p$ , the parameters of the corresponding VAR( $p$ ) model can be calculated from the Yule-Walker equations. Now note that the covariancematrix (3.4.45) is equivalent to the covariancematrix  $\Phi_0$  in (3.2.38) for the VAR(1) version of a VAR( $p$ ) model if  $q=p-1$ . Also  $\Phi_0$  reduces to the matrix in (3.4.39) in the univariate case. So, requiring that these matrices are positive definite essentially is the same as

requiring that the model describes a valid stochastic process and causes that the Yule-Walker equations can be solved for the model parameters.

In practical applications of the YW estimation procedure, especially for high dimensional ( $n$ ) models of a high order ( $p$ ), numerical problems can occur when inverting the covariance matrix in (3.2.31), even when the biased covariance estimators are used. In that case, the high dimensional  $np \times np$  covariance matrix  $\Omega_0$  may have very small negative eigenvalues which cause its inversion to fail because it is no longer a positive definite matrix as. A simple solution is to increase all the diagonal elements of the covariance matrix by a small factor which causes the matrix to become diagonal dominant and have only positive eigenvalues. Often, only a very small factor is needed (say 0.01%) and therefore the covariance matrix and hence the corresponding VAR model estimates will not have changed significantly<sup>25</sup>.

Because of the purely mathematical conversion of the estimated autocovariances into the model parameters, the YW method can be said to have a lesser statistical foundation than the OLS method. For example using  $t$ -tests to assess the statistical significance of the estimates is strictly speaking no longer valid. However, statistical testing procedures which do not specifically rely on the LS assumption, such as testing for Normality and autocorrelation in the residuals, are still possible. Furthermore of course the estimates for the mean, variance and autocorrelations, which are input to the YW procedure, have statistical properties. For empirical autocorrelations for example it is possible to specify confidence intervals which can be used to test whether they are statistically significant different from zero or not. The mathematical transformations therefore transform the primal uncertainty in these estimates into the statistical uncertainty of the YW estimates of the model parameters. More on the issue of testing significance in VAR models estimated by the YW method will be said in section 4.8. Note that unlike the OLS approach, for the YW method it has no use and also is not possible to perform parameter elimination techniques as discussed in 3.4.3 to increase the efficiency of the estimates. The YW procedure always calculates all parameters of a VAR model in such a way that the properties of the stochastic process it describes are by definition equal to the historical estimates of these properties. This property of the YW estimation procedure makes it especially useful for ALM or general risk management purposes. Boender and Romeijn (1991) describe an early ALM application of the YW estimation procedure for VAR models. The example in section 1.2 is another ALM application of this procedure.

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<sup>25</sup> Thanks go to Lucas Vermeulen for pointing out this simple solution.

Besides the direct matrix inversion, there also exists an other iterative procedure to solve the (univariate) YW equations. If  $\beta_{p,i}$  is defined as the  $i$ -th autoregressive parameter (i.e. with lag  $i$ ) from an AR( $p$ ) model, Appendix B.1 shows that these satisfy the relations

$$\begin{aligned}\beta_{p,i} &= \beta_{p-1,i} - \beta_{p,p}\beta_{p-1,p-i} \quad \text{for } p > 0 \text{ and } i < p \\ \beta_{p,p} &= \frac{\gamma_p - \sum_{j=1}^{p-1} \gamma_{p-j}\beta_{p-1,j}}{\sigma_{p-1}^2} \\ \sigma_p^2 &= \sigma_{p-1}^2 (1 - \beta_{p,p}^2)\end{aligned}\tag{3.4.48}$$

Here the variance of an AR( $p$ ) model is  $\sigma_p^2$  and the  $\gamma_k$  as usually indicate the autocovariances of the stochastic process described by the model. This recursion is also called the *Levinson recursion* and is formally derived in Levinson (1947) and Durbin (1960). By starting from  $\sigma_0^2 = \gamma_0$  in each step only the parameter  $\beta_{p,p}$  has to be calculated to know all other parameters  $\beta_{p,1}, \dots, \beta_{p,p-1}$  of the AR( $p$ ) model consistent with the autocovariances  $\gamma_0, \dots, \gamma_p$ . The parameters  $\beta_{i,i}$  for  $i=1, \dots, p$  are called the *reflection* or *partial correlation coefficients*. The last term comes from the fact that  $\beta_{i,i}$  describes the correlation between  $x_t$  and  $x_{t-i}$  corrected for the intermediate correlations with  $x_{t-1}, \dots, x_{t-i+1}$ . Also see Box and Jenkins (1976). The Levinson recursion can provide computational advantages when successively many models of increasing order have to be estimated. In the context of the research presented here, a more important property of the Levinson recursion is that it can be shown that an AR model is stationary if and only if the absolute value of all of the partial correlation coefficients is less than one. Using the biased estimators of the autocovariances when estimating by YW apparently implies that all partial correlation coefficients satisfy this stationarity condition. The next section describes an alternative estimation technique for univariate AR models that specifically makes use of the Levinson recursion and the stationarity condition in terms of the partial correlation coefficients.

### 3.4.5 Burg and Two sided Least Squares

In Chapter 4 the important concept of Maximum Entropy in the context of estimating spectral densities will be discussed. This concept was introduced by Burg (1967, 1975). For this purpose Burg (1968) also suggested another estimation technique for univariate AR models. Although the Maximum Entropy concept and this *Burg estimation technique* are strongly related, they may equally well be seen as two separate and distinct contributions from Burg's Maximum Entropy concept. A detailed description of the Burg method can be found in Andersen (1974), Haykin and Kessler (1976), Ulrych and Bishop (1975) and Ulrych et al. (1973). The method looks a lot like the OLS estimator while the Levinson recursion is incorporated to ensure that the estimated model satisfies the stationarity condition. Note that use of the Levinson recursion combined with an LS criterion is possible because (3.4.48) does not depend on the way the autocovariances are estimated. Unlike the YW technique, which as we know can also impose a stationarity restriction, the Burg method does not need a-priori estimates of the autocovariances. Instead it

guarantees stationarity by imposing that the absolute value of all partial correlation coefficients is less than one. To see how the algorithm works suppose the parameters  $\beta_{2,1}$  and  $\beta_{2,2}$  of some AR(2) model are already known and have to be used to estimate the parameters of an AR(3) model. Abstracting from a constant term, the AR(3) model looks like

$$x_t = \beta_{3,1}x_{t-1} + \beta_{3,2}x_{t-2} + \beta_{3,3}x_{t-3} + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(0, \sigma_3^2) \quad (3.4.49)$$

The Levinson recursion (3.4.48) implies that

$$\begin{aligned} \beta_{3,1} &= \beta_{2,1} - \beta_{3,3}\beta_{2,2} \\ \beta_{3,2} &= \beta_{2,2} - \beta_{3,3}\beta_{2,1} \\ \sigma_3^2 &= \sigma_2^2(1 - \beta_{3,3}^2) \end{aligned} \quad (3.4.50)$$

Substituting for  $\beta_{3,1}$ ,  $\beta_{3,2}$  and  $\sigma_3^2$  in (3.4.49) gives the following representation of the AR(3) model.

$$x_t = (\beta_{2,1} - \beta_{3,3}\beta_{2,2})x_{t-1} + (\beta_{2,2} - \beta_{3,3}\beta_{2,1})x_{t-2} + \beta_{3,3}x_{t-3} + \varepsilon_t \quad (3.4.51)$$

where  $\varepsilon_t \sim N(0, \sigma_2^2(1 - \beta_{3,3}^2))$ . Because  $\beta_{2,1}$  and  $\beta_{2,2}$  and  $\sigma_2^2$  are known from the previous iteration, the partial correlation coefficient  $\beta_{3,3}$  is the only parameter to be estimated. It now seems logical to estimate this parameter based on the LS criterion by minimizing the sum of squared prediction errors

$$V(\beta_{3,3}) = \sum_{t=4}^T (x_t - (\beta_{2,1} - \beta_{3,3}\beta_{2,2})x_{t-1} - (\beta_{2,2} - \beta_{3,3}\beta_{2,1})x_{t-2} - \beta_{3,3}x_{t-3})^2 \quad (3.4.52)$$

with respect to  $\beta_{3,3}$ . Given this estimate, the next partial correlation coefficient  $\beta_{4,4}$  of an AR(4) model can be estimated and so on. Box and Jenkins (1976) and Ulrych and Bishop (1975) show that this procedure leads to approximate ML estimates. Burg (1968) extends this procedure by noting that an univariate stationary process can equally well be read in both directions, forwards and backwards in time. Given that the dynamics of such a process are determined by its autocovariances  $\gamma_k$ , this is easily seen by noting that the autocovariances  $\gamma_k$  are symmetrical around  $k=0$  (i.e.  $\gamma_k = \gamma_{-k}$ ). By this reversal property, the parameter  $\beta_{3,3}$  in the example can also be estimated by simultaneously minimizing both the forward and backward prediction errors. That is, minimize

$$\begin{aligned} V^*(\beta_{3,3}) &= \sum_{t=4}^T (x_t - \beta_{3,1}x_{t-1} - \beta_{3,2}x_{t-2} - \beta_{3,3}x_{t-3})^2 \\ &\quad + \sum_{t=4}^T (x_{t-3} - \beta_{3,1}x_{t-2} - \beta_{3,2}x_{t-1} - \beta_{3,3}x_t)^2 \end{aligned} \quad (3.4.53)$$

in which  $\beta_{3,1}$  and  $\beta_{3,2}$  are substituted from (3.4.50). Andersen (1974) and Marple (1987) give detailed descriptions of an efficient recursive scheme to calculate these Burg estimators. Also the Burg estimators can be shown to give approximate ML

estimates. Furthermore it can be proved that all partial correlation coefficients estimated this way are in absolute value less than one, which guarantees that the related AR models are stationary. Just as with the YW method, the constant term can easily be estimated from (3.4.41) given values for  $\hat{\mu}, \hat{\beta}_1, \dots, \hat{\beta}_p$ .

Besides the univariate Levinson recursion and the Burg algorithm derived from it, also multivariate extensions exist. Wiggins and Robinson (1965) were the first to develop a recursive solution to the multivariate Yule-Walker equations. Because it is structured along the same lines as the procedure outlined here, this can be seen as a multivariate Levinson recursion. Nuttall (1976a, b) and Strand (1977) independently developed a multivariate version of the Burg algorithm while Nuttall (1977) shows how stationarity of the resulting VAR model is guaranteed. The univariate version of this Nuttall-Strand algorithm is indeed identical to the Burg algorithm. These multivariate algorithms are not discussed here. An implementation can be found in for example Marple (1987).

A final estimation technique is a straightforward generalization of the idea of Burg that any stationary univariate process can be read in both directions. If the constraint that the AR estimates have to satisfy the Levinson recursion is left out of the Burg procedure, a simple approach remains in which all parameters are simultaneously estimated by minimizing the sum of both the forward and backward prediction errors. Because this comes down to estimating an AR model in both directions by OLS, we shall call this estimation technique *Two sided Least Squares* (TLS). Just as the OLS technique, applying TLS does not guarantee that the resulting AR model satisfies the stationarity condition. A simple implementation of the TLS estimation technique for estimating an AR( $p$ ) model on a sample  $x_1, \dots, x_T$  of data, is by extending the classical regression model as

$$y = X\beta + \varepsilon \quad (3.4.54)$$

where now

$$y = \begin{bmatrix} x_{p+1} \\ \vdots \\ x_T \\ x_1 \\ \vdots \\ x_{T-p} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_p & x_{p-1} & \cdots & x_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T-1} & x_{T-2} & \cdots & x_{T-p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_2 & x_3 & \cdots & x_{p+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T-p+1} & x_{T-p+2} & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \nu \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$$

and subsequently estimating this model by the conventional OLS technique on a sample twice as big. Minor adjustments have to be made when testing for example for serial correlation because the residuals from the top and bottom  $T-p$  equations are not adjacent in time. A multivariate version of such a TLS procedure does not seem to exist. This is probably because a multivariate stochastic process is not as easy reversible as an univariate process. To see this note that the multivariate autocovariance matrices  $\Gamma_k$  are not symmetrical around  $k=0$  as indicated by (3.1.6).

### 3.5 Order selection criteria

As noted at the start of section 3.4, for estimating an AR( $p$ ) or VAR( $p$ ) model applying some estimation technique by itself is not enough. One also has to decide on the order  $p$  of the model. For this purpose several *order selection criteria* have been proposed in the literature. These criteria have in common that in some way they measure the “distance” between an estimated model and the “true” model. For each order a value of the relevant criterion can be calculated. The order for which the value of the criterion is minimized is seen as the model which is closest to the true model and is therefore the “optimal” order. Each of the criteria has a different theoretical motivation. Therefore different criteria will in general also lead to different orders.

This section describes five order selection from the literature. Each of these criteria has both an univariate and a multivariate version to determine the order of respectively an AR or VAR model. In all the formulae,  $p$  is the order of a model,  $m$  is some a-priori maximum order,  $n$  is the dimension of the model and  $T$  is the actual number of observations available for estimation. In general  $T$  is therefore equal to the sample size minus the number of lags ( $p$ ) included. Furthermore  $\hat{\sigma}_p^2$  and  $\hat{\Sigma}_p$  are the ML estimators (i.e. not corrected for the number of parameters estimated) of respectively the residual variance from an AR model and the residual covariance matrix from a VAR model. Each of the criteria assumes that the models are estimated including a constant term.

#### 1. Final Prediction Error (FPE)

Because often the purpose of estimating a time series model is to predict the future value of some variable, it seems logical to choose the order such that the prediction errors are minimized while putting a penalty on the number of parameters estimated. Based on this idea Akaike (1969, 1971) introduced the *Final Prediction Error (FPE) criterion* of which the univariate version is

$$FPE(p) = \frac{T + p + 1}{T - p - 1} \hat{\sigma}_p^2 \quad (3.5.1)$$

Its multivariate version is

$$FPE(p) = \left( \frac{T + np + 1}{T - np - 1} \right)^n |\hat{\Sigma}_p| \quad (3.5.2)$$

#### 2. Akaike Information Criterion (AIC)

The *Akaike Information Criterion (AIC)* as suggested by Akaike (1973, 1974) is based on an extension of the ML principle. The general univariate version is

$$AIC(p) = \ln(\hat{\sigma}_p^2) + \frac{2(p+1)}{T} = \ln(\hat{\sigma}_p^2) + \frac{2}{T} \{\# \text{ parameters}\} \quad (3.5.3)$$



Its multivariate version is

$$AIC(p) = \ln|\hat{\Sigma}_p| + \frac{2(pn^2 + n)}{T} = \ln|\hat{\Sigma}_p| + \frac{2}{T} \{\# \text{ parameters}\} \quad (3.5.4)$$

The AIC and the FPE criterion are asymptotically equivalent. Shibata (1976) shows that for final order AR processes, both the AIC and the FPE criterion have a positive probability for overestimating the true order of the model.

### 3. Schwarz Criterion (SC)

The *Schwarz Criterion* (SC) is derived by Schwarz (1978) based on a Bayesian approach. The general univariate version is

$$SC(p) = \ln(\hat{\sigma}_p^2) + \frac{(p+1)\ln(T)}{T} = \ln(\hat{\sigma}_p^2) + \frac{\ln(T)}{T} \{\# \text{ parameters}\} \quad (3.5.5)$$

Its multivariate version is

$$SC(p) = \ln|\hat{\Sigma}_p| + \frac{(pn^2 + n)\ln(T)}{T} = \ln|\hat{\Sigma}_p| + \frac{\ln(T)}{T} \{\# \text{ parameters}\} \quad (3.5.6)$$

Unlike the AIC and the FPE criterion, the SC does not have the tendency to overestimate the true model order.

### 4. Hannan and Quinn criterion (HQ)

A criterion that also does not overestimates the model order is the *Hannan and Quinn criterion* (HQ). Hannan and Quinn (1979) give for the univariate version

$$HQ(p) = \ln(\hat{\sigma}_p^2) + \frac{2(p+1)c \ln(\ln(T))}{T} = \ln(\hat{\sigma}_p^2) + \frac{2c \ln(\ln(T))}{T} \{\# \text{ parameters}\} \quad (3.5.7)$$

where  $c > 1$ . In applications of this criterion  $c=1.1$  will be assumed. Its multivariate version is given by Quinn (1980) as

$$HQ(p) = \ln|\hat{\Sigma}_p| + \frac{2(pn^2 + n)\ln(\ln(T))}{T} = \ln|\hat{\Sigma}_p| + \frac{2\ln(\ln(T))}{T} \{\# \text{ parameters}\} \quad (3.5.8)$$

Note the strong similarity between the HQ criterion and the SC.

### 5. Criterion for Autoregressive Transfer functions (CAT)

Besides the previous well known and rather similar four order selection criteria, a little different is the *Criterion for Autoregressive Transfer functions* (CAT) as introduced by Parzen (1974 and 1977a, b). Contrary to the other criteria, the CAT does not assume that the series necessarily satisfy an *exact* AR model. Instead Parzen assumes that by using a finite order AR model merely an adequate *approximation* is obtained to the possibly infinite order AR representation of a general (second) order stationary process. The optimal order given by the CAT therefore no longer indicates the order of some true model but merely the order of an AR model



which, in a sense, provides the best approximation to the observed time series process. The criterion is based on the spectral density  $S(\omega)$  of the true stochastic process and its estimated counterpart  $\hat{S}_p(\omega)$  where  $\omega$  is the frequency of fluctuations. More on spectral densities is said in Chapter 4. For now it is sufficient to know that a spectral density is merely another representation of a stochastic process which basically contains the same information on the dynamics of the process as the more familiar autocovariances. Specifically, the criterion is derived from choosing the order that minimizes the integrated relative mean squared error of the spectral estimate given by

$$I(p) = \int_0^{2\pi} E \left( \frac{\hat{S}_p(\omega) - S(\omega)}{S(\omega)} \right)^2 d\omega \quad (3.5.9)$$

By minimizing  $I(p)$  with respect to  $p$ , the “distance” between the true and the estimated stochastic process over all frequencies is minimized. Parzen shows that in the univariate case minimizing (3.5.11) is equivalent to minimizing

$$CAT(p) = \begin{cases} \left( \frac{1}{T} \sum_{j=1}^p \left( \frac{T-j-1}{T} \right) \frac{1}{\hat{\sigma}_j^2} \right) - \left( \frac{T-p-1}{T} \right) \frac{1}{\hat{\sigma}_p^2} & \text{for } p > 0 \\ - \left( 1 + \frac{1}{T} \right) & \text{for } p = 0 \end{cases} \quad (3.5.10)$$

with respect to  $p$ . Its multivariate version is

$$CAT(p) = trace \left\{ \left( \frac{n}{T} \sum_{j=1}^p \left( \frac{T}{T-nj-1} \hat{\Sigma}_j \right)^{-1} \right) - \left( \frac{T}{T-np-1} \hat{\Sigma}_p \right)^{-1} \right\} \quad (3.5.11)$$

The *trace* operator is defined as (A.3.3). Lütkepohl (1985) performed an extensive Monte Carlo experiment to investigate the quality of the various order selection criteria. His results prefer the SC for small and middle size samples. The SC most often chooses the right order and the resulting models have the best predicting qualities. Furthermore it is known that

$$\begin{aligned} SC(p) &\leq AIC(p) \quad \text{for } T \geq 8 \\ SC(p) &\leq HQ(p) \quad \text{for } T \geq 1 \\ HQ(p) &\leq AIC(p) \quad \text{for } T \geq 16 \end{aligned} \quad (3.5.12)$$

which implies

$$SC(p) \leq HQ(p) \leq AIC(p) \quad \text{for } T \geq 16 \quad (3.5.13)$$

### 3.6 Confidence intervals

When using VAR models for prediction or simulation purposes, often there is a need for confidence intervals. Such intervals describe for each future point in time the upper and lower boundaries of the potential values which a variable from the model can attain with a certain confidence level, given the stochastic process as described by the model. One way of obtaining such intervals is by performing a large number of simulations of the model and then calculating the relevant percentiles from these simulations, for example the 2.5% and 97.5% percentile for a 95% confidence interval. A more efficient method is to first calculate the conditional expectations and variances for each future point in time from the model parameters, whereafter the percentiles can easily be calculated from these expectations and variances using the known percentiles of the standard Normal distribution. This section describes the relevant formulae for calculating confidence intervals from a given VAR model. This is done for a number of special cases which are used in Part IV for the development of alternative scenario models.

#### 3.6.1 MA representation

The calculation of conditional variances and hence confidence intervals makes use of the Moving Average (MA) representation of a VAR model. An  $n$  dimensional VAR( $p$ ) model

$$x_t = v + A_1 x_{t-1} + A_2 x_{t-2} + \dots + A_p x_{t-p} + \varepsilon_t \quad (3.6.1)$$

has the following MA representation

$$x_t = \mu + \Phi_0 \varepsilon_t + \Phi_1 \varepsilon_{t-1} + \dots = \mu + \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i} \quad (3.6.2)$$

in which the  $\Phi_i$  coefficients are  $n \times n$  matrices. Lütkepohl (1991, p. 18) shows that these matrices can be obtained from the original parameters using the recursion

$$\begin{aligned} \Phi_0 &= I_n \\ \Phi_i &= \sum_{j=1}^i \Phi_{i-j} A_j \quad i = 1, 2, \dots \end{aligned} \quad (3.6.3)$$

The expected value  $\mu$  is as in (3.2.36) equal to

$$\mu = E(x_t) = (I - A_1 - A_2 \dots - A_p)^{-1} v \quad (3.6.4)$$

### 3.6.2 Interpolation

As indicated in the introduction, confidence intervals can be calculated based on the conditional expectations and the conditional variances. To see how these can be calculated from the MA representation (3.6.2), assume a general situation in which a VAR model is first used to simulate values for  $K$  future time periods. Next, for  $T-1$  time periods between each pair of simulated values the value is determined using linear interpolation between the adjacent simulated values.  $T$  can be seen as a kind of observation frequency, one observation for each  $T$  time periods. Note that setting  $T=1$  results in a standard simulation without interpolation.

From a standard simulation of a VAR model with a desired horizon of  $KT$  time periods, the values available are  $x_0, x_T, x_{2T}, \dots, x_{KT}$ . For simplicity of notation assume  $x$  is just a single variable which allows us to write  $E(xx')=E(x^2)$ . Using the earlier notation results in the multivariate case. Based on linear interpolation the intermediate values are

$$\begin{aligned} x_{(j-1)T+i} &= x_{(j-1)T} + \frac{i}{T}(x_{jT} - x_{(j-1)T}) \quad i = 1, \dots, T-1 \quad j = 1, \dots, K-1 \\ &= \left(1 - \frac{i}{T}\right)x_{(j-1)T} + \frac{i}{T}x_{jT} \end{aligned} \quad (3.6.5)$$

If we assume without loss of generality that  $E(x_i)=0$  then the variance of these interpolated values is

$$\begin{aligned} E(x_{(j-1)T+i}^2) &= \left(1 - \frac{i}{T}\right)^2 E(x_{(j-1)T}^2) + \left(\frac{i}{T}\right)^2 E(x_{jT}^2) \\ &\quad + 2\left(1 - \frac{i}{T}\right)\left(\frac{i}{T}\right)E(x_{(j-1)T}x_{jT}) \end{aligned} \quad (3.6.6)$$

Of course this still requires the conditional variances of the originally simulated time periods  $(j-1)T$  and  $jT$ . These conditional variances can be calculated from the MA representation (3.6.2). To see this, suppose  $x_t(h)$  is the  $h$  time periods ahead predicted value based on all information until and including time  $t$ . This predicted value is equal to the conditional expected value which can be easily calculated by recursive substitution in the VAR representation (3.6.1) with  $\varepsilon_t=0$ , starting from the last  $p$  observations  $t-p+1, \dots, t$ . Using the MA representation, the deviations from these conditional expectations are

$$q_h = x_{t+h} - x_t(h) = \mu + \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t+h-i} - \mu - \sum_{i=h}^{\infty} \Phi_i \varepsilon_{t+h-i} = \sum_{i=0}^{h-1} \Phi_i \varepsilon_{t+h-i} \quad (3.6.7)$$

with moments

$$E(q_h) = 0 \quad (3.6.8)$$

and

$$E(q_h q'_h) = \sum_{i=0}^{h-1} \Phi_i \Sigma \Phi'_i \quad (3.6.9)$$

For the time periods for which interpolation is not necessary, the diagonal elements of (3.6.9) directly give the conditional variances of the variables in the model for the various future time periods  $h$ . From (3.6.6) we see that for the intermediate time periods, the conditional autocovariance between each pair of originally simulated time periods is also required. Based on

$$q_{h-1} = \sum_{i=0}^{h-2} \Phi_i \varepsilon_{t+h-i} \quad (3.6.10)$$

these autocovariances are

$$E(q_h q'_{h-1}) = \sum_{i=1}^{h-1} \Phi_i \Sigma_\varepsilon \Phi'_{i-1} \quad (3.6.11)$$

Substituting the appropriate conditional variances from (3.6.9) and conditional autocovariances from (3.6.11) into (3.6.6) yields the conditional variances of the variables for intermediate time periods.

Given the conditional expectation  $x_{k,t}(h)$  of the  $k$ -th variable from the model for  $h$  time periods ahead and its conditional variance  $\sigma_k(h)$ , the  $(1-\alpha)\%$  confidence interval is

$$[x_{k,t}(h) - z_{0.5\alpha} \sigma_k(h), x_{k,t}(h) + z_{0.5\alpha} \sigma_k(h)] \quad (3.6.12)$$

Here  $z_\alpha$  is the  $\alpha$  percent percentile of the standard Normal distribution with Cumulative Distribution Function  $\Phi(\cdot)$  and  $\Phi(z_\alpha) = \alpha$ .

### 3.6.3 Integration

Now suppose that confidence intervals are required for the level of a logarithmic index variable which is calculated from simulated period to period growth of the index. After the logarithmic index levels have been calculated, intermediate values are obtained through interpolation. Note that the integration into levels takes place *before* the interpolation. The other way around, first interpolating the first differences and then integrating the resulting numbers, makes little sense since it results in  $T$  time period changes which are inconsistent with the originally simulated numbers. What matters is that the interpolation takes place on the levels of the variable and not on the first order differences.

Compared to the situation from the previous section the difference is that here the VAR model is specified and estimated in terms of the first order differences of the logarithmic index while the confidence intervals are required in terms of the levels of the logarithmic index. Hence we are interested in the conditional expectations and variances of the integrated process

$$y_t = \sum_{i=0}^{\infty} x_{t-i} \quad (3.6.13)$$

with

$$x_t = \mu + \sum_{i=0}^{\infty} \Phi_i \varepsilon_{t-i}$$

By substituting for  $x_t$  in (3.6.13) for several steps in the summation, it can be seen that

$$y_t = \mu^* + \sum_{i=0}^{\infty} \Phi_i^* \varepsilon_{t-i} \quad (3.6.14)$$

with

$$\mu^* = \sum_{i=0}^{\infty} \mu \quad \text{and} \quad \Phi_i^* = \sum_{j=0}^i \Phi_j$$

The integrated process  $y_t$  therefore also has an MA representation, just as the underlying model for  $x_t$ , only with a different expected value and different MA coefficients. Because of this similarity, the formulae from the previous section can equally well be applied after first having calculated the adjusted expected value and MA coefficients.

### 3.6.4 Differencing

Now suppose that confidence intervals are required for the first order differences of a logarithmic index variable which are calculated from a simulated (detrended) logarithmic index. Intermediate values of the logarithmic index are obtained through interpolation. An obvious example is when a (detrended) price index has been simulated and one wants to calculate the confidence intervals for the price inflation. Note that the differencing takes place *after* the interpolation of the levels. The other way around, first differencing the levels and then interpolating the resulting numbers, makes little sense since it results in very different numbers than the intended period to period changes. What matters is that the interpolation takes place on the levels of the variable and not on the first order differences.

As in section 3.6.2, assume that from a standard simulation of a VAR model with a desired horizon of  $KT$  time periods, the values available are  $x_0, x_T, x_{2T}, \dots, x_{KT}$ . Interpolating these for the intermediate time periods and then calculating the period to period changes (and not the  $T$  period changes) results in

$$x_{(j-1)T+i} = \frac{1}{T}(x_{jT} - x_{(j-1)T}) \quad i = 1, \dots, T \quad j = 1, \dots, K \quad (3.6.15)$$

If we assume without loss of generality that  $E(x_i)=0$  then the variance of these first order differences is

$$E(x_{(j-1)T+i}^2) = \left(\frac{1}{T}\right)^2 (E(x_{(j-1)T}^2) + E(x_{jT}^2) - 2E(x_{(j-1)T}x_{jT})) \quad (3.6.16)$$

The second term is the variance of the total change between two successive originally simulated time periods which are  $T$  time periods apart. The variances and the autocovariance terms can be calculated as described in section 3.7.2. Finally note that in this case only the simulated values need to be differenced and differencing of the original VAR model in its MA representation is not required.

### 3.7 Eigenvalue restricted VAR models

From section 3.3.2 we know that the eigenvalues and eigenvectors of the parameter matrices of a VAR model are very important in determining the dynamic stochastic properties of the model. In a sense one could say that these eigenvalues and eigenvectors are more structural or fundamental types of parameters than the conventional parameters from which it is hard to tell anything at all about the dynamics of the model. This section is entirely devoted to the specification and estimation of a new class of VAR models: *EigenValue Restricted (EVR) VAR models*<sup>26</sup>. Just as the other types of VAR models mentioned in section 2.2.7, they aim to reduce the problem of imprecise estimates of VAR model parameters caused by the fact that the number of parameters is often quite substantial relative to the available sample size or time series length. To reduce this potential over-parametrization, (valid) restrictions are imposed on the model parameters in order to increase the estimation efficiency. The underlying idea of EVR VAR models is to impose these restrictions on essential (transformations of) parameters of the model that determine its dynamics: the eigenvalues. For low order VAR models this is done by enforcing specific eigenvalues, based on some prior knowledge, and for high order VAR models by enforcing a limited number of different eigenvalues and thereby preventing overfitting of the model. The framework can relatively easy be adjusted to incorporate restrictions on the eigenvectors. EVR VAR models are one of the components of the new scenario framework described and tested in Chapter 20.

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<sup>26</sup> As described in section 3.7.3, the term EVR might also stand for an *eigenvector* restricted VAR model. Although applications of restricting the eigenvectors are certainly not unthinkable, the case of restricting the eigenvalues is the most obvious because of the more fundamental role of the eigenvalues in determining the dynamics of a VAR model. Because these are also the types of models used in the research presented here they are used in the terminology.

This section makes use of vector and matrix manipulations and differentiation. Appendix A of Lütkepohl (1991) gives a useful summary of matrix and vector algebra and also gives further references. Some of the basic matrix operations are also given in Appendix A.3.

### 3.7.1 Multivariate Least Squares

The specification and estimation of EVR VAR models is based on the Multivariate Least Squares (MLS) estimation of a VAR model. A VAR(1) model for  $t=1, \dots, T$  without a constant term is

$$y_t = A_1 y_{t-1} + \varepsilon_t \quad (3.7.1)$$

with

$$y_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n,t} \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix}, E(\varepsilon_t) = 0, E(\varepsilon_t \varepsilon_s') = \begin{cases} \Sigma & \text{for } s = t \\ 0 & \text{for } s \neq t \end{cases}$$

This can equally well be the VAR(1) representation of a VAR( $p$ ) model as given by (3.2.27). The constant term has been excluded deliberately for reasons that will become clear in the next section. This is not a limitation of the models. The time series used for estimation only have to be corrected for their mean value while based on (3.2.36) a constant vector can be added after the estimation of the other parameter matrices to achieve the desired mean value of the model, for example the historical means. Using the following notation from Lütkepohl (1991, section 3.2)

$$\begin{aligned} y_t &= (y_{1,t}, \dots, y_{n,t})' & t &= 0, \dots, T \\ Y &= (y_1, \dots, y_T) & n \times T \\ B &= A_1 & n \times n \\ Z &= (y_0, \dots, y_{T-1}) & n \times T \\ U &= (\varepsilon_1, \dots, \varepsilon_T) & n \times T \\ \bar{y} &= \text{vec}(Y) & nT \times 1 \\ \bar{\beta} &= \text{vec}(B) & n^2 \times 1 \\ \bar{u} &= \text{vec}(U) & nT \times 1 \end{aligned}$$

it is possible to rewrite the VAR(1) model as

$$Y = BZ + U \quad (3.7.2)$$

or

$$\text{vec}(Y) = \text{vec}(BZ) + \text{vec}(U) = (Z' \otimes I_n) \text{vec}(B) + \text{vec}(U) \quad (3.7.3)$$

and thus

$$\bar{y} = (Z' \otimes I_n) \bar{\beta} + \bar{u} \quad (3.7.4)$$



For the new error terms holds

$$E(\bar{u}) = 0, E(\bar{u}\bar{u}') = \Sigma_u = I_T \otimes \Sigma \quad (3.7.5)$$

The MLS estimator gives the value for  $\bar{\beta}$  that minimizes the multivariate Residual Sum of Squares (RSS) which are weighed according to the inverse of the  $\Sigma_u$  matrix.

$$\begin{aligned} RSS(\bar{\beta}) &= \bar{u}' \Sigma_u^{-1} \bar{u} = (\bar{y} - (Z' \otimes I_n) \bar{\beta})' (I_T \otimes \Sigma)^{-1} (\bar{y} - (Z' \otimes I_n) \bar{\beta}) \\ &= \bar{y}' (I_T \otimes \Sigma^{-1}) \bar{y} + \bar{\beta}' (Z \otimes I_n) (I_T \otimes \Sigma^{-1}) (Z' \otimes I_n) \bar{\beta} \\ &\quad - 2 \bar{\beta}' (Z \otimes I_n) (I_T \otimes \Sigma^{-1}) \bar{y} \\ &= \bar{y}' (I_T \otimes \Sigma^{-1}) \bar{y} + \bar{\beta}' (ZZ' \otimes \Sigma^{-1}) \bar{\beta} - 2 \bar{\beta}' (Z \otimes \Sigma^{-1}) \bar{y} \end{aligned} \quad (3.7.6)$$

The row vector with the first order partial derivatives for all the  $n^2$  elements of  $\bar{\beta}$  is

$$\frac{\partial RSS(\bar{\beta})}{\partial \bar{\beta}'} = 2 \bar{\beta}' (ZZ' \otimes \Sigma^{-1}) - 2 \bar{y}' (Z' \otimes \Sigma^{-1}) \quad (3.7.7)$$

Setting these derivatives equal to zero (first order conditions) and solving for  $\bar{\beta}$  yields the MLS estimator

$$\bar{\beta}_{MLS}' = \bar{y}' (Z' \otimes \Sigma^{-1}) ((ZZ')^{-1} \otimes \Sigma) = \bar{y}' (Z' (ZZ')^{-1} \otimes I_n) \quad (3.7.8)$$

Here we used some simple rules for the Kronecker product given in Appendix A.3. The second order partial derivative or Hessian matrix is

$$\frac{\partial^2 RSS(\bar{\beta})}{\partial \bar{\beta}' \partial \bar{\beta}} = 2 (ZZ' \otimes \Sigma^{-1}) \quad (3.7.9)$$

which is a positive definite matrix (second order conditions) indicating that the estimator (3.7.8) indeed minimizes the multivariate RSS (3.7.6). The MLS estimator is equivalent to applying the OLS estimator (3.4.4) for each of the  $n$  equations separately. An estimator for the covariancematrix of the error terms is

$$\begin{aligned} \Sigma_{MLS} &= \frac{1}{T} U U' = \frac{1}{T} (Y - B_{MLS} Z) (Y - B_{MLS} Z)' \\ &= \frac{1}{T} (Y - Y Z' (ZZ')^{-1} Z) (Y - Y Z' (ZZ')^{-1} Z)' \\ &= \frac{1}{T} Y (I_T - Z' (ZZ')^{-1} Z) Y' \end{aligned} \quad (3.7.10)$$

in which

$$B_{MLS} = YZ'(ZZ')^{-1} \quad (3.7.11)$$

is another representation of the MLS estimator in terms of the parameter matrix  $B$  instead of  $vec(B)$  as in (3.7.8).

### 3.7.2 Eigenvector derivatives

Suppose we now want to fix the eigenvalues of a VAR( $p$ ) model at some specific values and find the eigenvectors that minimize the multivariate RSS (3.7.6) in order to estimate the parameter matrices with restrictions on the eigenvalues. If we use the VAR(1) representation (3.2.37) of the VAR( $p$ ) model and exclude the constant vector, without loss of generality we can remain in the context of the previous section. Using the crucial relation (3.3.15) we can then write

$$\bar{\beta} = vec(B) = vec(A_1) = vec(PA_vP^{-1}) \quad (3.7.12)$$

where

$$A_v = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \text{ and } P = \begin{bmatrix} \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{1,n} \end{pmatrix} & \dots & \begin{pmatrix} v_{n,1} \\ \vdots \\ v_{n,n} \end{pmatrix} \end{bmatrix}.$$

So  $A_v$  is the zero matrix with the  $n$  eigenvalues  $\lambda_i$  of  $A_1$  on the main diagonal and  $P$  is the matrix having the  $n$  eigenvectors  $v_1, \dots, v_n$  of  $A_1$  as its columns. Because of the restrictions on the eigenvalues, the matrix  $A_v$  is fixed while the matrix  $P$  contains the parameters that have to be estimated. In order to find the values of these parameters that minimize the multivariate RSS the usual way to proceed is to calculate the first order partial derivatives, set these equal to zero (first order conditions) and hope that these equations can be solved for the parameters of the  $P$  matrix. It is crucial to note that the rewriting (3.7.12) of the parameter matrix and hence the total procedure described here is only possible when (a) the VAR(1) representation of a VAR( $p$ ) model is used and (b) the constant vector is excluded.

The chain rule for matrix differentiation is

$$\frac{\partial h(g(\delta))}{\partial \delta'} = \frac{\partial h(\alpha)}{\partial \alpha'} \frac{\partial g(\delta)}{\partial \delta'} \quad (3.7.13)$$

where  $\alpha$  and  $\delta$  are  $m \times 1$  and  $q \times 1$  parameter vectors,  $\alpha = g(\delta)$ , and  $h(\alpha)$  and  $g(\delta)$  are  $r \times 1$  and  $m \times 1$  vector valued functions of  $\alpha$  and  $\delta$ . Applying this rule with  $m=q=n^2$  and  $r=1$ , the row vector with partial derivatives of the multivariate RSS for all the parameters of the  $P$  matrix, given the  $A_v$  matrix, is

$$\frac{\partial RSS(vec(P))|_{A_v}}{\partial vec(P)'} = \frac{\partial RSS(vec(PA_vP^{-1}))}{\partial vec(PA_vP^{-1})'} \frac{\partial vec(PA_vP^{-1})}{\partial vec(P)'} \quad (3.7.14)$$

The first of the two expressions on the right hand side is simply the derivative (3.7.7) with  $\bar{\beta}$  replaced by  $\text{vec}(PA_v P^{-1})$ .

$$\frac{\partial \text{RSS}(\text{vec}(PA_v P^{-1}))}{\partial \text{vec}(PA_v P^{-1})'} = 2\text{vec}(PA_v P^{-1})'(ZZ'\otimes\Sigma^{-1}) - 2\bar{y}'(Z'\otimes\Sigma^{-1}) \quad (3.7.15)$$

The second expression can be determined by defining the function

$$B(P) = PA_v P^{-1} \quad (3.7.16)$$

and then writing

$$\begin{aligned} B(P)P &= PA_v \\ \text{vec}(B(P)P) &= \text{vec}(PA_v) \end{aligned} \quad (3.7.17)$$

Differentiating both sides with respect to  $\text{vec}(P)'$  yields

$$\frac{\partial \text{vec}(B(P)P)}{\partial \text{vec}(P)'} = \frac{\partial \text{vec}(PA_v)}{\partial \text{vec}(P)'} \quad (3.7.18)$$

Using the relation

$$\text{vec}(ABC) = (I \otimes AB)\text{vec}(C) = (C' B' \otimes I)\text{vec}(A) \quad (3.7.19)$$

the right hand side of (3.7.17) is equal to

$$\text{vec}(PA_v) = (A_v' \otimes I_n)\text{vec}(P) \quad (3.7.20)$$

and thus the right hand side of (3.7.18) is

$$\frac{\partial \text{vec}(PA_v)}{\partial \text{vec}(P)'} = (A_v' \otimes I_n) \quad (3.7.21)$$

For the left hand side of (3.7.18) we use the following product rule for vector differentiation.

$$\frac{\partial \text{vec}(AC)}{\partial \delta'} = (I_q \otimes A) \frac{\partial \text{vec}(C)}{\partial \delta'} + (C' \otimes I_r) \frac{\partial \text{vec}(A)}{\partial \delta'} \quad (3.7.22)$$

where  $\delta$  is a  $m \times 1$  parameter vector and  $A(\delta)$  and  $C(\delta)$  are  $r \times s$  and  $s \times q$  matrix functions of  $\delta$ . Using  $\delta = \text{vec}(P)$ ,  $A(\delta) = B(P)$ ,  $C(\delta) = P$ ,  $m = n^2$  and  $r = s = q = n$  gives for the left hand side of (3.7.18)

$$\begin{aligned} \frac{\partial \text{vec}(B(P)P)}{\partial \text{vec}(P)'} &= (I_n \otimes B(P)) \frac{\partial \text{vec}(P)}{\partial \text{vec}(P)'} + (P' \otimes I_n) \frac{\partial \text{vec}(B(P))}{\partial \text{vec}(P)'} \\ &= (I_n \otimes B(P)) + (P' \otimes I_n) \frac{\partial \text{vec}(B(P))}{\partial \text{vec}(P)'} \end{aligned} \quad (3.7.23)$$

Substituting (3.7.21) and (3.7.23) into (3.7.18) gives

$$(I_n \otimes B(P)) + (P' \otimes I_n) \frac{\partial \text{vec}(B(P))}{\partial \text{vec}(P)'} = (A_v' \otimes I_n) \quad (3.7.24)$$

from which can be solved

$$\begin{aligned} \frac{\partial \text{vec}(B(P))}{\partial \text{vec}(P)'} &= (P' \otimes I_n)^{-1} \{ (A_v' \otimes I_n) - (I_n \otimes B(P)) \} \\ &= (P' \otimes I_n)^{-1} \{ (A_v' \otimes I_n) - (I_n \otimes P A_v P^{-1}) \} \end{aligned} \quad (3.7.25)$$

Substituting (3.7.25) and (3.7.15) into the chain rule (3.7.14) finally gives the row vector of first order partial derivatives we were looking for.

$$\begin{aligned} \frac{\partial \text{RSS}(\text{vec}(P)) \parallel A_v}{\partial \text{vec}(P)'} &= \{ 2 \text{vec}(P A_v P^{-1})' (Z Z' \otimes \Sigma^{-1}) - 2 \bar{y}' (Z' \otimes \Sigma^{-1}) \} \\ &\quad (P' \otimes I_n)^{-1} \{ (A_v' \otimes I_n) - (I_n \otimes P A_v P^{-1}) \} \end{aligned} \quad (3.7.26)$$

Unfortunately the equation that results by setting this derivative equal to zero (first order conditions) cannot be solved for the  $P$  matrix. Therefore an analytical expression for the restricted estimator of the eigenvectors cannot be found. We have to resort to numerical procedures on which more will be said in section 3.7.5. In the remainder of this section, five important extensions of the derivative (3.7.26) are given which are needed in the numerical optimizations.

### 1. *Logarithm*

In numerical optimization procedures it is often better to optimize the natural logarithm of the RSS instead of the RSS itself in order to achieve a more smooth objective function. The derivative (3.7.26) can still be used by applying the chain rule.

$$\frac{\partial \ln(\text{RSS}(\text{vec}(P))) \parallel A_v}{\partial \text{vec}(P)'} = \frac{1}{\text{RSS}(\text{vec}(P)) \parallel A_v} \frac{\partial \text{RSS}(\text{vec}(P)) \parallel A_v}{\partial \text{vec}(P)'} \quad (3.7.27)$$

## 2. Normalization

The VAR(1) representation of a VAR( $p$ ) model has dimensions  $np \times np$ . Since the matrix  $P$  with the eigenvectors has the same dimensions this would mean that  $\text{vec}(P)$  consists of  $(np)^2$  parameters which have to be estimated. This is not true for two reasons. The first is that eigenvectors are only unique up to a constant factor. Because of this property the only true parameters within an eigenvector are the relative proportions between its elements. By normalizing each eigenvector on its first element, for each eigenvector there are not  $np$  but  $np-1$  parameters. The total number of parameters in  $\text{vec}(P)$  then becomes  $(np)^2 - np$ . For an example with  $n=2$  and  $p=3$  this means the change from the matrix

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,6} \\ \vdots & & \vdots \\ p_{6,1} & \cdots & p_{6,6} \end{bmatrix} \quad (3.7.28)$$

into the normalized matrix

$$P^* = \begin{bmatrix} 1 & \cdots & 1 \\ p_{2,1}^* & \cdots & p_{2,6}^* \\ \vdots & & \vdots \\ p_{6,1}^* & \cdots & p_{6,6}^* \end{bmatrix} \quad (3.7.29)$$

where

$$p_{i,j}^* = \frac{p_{i,j}}{p_{1,j}} \text{ for } i = 1, \dots, 6 \text{ and } j = 1, \dots, 6$$

For the derivative vector (3.7.27) this means that the  $P$  matrix that has to be substituted into the expression has to be normalized first by dividing each of its rows by the first row. Once the derivative vector has been calculated the elements corresponding to the first row can be deleted to obtain the derivatives for the remaining  $(np)^2 - np$  true parameters.

## 3. Chain rule

The second reason why the  $\text{vec}(P)$  vector contains less than  $(np)^2$  parameters are the special proportions within the eigenvectors of a VAR( $p$ ) model with  $p > 1$  as indicated by (3.3.12). Assuming that the eigenvectors have already been normalized on their first elements, for the example with  $n=2$  and  $p=3$  the  $P$  matrix has in fact the following structure.

$$P = \begin{bmatrix} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & \cdots & \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_1} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & & \frac{1}{\lambda_6} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_1^2} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & \cdots & \frac{1}{\lambda_6^2} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \end{bmatrix} \quad (3.7.30)$$

where  $\lambda_i$  are the  $1, \dots, np$  eigenvalues. First, this shows that there are in fact only  $np(n-1)$  true parameters in the  $P$  matrix. For each of the  $np$  eigenvectors  $n-1$  parameters. In the example these are the  $2 \times 3 \times (2-1) = 6$  parameters  $p_{2,1}, \dots, p_{2,6}$  from the second row. Second, (3.7.30) shows how the partial derivatives for these six true parameters can be calculated from the original derivatives for all the elements of  $P$  as given by (3.7.26). Based on the chain rule it holds for example that

$$\begin{aligned} \frac{\partial}{\partial p_{2,1}^*} &= \frac{\partial}{\partial p_{2,1}} + \frac{\partial}{\partial p_{3,1}} \frac{\partial p_{3,1}}{\partial p_{2,1}} + \frac{\partial}{\partial p_{6,1}} \frac{\partial p_{6,1}}{\partial p_{2,1}} \\ &= \frac{\partial}{\partial p_{2,1}} + \frac{\partial}{\partial p_{3,1}} \frac{1}{\lambda_1} + \frac{\partial}{\partial p_{6,1}} \frac{1}{\partial \lambda_1^2} \end{aligned} \quad (3.7.31)$$

where  $\partial/\partial p_{2,1}$ ,  $\partial/\partial p_{3,1}$  and  $\partial/\partial p_{6,1}$  are the original partial derivatives from the vector (3.7.26).

#### 4. Complex eigenvalues

In case a VAR( $p$ ) model has complex roots its eigenvalues and corresponding eigenvectors contain one or more complex conjugate pairs. If numerical optimization routines cannot handle complex numbers, the calculations need to be done in terms of real numbers being the real and imaginary parts of the complex numbers. What we need then are the partial derivatives for these real numbers. How can these be obtained from the in that case complex valued expression (3.7.26)? By comparing these analytical first order derivatives in complex terms with the separate derivatives for the real and imaginary parts of the eigenvectors as obtained by numerical differentiation, it turns out that these derivatives can easily be read from the complex valued derivatives.

Suppose one complex element from the  $P$  matrix is

$$p_{i,j} = a + bi \quad (3.7.32)$$

and that the complex valued partial derivative of the real valued RSS for this non conjugate element is

$$\frac{\partial}{\partial p_{i,j}} = c + di \quad (3.7.33)$$

Then the partial derivatives for the real and imaginary parts of  $p_{i,j}$  turn out to be

$$\frac{\partial}{\partial a} = 2c \quad \text{and} \quad \frac{\partial}{\partial b} = -2d \quad (3.7.34)$$

A simple example further supports this relation. Assume the simple function of two complex numbers

$$f(c_1, c_2) = c_1 c_2 \quad (3.7.35)$$

with

$$\begin{aligned} c_1 &= a + bi \\ c_2 &= c + di \end{aligned}$$

Then the first order partial derivatives are

$$\frac{\partial f}{\partial c_1} = c_2 = c + di \quad \text{and} \quad \frac{\partial f}{\partial c_2} = c_1 = a + bi \quad (3.7.36)$$

In case of the multivariate RSS we know that will hold  $c_2 = c_1^*$  because for each complex eigenvector, its complex conjugate will also be an eigenvector. This causes the function  $f(\cdot)$  to become real valued because

$$f(a, b) = c_1 c_1^* = (a + bi)(a - bi) = a^2 + b^2 \quad (3.7.37)$$

The first order partial derivatives for the real and imaginary parts of  $c_1$ ,  $a$  and  $b$ , are then

$$\frac{\partial f}{\partial a} = 2a \quad \text{and} \quad \frac{\partial f}{\partial b} = 2b \quad (3.7.38)$$

while the complex valued derivative for the non conjugate  $c_1$  becomes

$$\frac{\partial f}{\partial c_1} = a - bi \quad (3.7.39)$$

Comparing (3.7.38) and (3.7.39) confirms the relation between (3.7.33) and (3.7.34) for obtaining the partial derivatives for the real and imaginary parts of the eigenvectors from the original complex valued first order partial derivative vector. Note that two first order partial derivatives for two complex conjugate elements from a pair of eigenvectors also result in two real valued partial derivatives for the real and imaginary parts. The number of parameters in the derivative vector therefore does not change because of the switch from complex to real valued derivatives.

### 5. Second order partial derivatives

An analytical expression for the (Hessian) matrix

$$\frac{\partial^2 \text{RSS}(\text{vec}(P))}{\partial \text{vec}(P) \partial \text{vec}(P)^T} \Big|_{A_v} \quad (3.7.40)$$

of second order partial derivatives for the eigenvector parameters is hard to determine. This is even more so when the normalization of the eigenvectors, the chain rule in case  $p > 1$  and the switch from complex to real valued derivatives have to be taken into account. That is why this Hessian matrix can best be determined by (centred) numerical differentiation for each of the final  $np(n-1)$  true parameters of the first order partial derivative vector. The resulting Hessian matrix can be used to determine whether numerical optimization procedures have indeed found a (local)



minimum of the multivariate RSS by checking whether it is positive definite (second order conditions). It can also be used in some numerical optimization procedures such as the Newton-Raphson method which searches for the root of equating the first order partial derivative vector to zero (first order conditions).

### 3.7.3 Eigenvalue derivatives

Suppose now, the other way around from the previous section, that we want to fix the *eigenvectors* of a VAR( $p$ ) model at some specific values and find the *eigenvalues* that minimize the multivariate RSS (3.7.6) in order to estimate the parameter matrices with restrictions on the *eigenvectors*. So in this case EVR VAR actually stands for *eigenvector* restricted VAR model. For estimating such a model the following row vector of first order partial derivatives is required.

$$\frac{\partial \text{RSS}(\text{vec}(A_v))'P}{\partial \text{vec}(A_v)'} = \frac{\partial \text{RSS}(\text{vec}(PA_v P^{-1}))}{\partial \text{vec}(PA_v P^{-1})'} \frac{\partial \text{vec}(PA_v P^{-1})}{\partial \text{vec}(A_v)'} \quad (3.7.41)$$

The first expression on the right hand side is given by (3.7.15). The second expression can be obtained along similar lines as the derivative for  $\text{vec}(P)'$  in the previous section. Define the function

$$B(A_v) = PA_v P^{-1} \quad (3.7.42)$$

for which holds

$$\begin{aligned} B(A_v)P &= PA_v \\ \text{vec}(B(A_v)P) &= \text{vec}(PA_v) \\ (P' \otimes I_n) \text{vec}(B(A_v)) &= (I_n \otimes P) \text{vec}(A_v) \end{aligned} \quad (3.7.43)$$

Differentiating both sides for  $\text{vec}(A_v)$  yields

$$\begin{aligned} \frac{\partial \text{vec}(B(A_v)P)}{\partial \text{vec}(A_v)'} &= \frac{\partial \text{vec}(PA_v)}{\partial \text{vec}(A_v)'} \\ (P' \otimes I_n) \frac{\partial \text{vec}(B(A_v))}{\partial \text{vec}(A_v)'} &= (I_n \otimes P) \end{aligned} \quad (3.7.44)$$

from which can be solved

$$\frac{\partial \text{vec}(B(A_v))}{\partial \text{vec}(A_v)'} = (P' \otimes I_n)^{-1} (I_n \otimes P) \quad (3.7.45)$$

Substituting (3.7.15) and (3.7.45) into (3.7.41) yields the required row vector of first order partial derivatives.

$$\begin{aligned} \frac{\partial \text{RSS}(\text{vec}(A_v))}{\partial \text{vec}(A_v)'} P &= \left\{ 2 \text{vec}(P A_v P^{-1})' (Z Z' \otimes \Sigma^{-1}) - 2 \bar{y}' (Z' \otimes \Sigma^{-1}) \right\} \\ &\quad (P' \otimes I_n)^{-1} (I_n \otimes P) \\ &= \left\{ 2 \text{vec}(A_v)' (P^{-1} \otimes P') (Z Z' \otimes \Sigma^{-1}) - 2 \bar{y}' (Z' \otimes \Sigma^{-1}) \right\} \\ &\quad (P' \otimes I_n)^{-1} (I_n \otimes P) \end{aligned} \quad (3.7.46)$$

Equating this expression to zero (first order conditions) and solving for  $A_v$  yields only the unwanted trivial solution  $A_v=0$  so that we have to resort to numerical optimization procedures here as well. The (Hessian) matrix of second order partial derivatives for the eigenvalue parameters is

$$\frac{\partial \text{RSS}(\text{vec}(A_v))}{\partial \text{vec}(A_v)'} P = 2(P^{-1} \otimes P') (Z Z' \otimes \Sigma^{-1}) (P' \otimes I_n)^{-1} (I_n \otimes P) \quad (3.7.47)$$

With respect to the derivatives (3.7.46) and (3.7.47) for the elements from the  $A_v$  matrix a number of remarks need to be made.

1.  $A_v$  is a diagonal matrix so that only  $n$  of the  $n^2$  elements of the derivative vector (3.7.47) and  $n^2$  of the  $n^4$  elements of the derivative matrix (3.4.47) are relevant. These elements can easily be selected from the total vector and matrix of partial derivatives.
2. Unfortunately the use of the first and second order partial derivatives (3.7.46) and (3.7.47) is limited to the special case of  $p=1$ . If  $p>1$ , because of the special structure of the eigenvectors of the VAR(1) representation of a VAR( $p$ ) model as given by (3.3.12), the eigenvectors in the  $P$  matrix are a function of the  $n-1$  elements on the second until the  $n$ -th row but also of the eigenvalues. The  $P$  matrix is then also a function of the diagonal elements of the  $A_v$  matrix which causes that the  $P$  matrix can no longer be assumed constant which renders the two derivative expressions incorrect because they are derived assuming a fixed  $P$  matrix. For the time being it is not clear if and how these derivatives can be calculated in an analytical sense. Of course, numerical differentiation is always possible.
3. For obtaining the partial derivatives for the real and imaginary parts of the eigenvalues from an originally complex valued first order partial derivative vector exactly the same relations hold as for the eigenvector derivatives as described with point 4. in the previous section.
4. In the case of optimizing in terms of the eigenvalues it can for example be necessary to enforce that the search is limited to the stationary region in terms of the modulus or that, in case of complex eigenvalues, the search is limited to a certain frequency range. Such restrictions are difficult to formulate in terms of the real and imaginary parts of the eigenvalues. In terms of their polar coordinates it is much easier. Once the first order partial derivatives for the real part  $a$  and imaginary part  $b$  of a complex eigenvalue  $c=a+bi$  are known as  $\partial/\partial a$  and  $\partial/\partial b$ , the derivatives for the modulus  $R$  and the argument  $\omega$  can be calculated using the chain rule as

$$\begin{aligned}
\frac{\partial}{\partial R} &= \frac{\partial}{\partial a} \frac{\partial a}{\partial R} + \frac{\partial}{\partial b} \frac{\partial b}{\partial R} \\
\frac{\partial}{\partial \omega} &= \frac{\partial}{\partial a} \frac{\partial a}{\partial \omega} + \frac{\partial}{\partial b} \frac{\partial b}{\partial \omega}
\end{aligned}
\tag{3.7.48}$$

with

$$\begin{aligned}
a &= R \cos(\omega 2\pi) \\
b &= R \sin(\omega 2\pi)
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\partial a}{\partial R} &= \cos(\omega 2\pi) \\
\frac{\partial b}{\partial R} &= \sin(\omega 2\pi) \\
\frac{\partial a}{\partial \omega} &= -R 2\pi \sin(\omega 2\pi) \\
\frac{\partial b}{\partial \omega} &= R 2\pi \cos(\omega 2\pi)
\end{aligned}$$

Derivatives for real eigenvalues remain unchanged because of this transformation.

5. If restrictions are imposed on the eigenvalues in a sense that some eigenvalues are a function of some other eigenvalues, the relevant first order partial derivatives can again be obtained by applying the chain rule. Suppose for example that for the polar coordinates of two complex eigenvalues has to hold

$$\begin{aligned}
R_2 &= f(R_1) \\
\omega_2 &= g(\omega_1)
\end{aligned}
\tag{3.7.49}$$

for some functions  $f(\cdot)$  and  $g(\cdot)$ . The derivatives for  $R_1$  and  $\omega_1$  are then

$$\begin{aligned}
\frac{\partial}{\partial R_1^*} &= \frac{\partial}{\partial R_1} + \frac{\partial}{\partial R_2} \frac{\partial f(R_1)}{\partial R_1} \\
\frac{\partial}{\partial \omega_1^*} &= \frac{\partial}{\partial \omega_1} + \frac{\partial}{\partial \omega_2} \frac{\partial g(\omega_1)}{\partial \omega_1}
\end{aligned}
\tag{3.7.50}$$

6. To enforce that the search is limited to for example the stationary region, for the modulus of all eigenvalues must hold  $R_i \in [0, 1]$ . To implement such a restriction in a numerical optimization procedure without bounds, the following transformation can be used to optimize over  $x^*$  instead of  $x$ .

$$x = \alpha + \frac{\beta - \alpha}{1 + \exp(-x^*)} \text{ and } x^* = \ln\left(\frac{x - \alpha}{\beta - x}\right) \quad (3.7.51)$$

with  $x \in [\alpha, \beta]$  and  $x^* \in \langle -\infty, \infty \rangle$ .

#### 3.7.4 Eigenvectors based on matrix approximation

In section 3.7.2 we have seen that unfortunately an analytical expression for the estimator of the  $P$  matrix of eigenvectors, given a fixed matrix  $A_v$  with eigenvalues, does not exist and that we have to resort to numerical optimization procedures. Given the potentially large number  $np(n-1)$  of parameters which have to be optimized, it is convenient to have a good starting solution. There are two possibilities:

1. Use the relevant elements from the  $P$  matrix from the unrestricted MLS estimator (3.7.8) or other estimation method. This works especially well when the restricted eigenvalues are close to the eigenvalues of the unrestricted estimates. Note that the  $n+1$ -th until the  $(np)^2$ -th row of the  $P$  matrix need to be recalculated based on the new restricted eigenvalues and the true  $n-1$  parameters according to the special structure (3.3.12).
2. Use the optimal solution for another type of objective function for which an analytical solution does exist. This objective function should in some sense be an approximation of the multivariate RSS (3.7.6). The basic idea is to find a  $P$  matrix that minimizes the element by element difference between the  $PA_vP^{-1}$  matrix and the unrestricted parameter matrix  $A_e$ . That is, try to find the eigenvectors of a matrix that is as close as possible to the unrestricted matrix that in turn minimizes the multivariate RSS.

Such a matrix based approximate solution for  $P$  can be derived from the residual sum of squares

$$RSS_{Matrix}(vec(P))|_{A_v} = vec(A_e - PA_vP^{-1})' vec(A_e - PA_vP^{-1}) \quad (3.7.52)$$

The subscript *matrix* distinguishes this RSS from the original, time series based, RSS (3.7.6). Just as in (3.3.15),  $A_e$  is the parameter matrix of the VAR(1) representation of a VAR( $p$ ) model in terms of the standard basis.  $A_v$  is the same matrix but now in terms of the basis consisting of the eigenvectors of  $A_e$ . Based on the chain rule, the vector of first order partial derivatives is

$$\frac{\partial RSS_{Matrix}(vec(P))|_{A_v}}{\partial vec(P)'} = \frac{\partial RSS_{Matrix}(vec(PA_vP^{-1}))}{\partial vec(PA_vP^{-1})'} \frac{\partial vec(PA_vP^{-1})}{\partial vec(P)'} \quad (3.7.53)$$

The second expression is given by (3.7.25). For the first expression we introduce

$$B(P) = PA_vP^{-1} \quad (3.7.54)$$

which results in

$$\begin{aligned}
 RSS_{Matrix}(vec(B(P))) &= vec(A_e - B(P))' vec(A_e - B(P)) \\
 &= \{vec(A_e)' - vec(B(P))'\} \{vec(A_e) - vec(B(P))\} \\
 &= vec(A_e)' vec(A_e) - vec(A_e)' vec(B(P)) \\
 &\quad - vec(B(P))' vec(A_e) + vec(B(P))' vec(B(P))
 \end{aligned} \tag{3.7.55}$$

and thus

$$\frac{\partial RSS_{Matrix}(vec(B(P)))}{\partial vec(B(P))'} = -2vec(A_e)' + 2vec(B(P))' \tag{3.7.56}$$

Substitution of (3.7.25) and (3.7.56) in (3.7.53) yields

$$\begin{aligned}
 \frac{\partial RSS_{Matrix}(vec(P))|_{A_v}}{\partial vec(P)'} &= \{2vec(PA_v P^{-1})' - 2vec(A_e)'\} \\
 &\quad (P' \otimes I_n)^{-1} \{(A_v' \otimes I_n) - (I_n \otimes PA_v P^{-1})\}
 \end{aligned} \tag{3.7.57}$$

Unfortunately, equating these first order partial derivatives to zero (first order conditions) and solving for  $P$  yields only the trivial solution  $P=0$ . Numerical procedures are required to find a non trivial solution. In that case it is more efficient to perform the numerical optimization directly in terms of the original multivariate RSS instead of this approximate matrix solution.

A slightly different formulation of the matrix RSS to be minimized *does* however result in an analytical solution as required. The element by element deviation  $\varepsilon$  as implicitly used in the RSS (3.7.52) come from the equation

$$A_e = PA_v P^{-1} + \varepsilon \tag{3.7.58}$$

Postmultiplying this by  $P$  gives

$$A_e P = PA_v + \varepsilon P \tag{3.7.59}$$

Now assume that the postmultiplying matrix for  $\varepsilon$  is a fixed  $B$  matrix. Then we get

$$A_e P = PA_v + \varepsilon B \tag{3.7.60}$$

For the VAR(1) representation of an  $n$  dimensional VAR( $p$ ) model all the matrices in this equation have dimensions  $np \times np$  and we could minimize the RSS with respect to all the  $(np)^2$  elements of the  $A_e$  matrix. However, from the structure of  $A_e$  as given by (3.2.37) we see that in fact only the first  $n$  rows of  $A_e$  contain true model parameters: the matrices  $A_1, A_2, \dots, A_p$  next to each other. In order to minimize the RSS only for these  $n^2 p$  elements we premultiply the equation by the  $n \times np$  matrix

$$K = [I_n | 0 \quad \dots \quad 0] \tag{3.7.61}$$

This results in the equation

$$A_{e,K}P = KPA_v + \varepsilon_K B \quad (3.7.62)$$

where

$$A_{e,K} = KA_e \text{ and } \varepsilon_K = K\varepsilon = (A_{e,K}P - KPA_v)B^{-1}$$

The vector of error terms is then

$$\begin{aligned} \text{vec}(\varepsilon_K) &= \text{vec}\{(A_{e,K}P - KPA_v)B^{-1}\} \\ &= (B^{-1} \otimes I_n)\text{vec}(A_{e,K}P - KPA_v) \\ &= (B^{-1} \otimes I_n)(\text{vec}(A_{e,K}P) - \text{vec}(KPA_v)) \\ &= (B^{-1} \otimes I_n)((I_n \otimes A_{e,K})\text{vec}(P) - (A_v' \otimes K)\text{vec}(P)) \\ &= (B^{-1} \otimes I_n)((I_n \otimes A_{e,K}) - (A_v' \otimes K))\text{vec}(P) \end{aligned} \quad (3.7.63)$$

Here we used different relations from Appendix A.3. The new matrix RSS becomes

$$\begin{aligned} \text{RSS}_{\text{Matrix}}(\text{vec}(P))|_{A_v} &= \text{vec}(\varepsilon_K)^{*'}\text{vec}(\varepsilon_K) \\ &= \text{vec}(P)^{*'}((I_n \otimes A_{e,K}) - (A_v' \otimes K))^{*'}(B^{-1} \otimes I_n)^{*'} \\ &\quad (B^{-1} \otimes I_n)((I_n \otimes A_{e,K}) - (A_v' \otimes K))\text{vec}(P) \end{aligned} \quad (3.7.64)$$

Here the operator  $(.)^{*'}$  denotes the transpose of the complex conjugate in order to have a real valued objective function in case of complex eigenvalues. The row vector of first order partial derivatives of this RSS is

$$\begin{aligned} \frac{\partial \text{RSS}_{\text{Matrix}}(\text{vec}(P))|_{A_v}}{\partial \text{vec}(P)'} &= \text{vec}(P)' 2((I_n \otimes A_{e,K}) - (A_v' \otimes K))^{*'}(B^{-1} \otimes I_n)^{*'} \\ &\quad (B^{-1} \otimes I_n)((I_n \otimes A_{e,K}) - (A_v' \otimes K)) \\ &= \text{vec}(P)' Q \end{aligned} \quad (3.7.65)$$

Note that, despite limiting the RSS to the first  $n$  rows of  $A_e$ , the RSS (3.7.64) is still a function of the entire  $P$  matrix. The symmetrical matrix  $Q$  therefore still has dimensions  $(np)^2 \times (np)^2$ . The only advantage is that the matrices involved in its calculation, such as  $A_{e,K}$ , are of lower dimensions. The (Hessian) matrix of second order partial derivatives is

$$\begin{aligned} \frac{\partial^2 \text{RSS}_{\text{Matrix}}(\text{vec}(P))|_{A_v}}{\partial \text{vec}(P)' \partial \text{vec}(P)} &= 2((I_n \otimes A_{e,K}) - (A_v' \otimes K))^{*'}(B^{-1} \otimes I_n)^{*'} \\ &\quad (B^{-1} \otimes I_n)((I_n \otimes A_{e,K}) - (A_v' \otimes K)) \end{aligned} \quad (3.7.66)$$

which is a positive definite matrix (second order conditions) indicating that equating (3.7.65) to zero and solving for  $P$  results in a (local) minimum for the matrix RSS.

In order to equate the first order partial derivatives to zero (first order conditions) and obtain a non trivial solution for the true  $np(n-1)$  parameters of  $P$ , several steps have to be taken. These steps can best be illustrated on an example for  $p=3$  and  $n=2$ . In that case

$$A_e = \begin{bmatrix} A_1 & A_2 & A_3 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix}, \quad A_v = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_6 \end{bmatrix} \quad (3.7.67)$$

and

$$P = \begin{bmatrix} 1 & \cdots & 1 \\ p_{2,1} & \cdots & p_{2,6} \\ \vdots & & \vdots \\ p_{6,1} & \cdots & p_{6,6} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & \cdots & \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_1} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & & \frac{1}{\lambda_6} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_1^2} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} & \cdots & \frac{1}{\lambda_6^2} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \end{bmatrix} \quad (3.7.68)$$

The transpose of the row vector (3.7.65) with symmetrical matrix  $Q$  then becomes

$$\begin{bmatrix} \partial/\partial p_{1,1} \\ \partial/\partial p_{2,1} \\ \partial/\partial p_{3,1} \\ \partial/\partial p_{4,1} \\ \partial/\partial p_{5,1} \\ \partial/\partial p_{6,1} \\ \vdots \\ \vdots \\ \partial/\partial p_{1,6} \\ \partial/\partial p_{2,6} \\ \partial/\partial p_{3,6} \\ \partial/\partial p_{4,6} \\ \partial/\partial p_{5,6} \\ \partial/\partial p_{6,6} \end{bmatrix} = \begin{bmatrix} q_{1,1} & \cdots & q_{1,36} \\ \vdots & & \vdots \\ q_{36,1} & \cdots & q_{36,36} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} \\ \frac{1}{\lambda_1} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} \\ \frac{1}{\lambda_1^2} \begin{pmatrix} 1 \\ p_{2,1} \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_6} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \\ \frac{1}{\lambda_6^2} \begin{pmatrix} 1 \\ p_{2,6} \end{pmatrix} \end{bmatrix} \quad (3.7.69)$$

1. The first step towards a non trivial solution of (3.7.65) equated to zero has already been taken into account in the right hand side vector of the example (3.7.69). By dividing all rows of  $P$  by the first row, the eigenvectors are normalized with respect to their first elements. This rules out the trivial solution  $P=0$  and does not limit the solution because eigenvectors are unique only up to a constant factor.
2. The second step has also already been taken into account in the right hand side vector of the example (3.7.69). This concerns the special relation between the  $p$  stacked  $n \times 1$  subvectors in the eigenvectors of the VAR(1) representation of a VAR( $p$ ) model as given by (3.3.12).



3. Next, the chain rule given by the example (3.7.31) has to be incorporated. These relations hold conditional on the substitution of the relations described in steps 1. and 2. For each eigenvector, the rows of the  $Q$  matrix corresponding to the second until the  $n$ -th row of the eigenvector have to be adjusted based on this chain rule. For each eigenvector this results in  $n-1$  equations for the partial derivative for the  $n-1$  true parameters of the eigenvector. The matrix  $Q$  has to be adjusted according to

$$Q[(i-1)np + j, \cdot] = Q[(i-1)np + j, \cdot] + \sum_{k=1}^{p-1} Q[(i-1)np + j + kn, \cdot] \frac{1}{\lambda_i^k} \quad (3.7.70)$$

for  $i=1, \dots, np$ , all eigenvectors, and  $j=2, \dots, n$ , all  $n-1$  true parameters for each eigenvector.

4. The fourth step is to select from the  $np$  rows corresponding to each of the  $np$  eigenvectors, the second until the  $n$ -th row from the modified  $Q$  matrix to isolate the equations corresponding to the derivatives for the  $n-1$  true parameters per eigenvector.  $Q$  then changes from a  $(np)^2 \times (np)^2$  matrix into a  $np(n-1) \times (np)^2$  matrix.
5. Create the new linear  $np(n-1) \times np(n-1)$  system  $Ab=c$  by first constructing the constant vector  $c$  as the negative of the sum of the columns of  $Q$  corresponding to the first (normalized) element of each of the eigenvectors. That is

$$c = - \sum_{i=1}^{(np)^2} Q[\cdot, i] \quad (3.7.71)$$

6. Create the  $np(n-1) \times np(n-1)$  matrix  $A$  from the system by adding for each true parameter per eigenvector the corresponding columns of  $Q$ . That is

$$A[\cdot, (i-1)(n-1) + j - 1] = \sum_{k=1}^p Q[\cdot, (i-1)np + (k-1)n + j] \quad (3.7.72)$$

for  $i=1, \dots, np$ , all eigenvectors, and  $j=2, \dots, n$ , all  $n-1$  true parameters for each eigenvector.

7. Solve the new system for  $b$  as  $b = A^{-1}c$
8. From the optimal solution  $b$  with dimensions  $np(n-1) \times 1$  and the  $np$  eigenvalues  $\lambda_1, \dots, \lambda_{np}$  calculate the full  $P$  matrix with dimensions  $(np)^2 \times (np)^2$  according to the special structure of the  $P$  matrix of the VAR(1) representation of a VAR( $p$ ) model as given by (3.3.12).

The optimal solution holds conditional on a fixed  $B$  matrix. From the original equation (3.7.59) we know that  $B=P$  must hold in order to obtain the true optimal solution with respect to the matrix  $A_e$ . Given the known optimal solution for a fixed  $B$  matrix it seems logical to first set  $B$  equal to the  $P$  matrix of the unrestricted matrix  $A_e$ . The resulting optimal  $P$  matrix can then be used as  $B$  in a next optimization and so on. Unfortunately this iterative procedure often does not converge. In that case the only possibility is to stick to a single application of the solution with either  $B=I$  or  $B=P$  from the unrestricted  $A_e$  matrix.

The matrix approximation as described here appears to have a special property that is worth mentioning. In case the diagonal of the fixed  $A_v$  matrix contains identical eigenvalues, the corresponding eigenvectors in the optimal  $P$  matrix turn out to be identical also. In that case the  $P$  matrix does not have full rank and therefore its inverse and hence  $A_e = PA_v P^{-1}$  cannot be calculated resulting in a failure to produce an adequate starting solution. The fact that identical eigenvalues lead to identical eigenvectors indicates that in fact some kind of independent sub-problems are being solved in the procedure. Examples indeed show a block diagonal structure for the  $A$  matrix in the system  $Ab=c$  that is solved in the end of the procedure. For example if  $p=2$  and  $n=6$  there are  $np=12$  eigenvalues and the  $A$  matrix has dimensions  $np(n-1) \times np(n-1) = 60 \times 60$  with 12  $(n-1) \times (n-1) = 5 \times 5$  matrices on its diagonal. In case of identical eigenvalues these, some of these  $5 \times 5$  sub-matrices are identical and therefore also produce identical optimal values for the  $n-1$  true parameters of the associated eigenvalues. Fortunately there is a simple solution for this problem. In case of identical eigenvalues it suffices to introduce only a small difference between these eigenvalues to prevent having identical eigenvectors in the solution. This can be implemented by multiplying each next identical eigenvalue in the  $A_v$  matrix by for example a factor 0.99.

### 3.7.5 Estimation

Three possible applications of the EVR VAR models seem interesting:

1. The first is fixing the eigenvalues and estimating the optimal corresponding eigenvectors. This can be either to enforce a certain type of dynamics on a low order VAR model or to reduce the dynamics of a high order VAR model. We will see a practical example of this type of EVR VAR models in section 20.5.
2. The second is finding the optimal eigenvalues with restrictions in terms of stationarity, frequency range or enforcing equality between certain eigenvalues while for each considered combination of eigenvalues the optimal corresponding set of eigenvectors is determined. The underlying idea is that the eigenvalues are the most fundamental parameters determining the dynamics of a VAR model while the eigenvectors “merely” give an optimal amplitude and phase adjustment to these fundamental dynamics for each of the variables. Estimating these types of EVR VAR models comes down to performing a numerical optimization in terms of the eigenvalues while for each RSS evaluation within this optimization yet another numerical optimization needs to be done in terms of the eigenvectors given the eigenvalues. Note that in this case derivatives can only be determined using numerical differentiation which further complicates the optimization. Although for low order, low dimension VAR models this can be done surprisingly well, for higher order and higher dimensional models such an estimation procedure becomes numerically infeasible. Putting restrictions on the eigenvectors might possibly reduce the computational effort. Also, having an analytical expression for the optimal eigenvectors, given a set of eigenvalues, would increase the range of feasible applications of this type of EVR VAR models. For now, these types of models will not be considered any further.

3. The third possible application is the case where restrictions are put on the eigenvectors while estimating the optimal corresponding eigenvalues. Although, as stated in the introduction of section 3.7, such applications are certainly not unthinkable, they are less obvious at the moment because of the more fundamental role of the eigenvalues in determining the dynamics of a VAR model. Furthermore, the application of the analytical derivatives in section 3.7.3 is limited to VAR models of order  $p=1$ . For now, these types of models will not be considered any further.

In the remainder of this section the focus is on the first of these three applications of EVR VAR model. From section 3.7.2 we know that an analytical solution for the optimal set of eigenvectors given some set of fixed eigenvalues does not exist. The only alternative is then to use a numerical procedure for finding the multidimensional root of the first order conditions equating (3.7.26) to zero. Together with the Hessian matrix of second order partial derivatives calculated by numerical differentiation of the vector of first order partial derivatives, this root finding problem can be solved with the Newton-Raphson algorithm as described in for example Burden and Faires (1989, section 10.2) or Press et al. (1989, section 9.6). However, for this algorithm it is well known that starting from a solution sufficiently close to an actual root is of crucial importance for actually finding the root instead of drifting away from the solution. It is therefore indispensable to first minimize the multivariate RSS (3.7.6) as far as possible using for example the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm and the Polak-Ribiere algorithm as described in for example respectively section 10.7 and 10.6 of Press et al. (1989). Both these algorithms make heavy use of the analytical first order partial derivatives for the eigenvectors as given by (3.7.26). Experimenting with the problem at hand learned that it works best to first apply the BFGS algorithm for a fast decrease in the objective function, second to apply the Polak-Ribiere algorithm for a slower but more precise further decrease of the objective function and finally to apply the Newton-Raphson algorithm for finding the actual optimal solution. In all cases it also appeared to be the most robust to use the natural logarithm of the multivariate RSS as the objective function instead of the RSS itself.

As indicated earlier, a good starting value for the matrix  $P$  is very welcome because of the potential high dimensionality of the problem. In any case it is best to first perform an unrestricted estimation using the MLS estimators described in section 3.7.1. If the desired restricted eigenvalues are close to the eigenvalues from this unrestricted estimate, it works well to use the eigenvalues from the unrestricted estimate as the starting values for  $P$ . Of course, in case of an order  $p>1$ , an update of the  $n+1$ -th until the  $np$ -th row according to the special structure (3.3.12) using the new set of eigenvalues is required. If the desired restricted eigenvalues are too different from the eigenvalues from the unrestricted estimate, for example if one contains only complex eigenvalues while the other also contains real eigenvalues, then a set of eigenvectors based on the matrix approximation as described in section 3.6.8 can be used with  $B=I$ .

With respect to this numerical estimation procedure of an EVR VAR model a number of further issues are worth discussing.

### 1. *The $\Sigma$ matrix*

The procedure requires an initial  $\Sigma$  matrix for the evaluation on the multivariate RSS (3.7.6). For this, the unrestricted MLS estimator (3.7.10) can be used. To be more precise, the inverse of this  $\Sigma$  matrix is required. In the case of the necessary VAR(1) representation of a VAR( $p$ ) model, the  $\Sigma$  matrix has the structure of the matrix  $\Psi$  in (3.2.37) which cannot be inverted. The MLS estimator for this matrix will typically have very small values at the zero positions in the matrix and can therefore be inverted. However, because of these small numbers, this inverse will typically have very large values, thereby disturbing the value of the multivariate RSS. The correct alternative is to first calculate the inverse of the normal  $n \times n$   $\Sigma$  matrix and then to put this in the top left corner of the  $np \times np$  matrix  $\Psi$  and defining this as  $\Psi^{-1}$  which goes into the multivariate RSS (3.7.6) as the  $\Sigma^{-1}$  matrix. In this way, in the multivariate RSS (3.7.6) the residuals of the  $n$  true variables are appropriately weighed by the normal  $n \times n$   $\Sigma^{-1}$  matrix while the residuals from the  $np - n$  artificial variables from the VAR(1) representation (3.2.37) are weighed by zero. The latter is without consequences because these residuals should be zero anyway by the definition of the VAR(1) representation (3.2.37).

Based on this initial  $\Sigma$  matrix, the restricted parameter matrices  $A_1, \dots, A_p$  can be estimated using the numerical procedure described previously. The corresponding  $\Sigma$  matrix can be simply estimated as the sample covariance matrix of the residuals calculated from the  $A_1, \dots, A_p$  estimates and the sample data of the variables in the model. Just as with the SUR estimator described in section (3.4.2), this new estimate of the  $\Sigma$  matrix can then be used for producing a new estimate for the  $A_1, \dots, A_p$  matrices from the numerical procedure which then in turn produces a new estimate of the  $\Sigma$  matrix. In principle this procedure can be repeated until convergence. However, in order to limit the computational effort and because experiments showed that the changes in the  $\Sigma$  matrix are only minor after the second estimate, two steps seem sufficient in practice.

### 2. *Identical eigenvalues*

At the end of the previous section we saw an undesirable property of the optimal  $P$  matrix based on the matrix approximation. In the numerical procedure outlined here, this problem of identical eigenvectors does not occur. Hence, several restricted eigenvalues can be set at exactly the same value without problems. Only the special case in which *all* eigenvalues are identical should be avoided. Not because the estimation would fail but because in this special case all the  $A_1, \dots, A_p$  matrices become diagonal matrices, no matter what the eigenvectors are. Clearly VAR models with this type of parameter matrices are unable to describe any cross correlations in time between its variables which will be considered to be too limiting in most circumstances. Including only one different eigenvalue is sufficient for having non diagonal  $A_1, \dots, A_p$  matrices and allowing for cross correlations in time.

### 3. *Adjusting model standard deviations*

Just as any other estimation procedure, apart from the Yule-Walker estimator, the standard deviations of the variables in an estimated EVR VAR model can differ from the sample standard deviations of these variables. If it is required that the model standard deviations are identical to the sample standard deviations, or to any other set of standard deviations, without changing the other statistical properties of the

model, its expected values and auto and cross correlations, this can easily be accomplished by using the Yule-Walker equations in the following steps.

- Use the Yule-Walker equations (3.2.33) and (3.2.30) to calculate the autocovariance matrices  $\Gamma_i$  for  $i=0, \dots, p$ . Also calculate the expected values from (3.2.36).
- Calculate the autocorrelation matrices  $P_i$  for  $i=0, \dots, p$  from the autocovariance matrices as

$$P_i = D^{-1} \Gamma_i D^{-1} \quad (3.7.73)$$

with

$$D^{-1} = \begin{bmatrix} 1/\sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/\sigma_n \end{bmatrix}$$

in which  $\sigma_i$  is the variance of the  $i$ -th variable, calculated as the square root of the  $i$ -th diagonal element of  $\Gamma_0$ .

- Calculate adjusted autocovariance matrices for  $i=0, \dots, p$  as

$$\Gamma_i^* = D^* P_i D^* \quad (3.7.74)$$

with

$$D^* = \begin{bmatrix} \sigma_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^* \end{bmatrix}$$

in which  $\sigma_i^*$  is the desired (sample) standard deviation of the  $i$ -th variable.

- Based on these adjusted autocovariance matrices calculate the corresponding parameter matrices  $A_1, \dots, A_p$  and  $\Sigma$  from the Yule-Walker equations (3.2.31). The corresponding constant vector is then

$$v = (I - A_1 - A_2 \dots - A_p)E(x_t) \quad (3.7.75)$$

Note that for a VAR(1) model this gives the following expressions for the adjusted parameters of the model

$$\begin{aligned} A_1 &= \Gamma_1 \Gamma_0^{-1} = (D^* P_1 D^*) (D^* P_0 D^*)^{-1} = D^* P_1 P_0^{-1} D^{*-1} \\ \Sigma &= \Gamma_0 - A_1 \Gamma_1' \\ v &= (I - A_1)E(x_t) \end{aligned} \quad (3.7.76)$$

These expressions show that both the parameter matrices, the covariance matrix of the error terms and the constant vector change by applying the new standard deviations in  $D^*$  and not only the covariance matrix of the error terms as one might have expected.

#### 4. Shape of the objective function

Because of the complex structure of an EVR VAR model in terms of the restricted eigenvalues and the eigenvectors that need to be estimated, it is interesting to have information on the shape of the objective function, the multivariate RSS (3.7.6), as a function of the parameters of the eigenvectors. Consider the case of a six dimensional VAR(1) model for which the six eigenvalues are fixed at three specific complex eigenvalues and their conjugates, describing a specific type of dynamics<sup>27</sup>. Given these eigenvalues, the  $P$  matrix with eigenvectors has been estimated using the numerical procedure described here. Starting from this optimal solution there are two possibilities of constructing one dimensional cross sections of the objective function in terms of each of the parameters of the eigenvectors. The first possibility is to view the real and imaginary parts of all eigenvector elements as being the parameters. The second possibility is to view the polar co-ordinates of all eigenvector elements as being the parameters.

For the first type of cross sections we varied the value of the real and imaginary parts of each of the five true parameters in the three primary eigenvectors, of which the complex conjugates are also eigenvectors, as

$$x^* = x + f \cdot x \quad (3.7.77)$$

with  $f \in \langle -1, 4 \rangle$  and  $x^* \in \langle 0, 5x \rangle$ . That is, the real and imaginary parts are varied between zero and five times their optimal value. When varying one of the parameters, all others remain unchanged at their optimal value. From the new combination of parameters the eigenvectors can be calculated from which an alternative parameter matrix  $A_1^*$  can be calculated using (3.3.15). This in turn can be used to calculate the logarithm of the associated multivariate RSS (3.7.6), keeping the  $\Sigma$  matrix unchanged. With a grid of 100 points in each cross section this results in  $3 \times 5 \times 2 = 30$  cross sections. Each picture in Figure 3.4 and 3.5 shows five cross sections corresponding to respectively the real and imaginary part of the five elements of one of the three primary complex eigenvectors. On the  $x$ -axis the value of  $f$  from (3.7.77) is plotted. The zero point corresponds to the optimal solution.

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<sup>27</sup> The model used here is the order  $p=1$  EVR VAR model estimated in section 20.5.



Figure 3.4: Cross sections from minimal value of  $\ln(\text{RSS})$  for real parts of five parameters of three primary complex eigenvectors of EVR VAR model of order  $p=1$ .

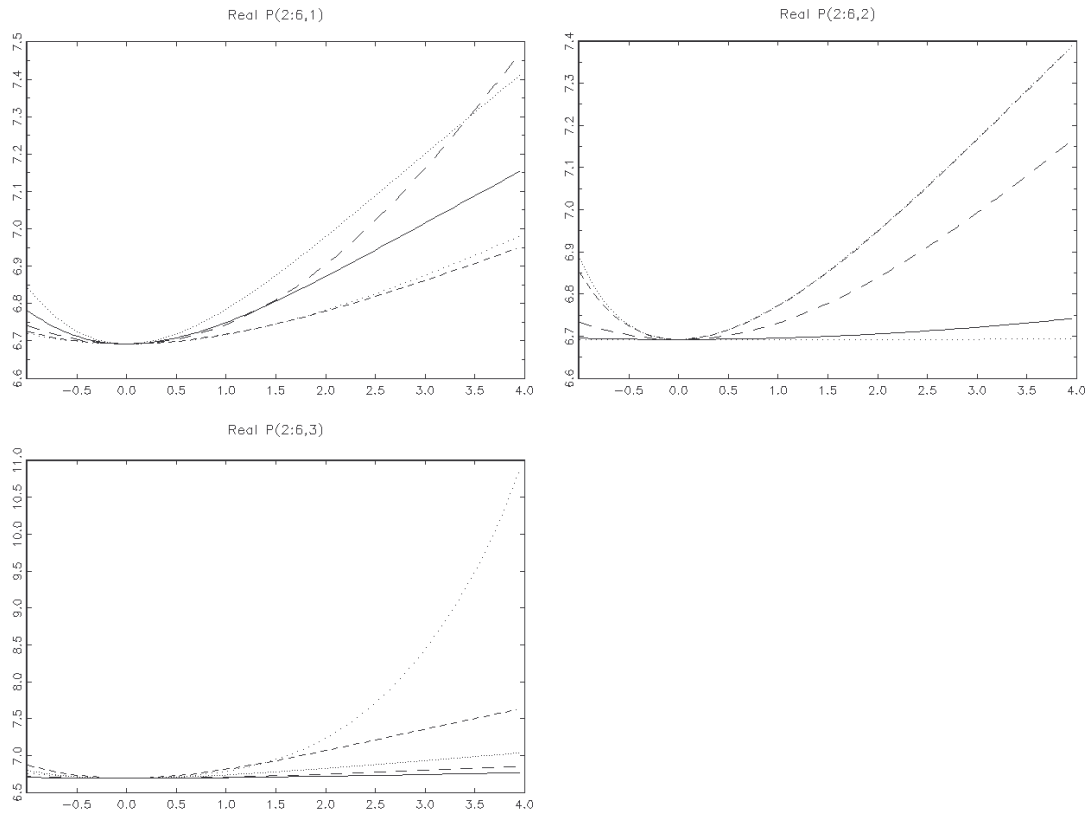
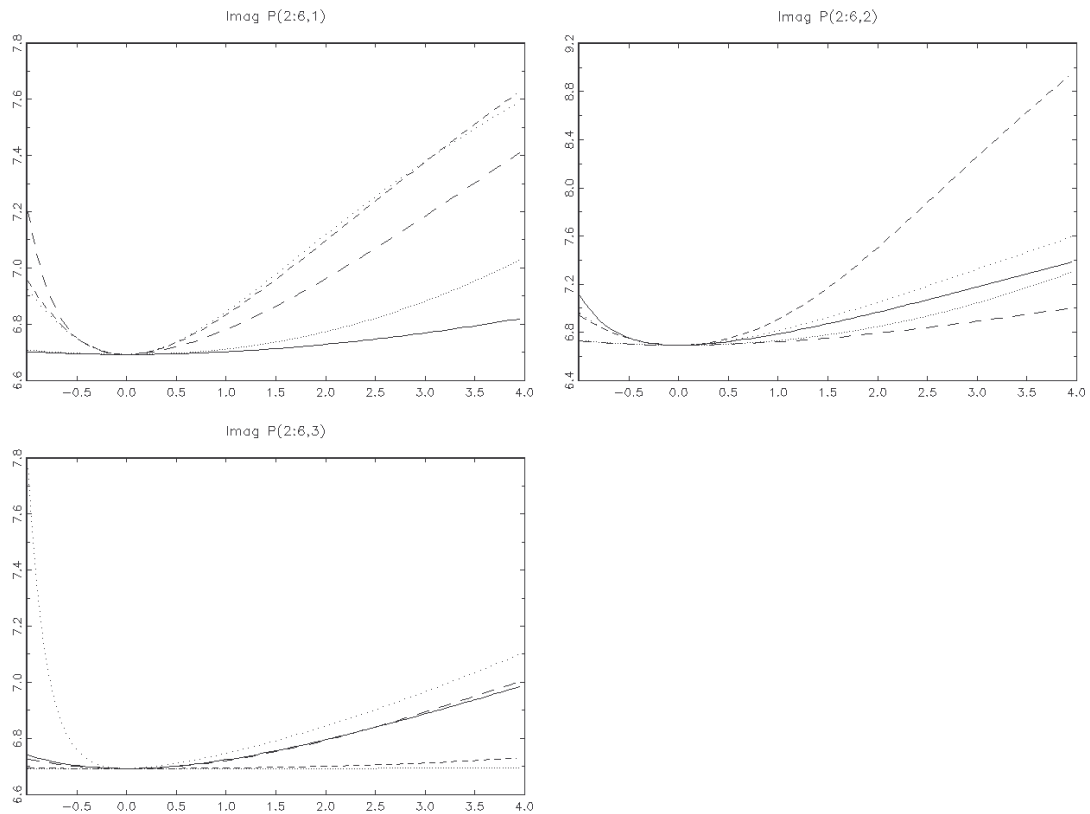


Figure 3.5: Cross sections from minimal value of  $\ln(\text{RSS})$  for imaginary parts of five parameters of three primary complex eigenvectors of EVR VAR model of order  $p=1$ .





From these cross sections in terms of the real and imaginary parts of the eigenvectors, the following can be observed.

- The solution found by means of the numerical optimization procedure seems indeed to be a global minimum. We can never be one hundred percent sure of this by looking in only a limited number of directions.
- In each considered direction, the objective function is convex without any local minima which guarantees that the numerical algorithms will find the global minimum.
- In some directions, the objective function can be rather flat which might cause a slow convergence of the numerical algorithms.

For the second type of cross sections in terms of the polar co-ordinates, the optimal eigenvalues were transformed into the  $\Delta$  (modulus) and  $\phi$  (argument) parameters as in section 3.3.2. For the  $\Delta$  parameters, we made cross sections using (3.7.77). For the  $\phi$  parameters, first the alias frequency in the range  $[-\pi/2, \pi/2]$  was determined. For this also see Appendix A.1. Next, they were expressed in terms of periods (years) by dividing them by the argument  $\omega$  of the corresponding eigenvalue. For this also see (A.1.8). Then the resulting lead was varied between minus one half a period length (lag) and plus half a period length using

$$x^* = x + f \cdot \frac{2\pi}{\omega} \quad (3.7.78)$$

with  $x^* \in \langle -\pi/\omega, \pi/\omega \rangle$ . The factor  $f$  should therefore be interpreted in terms of the number of period lengths change from the optimal phase and is truncated at such values that the resulting phase is not lower than minus half a period length and not higher than plus half a period length. Figure 3.6 and 3.7 show the cross sections in terms of the  $\Delta$  (modulus) parameters and transformed  $\phi$  (phase) parameters for each of the five parameters of the three eigenvectors. Just as with the cross sections in terms of the real and imaginary parts of the eigenvectors, there are  $3 \times 5 \times 2 = 30$  cross sections, the x-axis shows the value of  $f$  from respectively (3.7.77) and (3.7.78) while the zero point on the x-axis corresponds to the optimal solution.

Figure 3.6: Cross sections from minimal value of  $\ln(\text{RSS})$  for modulus of five parameters of three primary complex eigenvectors of EVR VAR model of order  $p=1$ .

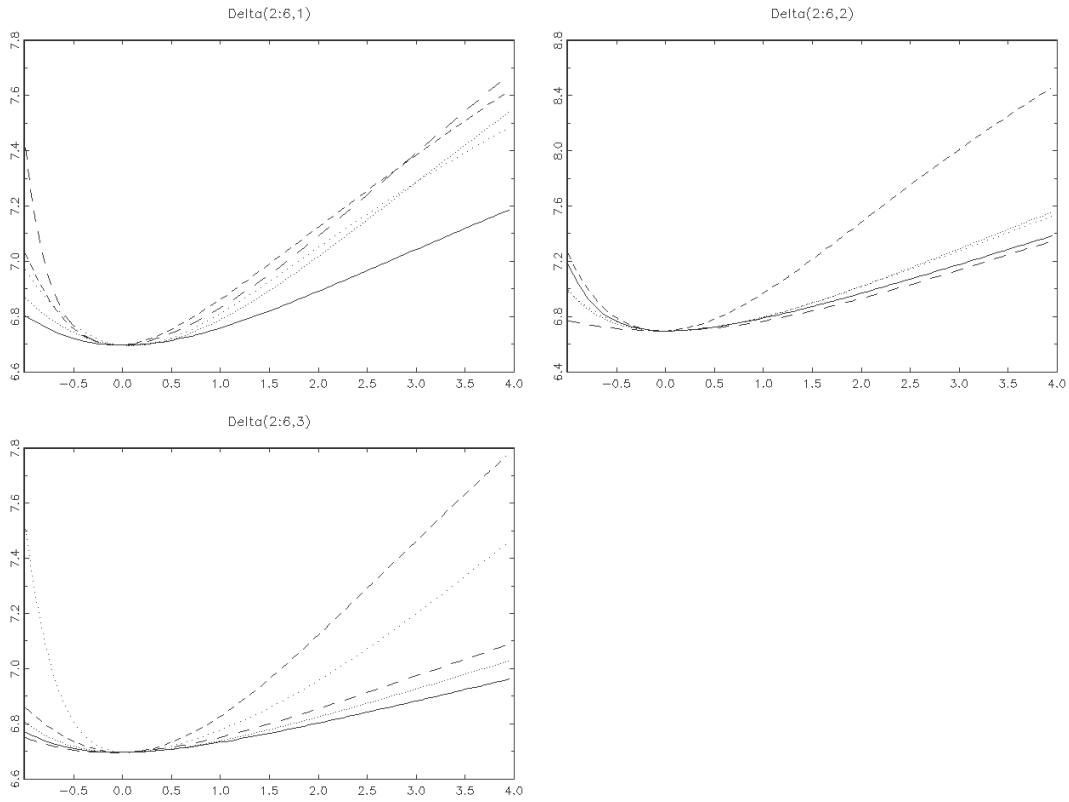
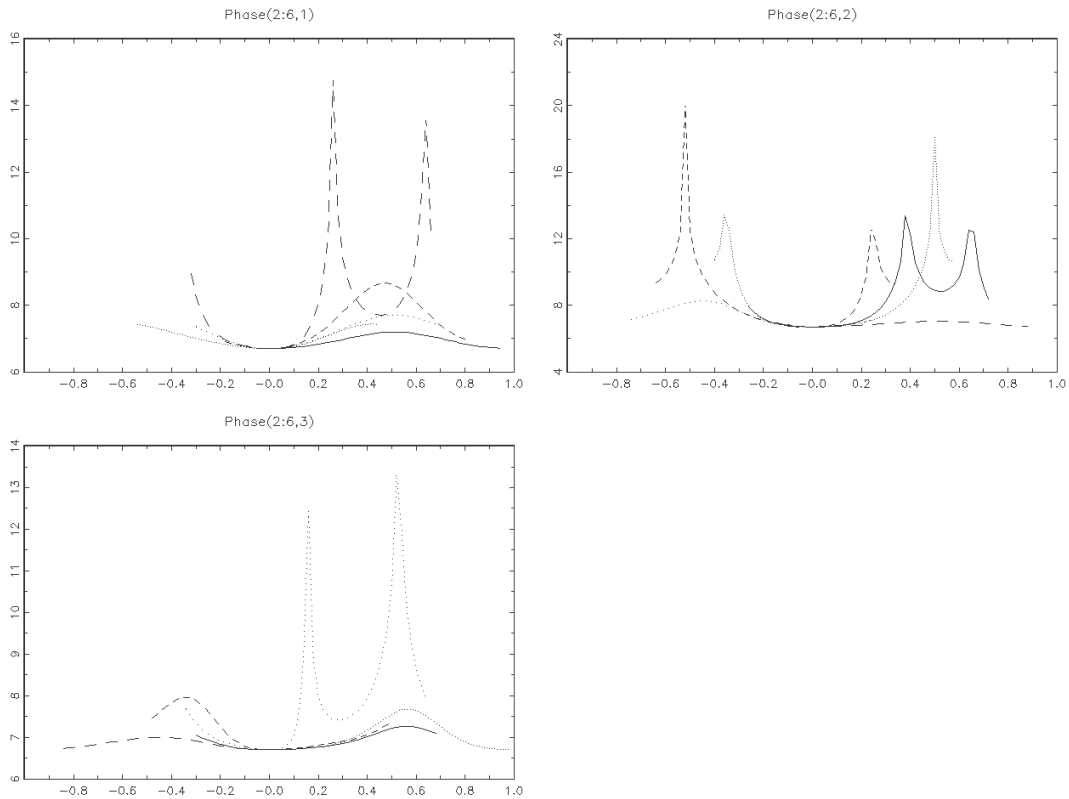


Figure 3.7: Cross section from minimal value of  $\ln(\text{RSS})$  for phase, in numbers of period lengths, of five parameters of three primary complex eigenvectors of EVR VAR model of order  $p=1$ .



From these cross sections in terms of the polar co-ordinates of the eigenvectors the following can be observed.

- The solution found by means of the numerical optimization seems indeed to be a global minimum.
- In terms of the modulus, the objective function is strictly (and more) convex, just as in case of the real and imaginary parts.
- In terms of the phase, the objective function is also convex in the direct neighbourhood of the global minimum. However at a greater distance of the optimal solution the objective function shows strong peaks which may enclose several local minima.

The peaks in the objective function are caused by situations in which the eigenvectors are (nearly) linear dependent. In such cases the  $P$  matrix no longer has full rank and the parameter matrix  $A_e = PA_v P^{-1}$  can no longer be calculated because of the ill conditioning of the  $P$  matrix and the associated problems in calculating its inverse. Apparently, in the cross sections in the directions of the real and imaginary parts such linear dependencies do not occur while in the special directions as determined by changing the phase they do occur. The latter directions are determined by the transformations from the real and imaginary parts of complex numbers into the polar co-ordinates and vice versa. The peaks and local minima in the objective function at least warn us that the logarithm of the multivariate RSS as a function of the eigenvectors can be difficult to minimize. A good starting solution for the  $P$  matrix can help to prevent that the numerical algorithms get stuck in a local minimum. Furthermore, restricting the eigenvalues to values close to the unrestricted estimates and using the corresponding eigenvectors as a starting solution seems a safe strategy, but of course limits the range of restricted eigenvalues. Finally, cross sections as those from Figure 3.6 and 3.7, starting from the optimal solution found, can be used to verify whether it is likely that a global minimum has been found or not. Again, such cross sections can never guarantee that a global minimum has been found because we can only look in a limited number of directions.

### 3.7.6 Relation to other types of VAR models

The objective of the EVR VAR models as presented in this section is to reduce the general problem of imprecise estimates of VAR model parameters caused by the fact that the number of parameters is often quite substantial relative to the available sample size or time series length. Of course EVR VAR models are not the only type of VAR models with this objective. In section 2.2.7 some alternative models are mentioned. Each of these models has its own specific way of imposing restrictions on the model parameters. This final section presents a short comparison of EVR VAR models with some of the most important alternative types of VAR models.

#### 1. VAR models with linear constraints

A first alternative for the EVR VAR models are VAR models on which linear constraints in terms of its conventional parameter matrices  $A_1, \dots, A_p$  are imposed. Such models and their estimation are described in for example Lütkepohl (1991, section 5.2). A special case of this type of models is a VAR model which has zero

constraints on its coefficients. These zero constraints can either come from prior information or from a statistical orientated procedure such as the Top-Down and Bottom-Up strategies described in section 3.4.3. Using linear constraints on the VAR model parameters, it is impossible to impose the strongly non-linear restriction in terms of eigenvalues or eigenvectors as done by the EVR VAR models. Contrary to the EVR VAR models, the linear restricted models can be estimated without use of numerical optimization procedures.

## 2. Reduced Rank VAR (RR VAR) models

A second alternative are the Reduced Rank VAR (RR VAR) models as described in Lütkepohl (1991, section 5.3). At first sight the RR VAR models show a resemblance to the EVR VAR models in the sense that also they impose non-linear restrictions on the VAR(1) representation of a VAR( $p$ ) model in terms of its eigenvalues. A rank  $r$ , lower than or equal to  $np$ , means that  $np-r$  eigenvalues are restricted to zero which is a special case of an EVR VAR model with fixed eigenvalues. An important first difference is however that in RR VAR models it is not possible to restrict the eigenvalues to other values than zero. An important second difference is that the resulting stochastic processes can be very different.

The following example illustrates the very different effects RR VAR models have on the dynamics of a VAR model. Consider the case of a six dimensional VAR(1) model which is estimated in three different ways. The first is an unrestricted estimate using the MLS estimator (3.7.8). The second is an EVR VAR model in which the six eigenvalues of the model are restricted to three specific complex eigenvalues and their complex conjugates, based on some prior information<sup>28</sup>. The third is an RR VAR model in which the parameter matrix is restricted to have rank two and hence can have one pair of complex conjugated eigenvalues or two real eigenvalues. Figure 3.8 shows the normalized auto-spectrum for the first of the six variables from the three estimated models. An auto-spectrum gives a very instructive description of the univariate dynamics of a stochastic variable. It describes the distribution of the total variance of the variable over cyclical movements with frequencies ranging from the lowest frequency zero until the highest frequency 0.5 (one cycle takes two time periods). The surface below the auto-spectrum integrates to one, representing the total variance, similar to a probability density function. A completely flat auto-spectrum corresponds to a (multivariate) white noise process. More information on an auto-spectrum and spectral analysis in general can be found in Chapter 4. From the results in Figure 3.8 we can observe the following:

- The unrestricted estimate shows a spectral peak around a frequency  $\omega=0.10$ , i.e. a period length of around ten years.
- The EVR estimate shows a sharper peak around the same frequency, thereby emphasizing the pseudo period behaviour of the variable and bringing it more in line with the auto-spectrum from an univariate Yule-Walker estimate which is also shown in the pictures.
- The RR VAR estimate shows a much weaker peak around the  $\omega=0.10$  frequency and a flatter auto-spectrum over the complete frequency range.

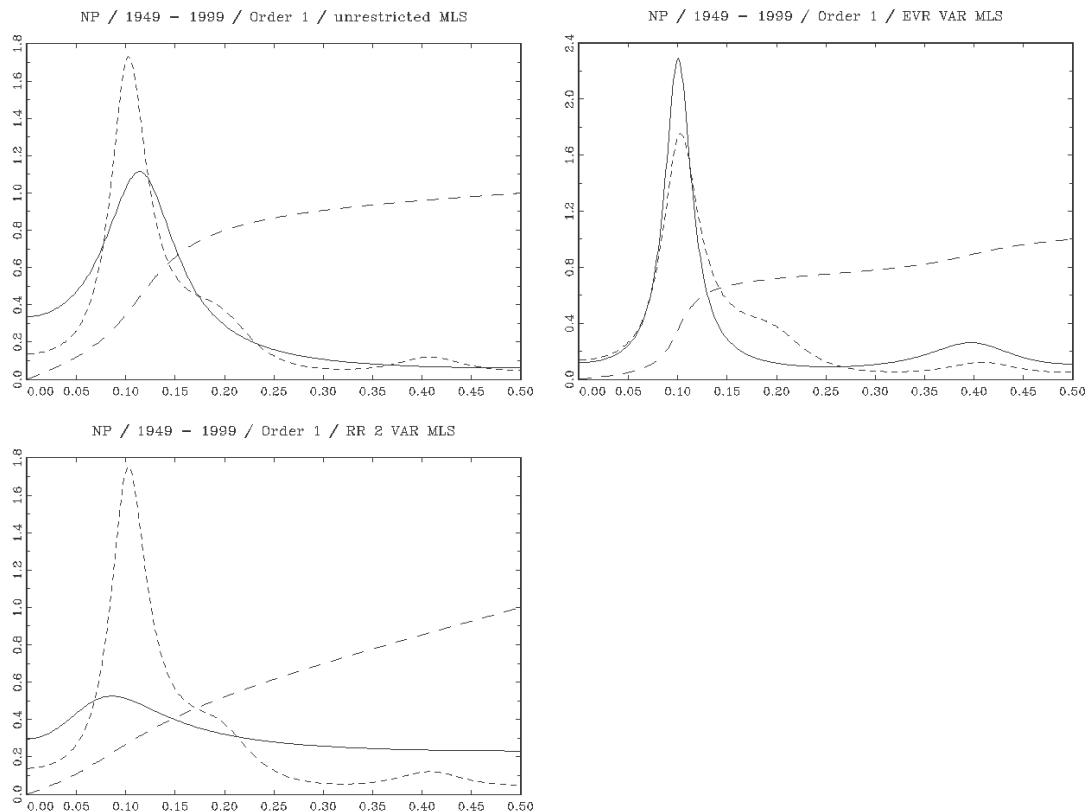
Although a set of two complex conjugate eigenvalues dominates the dynamics of the EVR estimate, the RR VAR model with rank two, that can also have two complex eigenvalues, shows much less dominant cyclical behaviour. Instead, because of its rather flat auto-spectrum, it describes much more of a very different white noise

<sup>28</sup> The model used here is the order  $p=1$  EVR VAR model estimated in section 20.5.

behaviour. Here we see that, although the RR VAR is restricted in terms of the number of non-zero eigenvalues, it is *not* restricted in the actual values these eigenvalues might have. Second, it holds that the lower the rank, the more eigenvalues are zero and the more the spectral densities, and hence the stochastic process described by the model, will resemble that of a (multivariate) white noise process described by the covariance matrix of the error terms. Contrary to the EVR VAR models, the RR VAR models can be estimated without use of numerical optimization procedures.

Figure 3.8: Normalized auto-spectrum of one variable from a six dimensional VAR(1) model.

*Top left:* unrestricted MLS estimate (rank six). *Top right:* EVR estimate with three fixed complex eigenvalues (rank six). *Bottom left:* RR estimate with rank two. The dotted auto-spectrum represents an univariate YW estimate.



### 3. Index VAR models

A third alternative for the EVR VAR models are index VAR models as described by Sargent and Sims (1977), Sims (1981) and Quah and Sargent (1993). Index VAR models show similarities with the RR VAR models. They will therefore probably also show the same differences with the EVR VAR models as the RR VAR models. Just like the EVR VAR models, the index VAR models have to be estimated using numerical optimization procedures.

### 4. Co-integration and error correction VAR models

A fourth alternative is the co-integration and error correction framework as introduced by Engle and Granger (1987) and further developed by for example Johansen (1988). The associated Vector Error Correction Model (VECM) is also a special class of a general VAR model.

The VAR( $p$ ) model

$$x_t = A_1 x_{t-1} + \dots + A_p x_{t-p} + \varepsilon_t \quad (3.7.79)$$

can be rewritten as

$$\Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{p-1} \Omega_i \Delta x_{t-i} + \varepsilon_t \quad (3.7.80)$$

with

$$\Pi = A_1 + \dots + A_p - I_n \text{ and } \Omega_i = - \sum_{j=i+1}^p A_j$$

The matrix  $\Pi$  can be written as

$$\Pi = \alpha \beta' \quad (3.7.81)$$

where  $\alpha$  and  $\beta$  are matrices with dimensions  $n \times r$  so that  $r$  is the rank of  $\Pi$ . The relations  $\beta' x_{t-1}$  are called the  $r$  co-integrating relations. If  $r=0$ , (3.7.80) becomes a stationary VAR( $p-1$ ) model on the stationary first order differences  $\Delta x_t$  and hence all  $n$  variables are integrated of order one. If  $r=n$ , (3.7.80) becomes the original VAR( $p$ ) model for the, non-integrated, level variables  $x_t$ . For all intermediate values  $0 < r < n$ , (3.7.80) described  $n$  integrated variables but together with  $r$  stationary co-integration relations. The only correspondence with an EVR VAR model is that a co-integrated VAR model is also determined by the rank and hence the eigenvalues of a matrix. However, in this case it is the very specific  $n \times n$  matrix  $\Pi$  within the VECM format instead of the  $np \times np$  matrix of the VAR(1) representation of a VAR model. Furthermore, just as with the RR VAR models, the restrictions on the eigenvalues can only be zero restrictions.

### 5. Bayesian VAR models

A fifth alternative for the EVR VAR models are Bayesian VAR models in which prior information on the model parameters is used in the estimates. In Bayesian VAR models the prior information is specified in terms of probability distributions instead of only fixed or equally likely ranges of values such as in the EVR VAR case. On the other hand, the prior information in Bayesian models in general holds in terms of the parameters of the conventional linear format of a VAR model and therefore, just as for the other models with linear constraints, does not allow for the specific non-linear restrictions in terms of the eigenvalues and eigenvectors.

### 6. Structural VAR (SVAR) models

A sixth alternative are the Structural VAR (SVAR) models in which one applies restrictions to arrive at “behavioral disturbances” that give a meaningful interpretation to the stochastic error components of the models. One than for example identifies permanent (supply) and transitory (demand) disturbances in the output process of the economy by applying the appropriate restrictions on a VAR model. The approach of incorporating structural information into a VAR model is

based on the fact that many theoretical models have an equivalent (restricted) VAR representation. These models are extensively used in the economic literature. There are two differences with the EVR VAR models. The first is that SVAR models focus on the disturbances while the EVR VAR models focus on the dynamics that are to a large extent determined by the autoregressive parameter matrices. The second difference is that the restrictions on SVAR models originate from economic theory. Although this can certainly also be the case for EVR VAR models, these will often also have restrictions with an empirical origin.

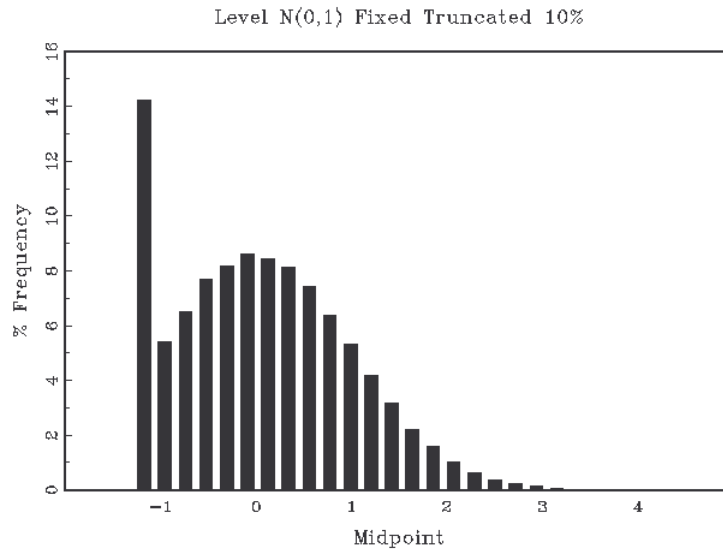
The conclusion of the list of comparisons of the EVR VAR models with other types of restricted VAR models must be that the EVR VAR models are indeed a separate class of VAR models with its very specific characteristics. Their most important feature is that they allow for imposing strongly non-linear restrictions on the parameter matrices in terms of the fundamental eigenvalues and eigenvectors which directly determine the dynamic properties of a VAR model. A complicating factor is the required numerical estimation of the models. Finally, some other types of restricted VAR models, such as the RR VAR models appear to be special cases of the EVR VAR models.

### 3.8 Truncated distributions

VAR models in general have Normally distributed error terms. The overall distribution of the variables described by the model is therefore a multivariate Normal distribution. Although using the Normal distribution has several advantages, one disadvantage might be the thin tails of this distribution. With small probabilities both very high and very low values can occur. In some applications this property can cause modeling problems. Think for example of an interest rate variable which for economic reasons should not fall below zero. However, for scenarios from a VAR model estimated on a sample of positive historical interest rates, this can very well be the case. This holds especially if the simulations start from a low end of sample value and / or have a large volatility. One simple way of preventing scenarios from a VAR model to fall below some critical value is to first simulate all values and then simply replace all simulated values that have fallen below the critical level by the critical level itself. The resulting distributions can look like the histogram in Figure 3.9 calculated from a  $N(0,1)$  distribution truncated at the 10% percentile of minus 1.282. From this we see that the entire 10% probability mass which was originally left of minus 1.282 has been transferred onto the truncation point ( $4\%+10\%=14\%$ ), thereby producing a rather undesirable probability distribution function (pdf). Another disadvantage of such an approach is that it changes the moments of the original distribution such as expected values, standard deviations and correlations.



Figure 3.9 Histogram of 10,000 draws from a  $N(0,1)$  distribution for which every value below the lower 10% percentile  $z_\alpha = -1.282$  is replaced by this truncation point.



In the following sections another way of truncating Normal distributions is described that does not show such an undesirable pdf and maintains the moments of the original distribution. Examples of other types of distributions with an absolute minimum value that might be used are the lognormal, Snedecor's F and Chi-Square distributions. Some compelling reasons for choosing the truncated Normal distribution are:

- Staying as close as possible to the Normal distribution of the VAR model which causes all the theory of VAR models as described in the preceding sections to still hold approximately.
- The preservation of the first two moments of the original distribution.
- A straightforward generalisation into a multivariate setting.
- Flexibility in choosing the truncation levels. This might just as well be an upper value instead of a lower value.

Section 3.8.1 starts by explaining the workings of truncating the levels of a Normally distributed univariate random variable. Section 3.8.2 extends this to truncating a special transformation of such a random variable: the first order differences. Section 3.8.3 closes with an explanation of how these principles can be applied on a more complex transformation of Normally distributed random variables: a VAR model. All theory here concerns the univariate case but can readily be generalised for the multivariate case. However, some remarks on the multivariate case will be made along the way. The basic principles of a truncated Normal distribution as given in the beginning of the next section are known from the literature. What is new here, is that we show how to achieve that the first and second moment of the truncated distribution are the same as these first and second moment of the original, non-truncated distribution. Furthermore, we extend this basic principle to maintaining the same first and second moment of transformation of Normal distributions such as the first order differences and finally to the new class of *Truncated VAR* (TVAR) models in section 3.8.3. TVAR models are one of the components of the new scenario framework described and tested in Chapter 20.

### 3.8.1 Normal distribution

Maddala (1983, p. 365) describes the expected value and variance of a Normally distributed variable  $X \sim N(\mu, \sigma)$  which is truncated on the left at  $X \geq c$  as

$$E(X) = \frac{\sigma \phi\left(\frac{c - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{c - \mu}{\sigma}\right)} + \mu \quad (3.8.1)$$

and

$$V(X) = \sigma^2 \left( 1 - \frac{\phi\left(\frac{c - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{c - \mu}{\sigma}\right)} \left( \frac{\phi\left(\frac{c - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{c - \mu}{\sigma}\right)} - \frac{c - \mu}{\sigma} \right) \right) \quad (3.8.2)$$

in which  $\phi(\cdot)$  and  $\Phi(\cdot)$  are respectively the standard Normal probability distribution function (pdf) and its Cumulative Distribution Function (CDF)

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right) \quad (3.8.3)$$

and

$$\Phi(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} u^2\right) du \quad (3.8.4)$$

The underlying idea of this type of distribution can best be explained by the sampling procedure. This is done by sampling from a usual  $N(\mu, \sigma)$  distribution but ignoring all sample values below the truncation level  $c$  and repeating the sampling until the required number of sample values greater than or equal to  $c$  has been achieved. So the probability mass left of  $c$  is redistributed to the right of  $c$  according to the same Normal distribution. In a multivariate setting this resampling procedure should be repeated for all variables at the same time in order to preserve the correlations between the variables. Furthermore, the resampling needs to be repeated until the values of all variables in all samples are greater than the truncation levels of the relevant marginal distributions. Depending on the dimensions of the model and the truncation level, this resampling may take a lot of time. In that case, importance sampling techniques can help to reduce the computational effort. In a univariate case with a lower (upper) truncation level, a simple alternative is to replace a negative (positive) disturbance by the same positive (negative) disturbance. In case this is not enough to prevent violating the truncating level, resampling needs to be done anyway.

Although correlations will in principle not change very much in this way, if  $c < \mu$  then (3.8.1) and (3.8.2) show that truncating the distribution on the left hand side increases the expected value and lowers the variance. If we do not want these moments to change, other values for the parameters  $\mu$  and  $\sigma$  need to be found which, given the truncation level  $c$ , result in the original values for  $E(X)$  and  $V(X)$  of the non-

truncated Normal distribution. The expressions (3.8.1) and (3.8.2) describe two non-linear equations in the two unknowns  $\mu$  and  $\sigma$  which can be solved by a numerical procedure. The value of the truncation level can be chosen as the  $\alpha$  percent percentile of the original Normal distribution with expected value  $E(X)$  and variance  $V(X)$  in order to in fact truncate  $\alpha$  percent of the distribution. It then holds that

$$\Phi\left(\frac{c - E(X)}{\sqrt{V(X)}}\right) = \alpha \quad (3.8.5)$$

and thus

$$c = z_\alpha \sqrt{V(X)} + E(X) \quad (3.8.6)$$

where  $z_\alpha$  is the  $\alpha$  percent percentile of the standard Normal distribution with

$$\Phi(z_\alpha) = \alpha \quad (3.8.7)$$

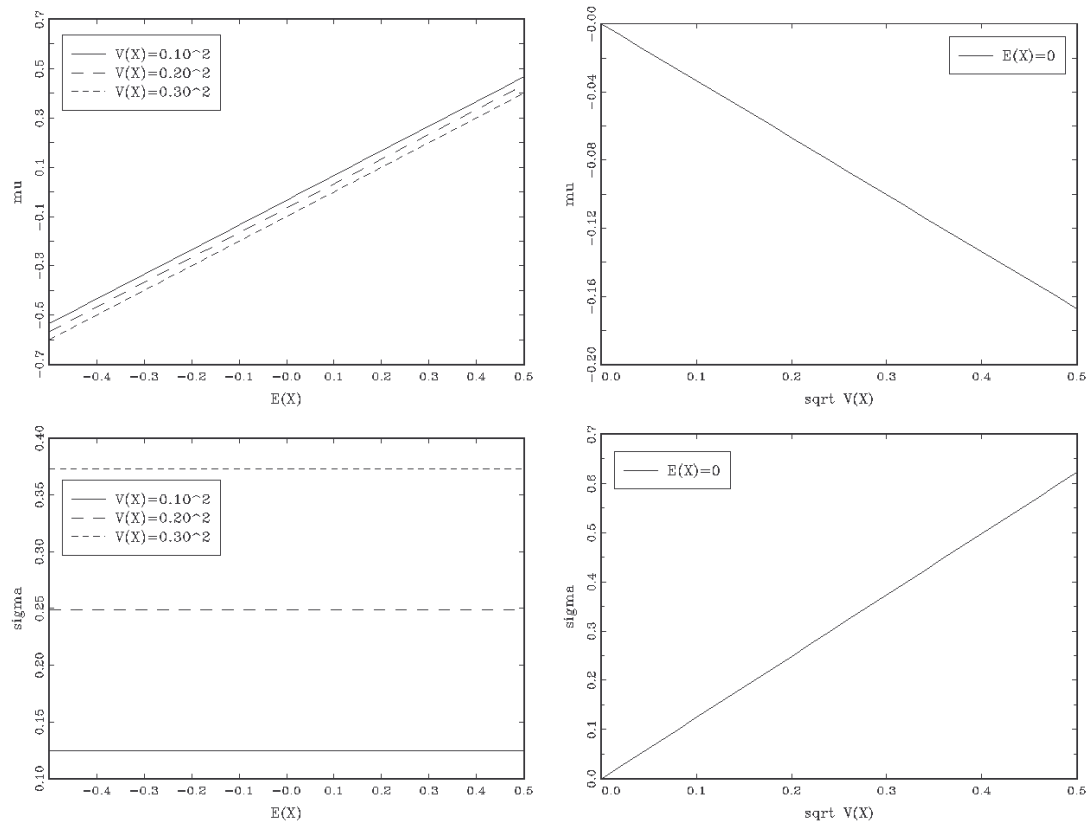
Suppose now that we solve the equations in terms of  $\mu$  and  $\sigma$  for

- $\alpha = 10\%, 5\%, 2.5\%$  and  $1\%$
- $E(X) = -50\%, -48\%, \dots, 48\%, 50\%$
- $\sqrt{V(X)} = 0.1\%, 2\%, 4\%, \dots, 50\%$

The idea is that once the corresponding values for  $\mu$  and  $\sigma$  have been tabulated, the values of  $\mu$  and  $\sigma$  needed to arrive at a truncated Normal distribution with some specific  $E(X)$ ,  $V(X)$  and  $\alpha$  percent truncation level, can be obtained from these pre-calculated tables by means of simple linear interpolation. As an example, Figure 3.10 shows the following cross sections from these tables for  $\alpha=5\%$ :

- The required  $\mu$  as a function of  $E(X)$  for  $\sqrt{V(X)} = 10\%, 20\%$  and  $30\%$ .
- The required  $\mu$  as a function of  $\sqrt{V(X)}$  for  $E(X) = 0\%$ .
- The required  $\sigma$  as a function of  $E(X)$  for  $\sqrt{V(X)} = 10\%, 20\%$  and  $30\%$ .
- The required  $\sigma$  as a function of  $\sqrt{V(X)}$  for  $E(X) = 0\%$ .

Figure 3.10 Cross sections of the required  $\mu$  and  $\sigma$  parameters for a truncated Normal distribution in order to achieve the indicated expected value  $E(X)$  and variance  $V(X)$  with the corresponding 5% truncation level.



From these cross sections we can observe the following with respect to the required  $\sigma$  parameter:

- It is independent of the value of  $E(X)$  (bottom left).
- It is positive linear dependent on the value of  $\sqrt{V(X)}$  (bottom right).
- The exact relation is  $\sigma = 1.2448 \times \sqrt{V(X)}$ .
- The coefficient in this linear relation shows that the required  $\sigma$  parameter is larger than  $\sqrt{V(X)}$  to compensate for the volatility lowering effect of truncating the distribution (above diagonal, bottom right).

Furthermore we can observe the following with respect to the required  $\mu$  parameter:

- The required  $\mu$  parameter is smaller than  $E(X)$  to compensate for the expected value increasing effect of truncating the distribution (below diagonal, top left).
- It is linear dependent on the value of  $E(X)$  with a coefficient of one (top left).
- It is negative linear dependent on the value of  $\sqrt{V(X)}$  (top right) because the expected value increases more when truncating the distribution for higher values of  $\sqrt{V(X)}$ .
- The exact relation is  $\mu = E(X) - 0.3341 \times \sqrt{V(X)}$ .

The interpolation approach we set out for is therefore not needed at all. Apparently, two simple linear equations of the following format are sufficient.

$$\mu = E(X) + \beta_{\mu}(\alpha)\sqrt{V(X)} \quad (3.8.8)$$

$$\sigma = \beta_{\sigma}(\alpha)\sqrt{V(X)} \quad (3.8.9)$$

Repeating this procedure for different values of  $\alpha$  reveals that the coefficients in these relations depend on the value of  $\alpha$ . The appropriate values as obtained by the numerical experiments are given in Table 3.1.

Table 3.1 Required coefficients in (3.8.8) and (3.8.9) for different values of  $\alpha$ .

$\alpha$	$\beta_{\mu}(\alpha)$	$\beta_{\sigma}(\alpha)$
1%	-0.0381	1.0434
2.5%	-0.1164	1.1082
5%	-0.3341	1.2448
10%	-1.6729	1.7731

Example histograms of the truncated Normal distribution are shown in Figure 3.11. The moments of the distribution are  $E(X)=0$  and  $V(X)=1$  and the truncation level is  $c=-1.645$  which is the  $\alpha=5\%$  percentile of the non-truncated Normal distribution with the same moments. The histograms resulting from 100.000 draws are plotted both for the original levels of  $X$  as for the first order differences of  $X$  calculated as the difference between consecutive (independent) draws. For comparison, Figure 3.11 also shows histograms of the corresponding non-truncated Normal distributions. Table 3.2 shows the sample mean, standard deviation, skewness and kurtosis.

Figure 3.11 Example histograms from 100.000 draws from a truncated Normal distribution with expected value  $E(X)=0$  and variance  $V(X)=1$  with the corresponding 5% truncation level, both for the original levels of  $X$  (bottom left) and the first order differences of  $X$  (bottom right) and together with the histograms of the corresponding non-truncated Normal distributions (top left and top right).

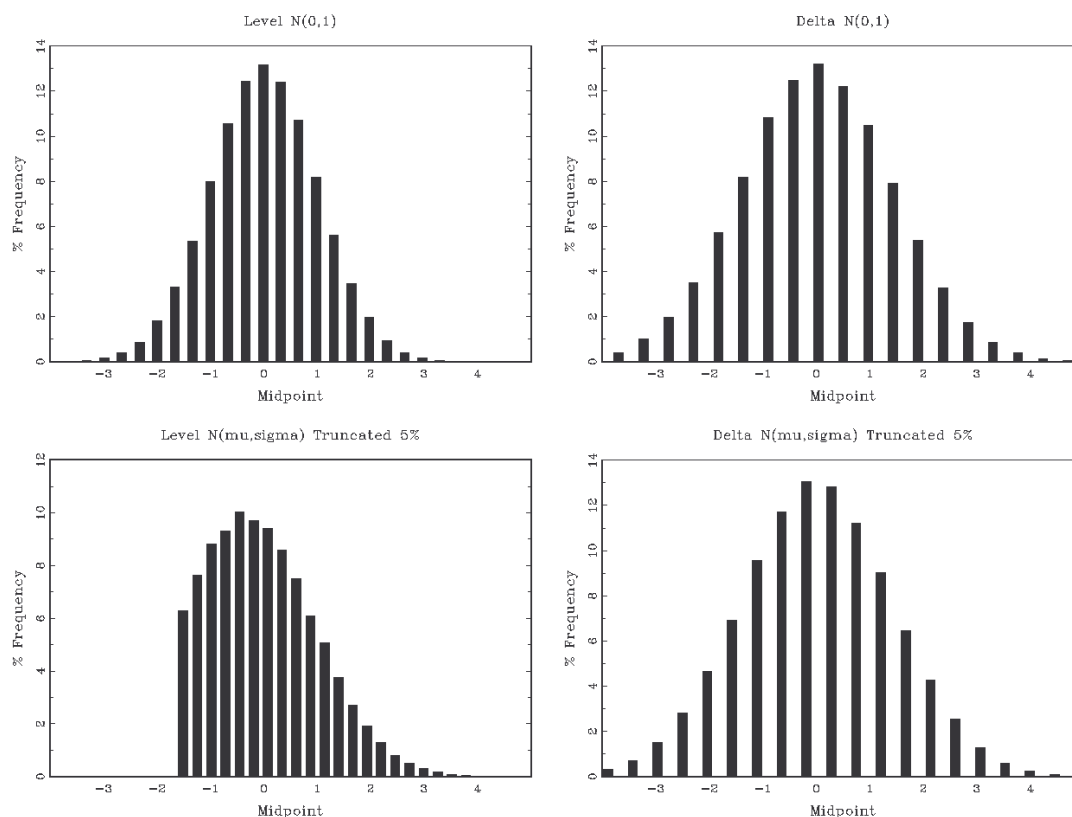


Table 3.2 Sample statistics corresponding to histograms in Figure 3.11.

	<i>Level N(0,1)</i>	<i>Level truncated 5%</i>	<i>Delta N(0,1)</i>	<i>Delta truncated 5%</i>
Mean	0.00	0.00	0.00	0.00
Stdev	1.00	1.00	1.41	1.42
Skew	0.00	0.57	0.00	-0.01
Kurt	3.00	2.98	3.00	2.98

From these results we can observe the following with respect to the distribution of the original levels of  $X$ :

- The first two moments of the original distribution ( $E(X)=0$  and  $V(X)=1$ ) are preserved by the truncated distribution of  $X$ .
- The truncation of the distribution at  $c=-1.645$  is clearly visible.
- Because of the truncation, the distribution has become skewed. So the third moment, and to a lesser extent also the fourth moment, are different from those of the corresponding non-truncated Normal distribution.

Furthermore we can observe the following with respect to the distribution of the first order differences of  $X$ :

- The first order differences of  $X$  are also Normally distributed but have a larger standard deviation of  $\sqrt{2} = 1.41$ .
- Truncating the level of  $X$  does *not* lead to the truncation of the first order differences of  $X$ . More general, truncating  $X$  does not truncate a transformation of  $X$ .
- The distribution of the first order differences of  $X$  has in fact hardly changed because of the truncating.

### 3.8.1 First order differences

In some applications it may be necessary not to truncate the distribution of a random variable  $X$  itself, but the distribution of a transformation of the random variable. The objective is then to sample from a distribution with some desired  $E(X)$  and  $V(X)$ , such that the distribution of the relevant transformation of  $X$  is truncated at some level  $c$ . A special type of transformation is the first order differences of consecutive values of  $X$ . In the previous section we saw that truncating the level of  $X$  does not lead to the truncation of the first order differences of  $X$ . An obvious approach is then not to resample if a sample value of  $X$  is below the truncation level  $c$ , but if the *difference* with the previous sample value of  $X$  is below the truncation level  $c$ . The value of  $c$  should be based on the non-truncated distribution of the first order differences which is

$$X_i - X_{i-1} \sim N(0, \sqrt{2 \cdot V(X) - 2 \cdot V(X) \cdot \rho}) \quad (3.8.10)$$

where  $\rho$  is the correlation between consecutive values of  $X$ . The truncation level for some value of  $\alpha$  then becomes

$$c = z_\alpha \sqrt{2 \cdot V(X) - 2 \cdot V(X) \cdot \rho} \quad (3.8.11)$$

Unfortunately the analytical expressions for  $E(X)$  and  $V(X)$  of this type of truncated Normal distribution are unknown which renders the solving of two equations approach from the previous section impossible. The solution is to find the required parameters  $\mu$  and  $\sigma$  by means of trial and error or by some numerical optimization procedure. The values of  $\mu$  and  $\sigma$  should for example be such that the sample estimators of  $E(X)$  and  $V(X)$ , as determined by simulation, are identical to those of the original distribution, given some truncation level of the first order differences at the corresponding  $\alpha$  percentile. The search is simplified by the fact that changing  $\mu$  does not change the variance of  $X_i - X_{i-1}$  and hence also not the extent to which the truncation level is violated. Therefore one can first find the  $\sigma$  that results in the required  $V(X)$  and then independently set  $E(X)$  by changing  $\mu$ . For an example with  $E(X)=0$ ,  $V(X)=1$ ,  $\rho=0$ ,  $\alpha=5\%$  and  $c=-2.326$ , this results in  $\beta_\mu=-0.14$  and  $\beta_\sigma=1.08$  in the earlier equations (3.8.8) and (3.8.9). Note that in absolute terms both these values are smaller than the corresponding ones from Table 3.1 when truncating the original distribution. Figure 3.12 shows the histograms for this example and Table 3.3 shows the corresponding sample mean, standard deviation, skewness and kurtosis.



Figure 3.12 Example histograms from 100.000 draws from a truncated Normal distribution of the first order differences with expected value  $E(X)=0$ , variance  $V(X)=1$ ,  $\rho=0$  with the corresponding 5% truncation level, both for the original levels of  $X$  (bottom left) and the first order differences of  $X$  (bottom right) and together with the histograms of the corresponding non-truncated Normal distributions (top left and top right).

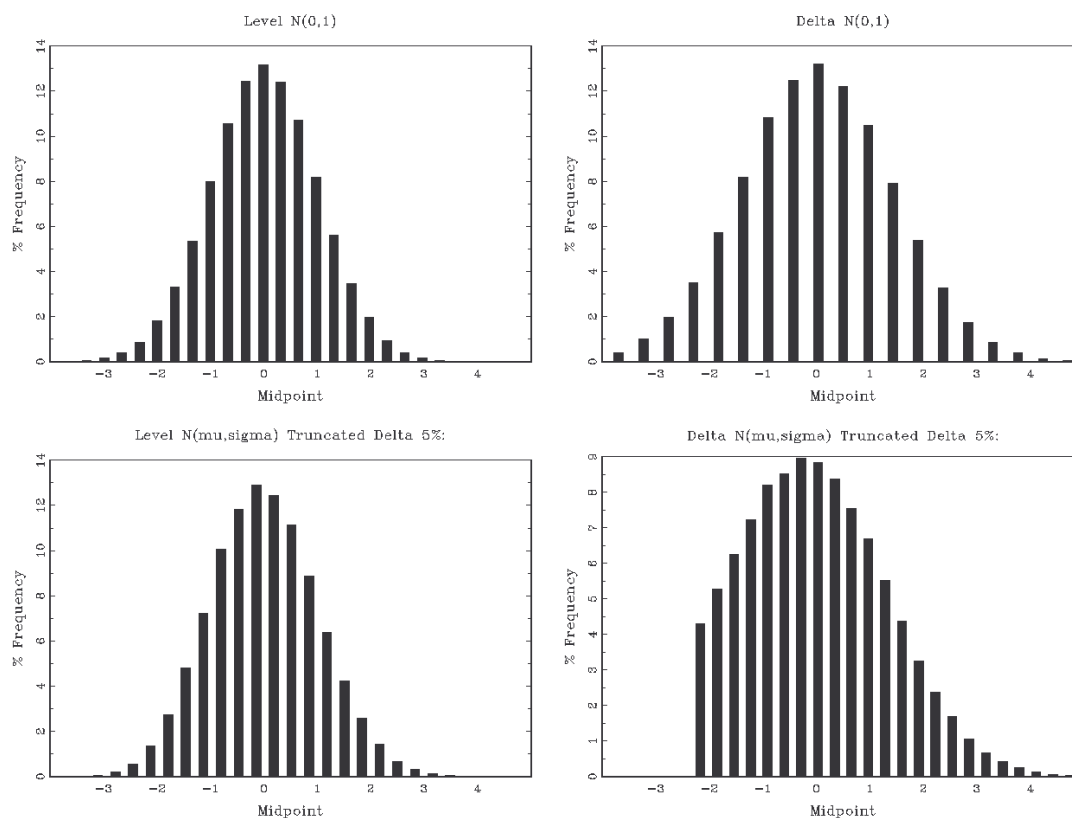


Table 3.3 Sample statistics corresponding to histograms in Figure 3.12.

	Level $N(0,1)$	Level truncated 5%	Delta $N(0,1)$	Delta truncated 5%
Mean	0.00	0.00	0.00	0.00
Stddev	1.00	1.00	1.41	1.29
Skew	0.00	0.12	0.00	0.39
Kurt	3.00	2.93	3.00	2.72

From these results we can observe the following:

- The first two moments of the original distribution ( $E(X)=0$  and  $V(X)=1$ ) are preserved by the truncated distribution of the first order differences of  $X$ .
- The truncation of the distribution of the first order difference at  $c=-2.326$  is clearly visible.
- In the case of truncating  $X$ , the distribution of the first order differences of  $X$  hardly changed. Now, it is just the other way around. The distribution of  $X$  has hardly changed, although it also has become a little skewed, while the distribution of the first order differences of  $X$  is truncated, has become skewed and has a lower standard deviation. Note that the lower standard deviation is the case because the objective was to maintain the same first two moments for the distribution of  $X$  and not of the first order differences of  $X$ . Maintaining the same first two moments of the first order differences, or any other transformation, of  $X$  is also possible in this numerical approach. Doing both is impossible, because

then we have two unknowns ( $\beta_\mu$  and  $\beta_\sigma$ ) that need to solve four equations (the first two moments of  $X$  and the first order differences of  $X$ ).

### 3.8.2 Truncated VAR models

Finally, the procedure outlined in the previous section for truncating the distribution of the first order differences of a Normally distributed random variable without changing the first two moments of the original distribution can be generalized for truncating the distributions of a VAR model, or even transformations of a VAR model. We shall call the resulting models *Truncated VAR (TVAR) models*. The MA representation (3.6.2) shows that a VAR model is just another type of transformation of Normally distributed random variables, being the error terms of the model. Again, truncating the distribution of these error terms will not truncate the distribution of the transformation. By truncating the error terms, the conditional distribution with an horizon of one time period is in fact truncated because for the AR( $p$ ) model (3.2.3) it holds for example that

$$x_{t+1} \sim N(\nu + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_p x_{t-p}, \sigma) \quad (3.8.12)$$

Truncating this conditional distribution of  $x$  does not truncate the unconditional distribution of  $x$ . The same holds for the first order differences of  $x$  since this is just another transformation of the error terms. To see this note that the MA representation (3.6.2) of an  $n$  dimensional VAR model implies

$$x_t - x_{t-1} = \sum_{i=0}^{\infty} \Phi_i^* \varepsilon_{t-i} \quad (3.8.13)$$

with

$$\begin{aligned} \Phi_0^* &= \Phi_0 = I_n \\ \Phi_i^* &= \Phi_i - \Phi_{i-1} \quad i = 1, 2, \dots \end{aligned}$$

which is also a MA representation but only with different weights. Just as with the simpler transformation of the first order differences of a Normally distributed random variable  $X$ , the solution is to follow a trial and error approach or some numerical optimization procedure. The objective of this procedure is to arrive at being able to sample from a VAR model for which the marginal distributions of the variables, or transformations of these marginal distributions, has been truncated at some  $\alpha$  percentile level, without changing the first two moments of the original VAR model. Note that because of the resampling approach, the dynamic properties of the VAR model, in terms of its autocorrelation structure, should also not change very much.

The procedure consists of the following steps:

1. Determine the expected value  $E(X)$  and variance  $V(X)$  of the unconditional distribution of (some transformation of) the VAR model by means of the Yule-Walker equations described in section 3.2.2. Using the same equations, also determine the autocovariance matrices until order  $p$  in case of a VAR( $p$ ) model.
2. Determine the conditional expected value and conditional variance for each period for a sufficiently long horizon. This can be done based on the procedure described in section 3.6.2. In case the distribution of the first order differences needs to be truncated, there are two alternatives. The first is to base the conditional moments directly on the MA representation (3.8.13) of the first order differences. The second is to calculate them indirectly from the conditional moments of the original variables as

$$\mu_{\text{delta},h} = \mu_h - \mu_{h-1} \quad (3.8.14)$$

and

$$\sigma_{\text{delta},h} = \sqrt{\sigma_h^2 + \sigma_{h-1}^2 - 2 \cdot \rho_1 \cdot \sigma_h \cdot \sigma_{h-1}} \quad (3.8.15)$$

where  $\mu_h$ ,  $\sigma_h$  and  $\rho_1$  are respectively the conditional expected value for horizon  $h$ , the conditional standard deviation for horizon  $h$  and the first order autocorrelation of the relevant variable, that is the correlation between time  $h$  and  $h-1$ .

3. Based on the conditional expected values  $\mu_h$  and conditional standard deviations  $\sigma_h$ , calculate the truncation level  $c_h$  for each horizon as the  $\alpha$  percent percentile of the conditional distributions, similar to (3.8.6), as

$$c_h = Z_\alpha \sigma_h + \mu_h \quad (3.8.16)$$

4. Adjust the model unconditional expected value vector and autocovariance matrices from step 1 according to the relations (3.8.8) and (3.8.9) for each variable based on some initial values for  $\beta_\mu$  and  $\beta_\sigma$ , one combination for each of the  $n$  variables from the model. Adjust the parameters of the VAR model accordingly by using the Yule-Walker equations as in the procedure described in section 3.7.5. By adjusting the moments of the unconditional distribution of the model in this way, the moments of the conditional distributions will change in a similar fashion.
5. Sample a great number of scenarios from the adjusted VAR model with a sufficiently long horizon while resampling each next value in case it has fallen below the truncation level for the relevant horizon until all scenarios are above the truncation line defined by the  $c_h$  values.
6. Estimate the unconditional mean and variance of (some transformation of) the truncated model by the sample moments from say the last 10 years of 1000 scenarios with a 50 year horizon. The time period used should be such that the model has arrived at its unconditional distribution, hence the scenarios show a stable expected value and variance.

7. In case the first two sample moments are different from the required  $E(X)$  and  $V(X)$  values, change the  $\beta_\mu$  and  $\beta_\sigma$  parameters and restart from step 4 until the first two sample moments are equal to the required  $E(X)$  and  $V(X)$  values. Note that especially for high dimensional VAR models it is worth the effort of automating this search by means of some numerical optimization procedure.

An example application of this type of truncated VAR model is described in section 20.4 where the first order differences of a univariate AR model for a logarithmic consumer price index are truncated. The objective there is to prevent the model describing negative inflation rates without changing the moments of the distribution of the index itself. There, it is also shown that the dynamics of a truncated VAR model, in terms of its autocorrelation structure, indeed do not change very much because of the truncating procedure, provided that a sufficiently small level of  $\alpha$  is used for determining the truncation level.

## 4 Spectral Analysis

Chapter 2 includes a general introduction on spectral analysis. Section 2.2.11 contains not only an overview of some of its most important concepts such as the periodogram and the spectral density, it also contains some historical background and references to further readings. This second chapter of Part II on the theoretical background contains some actual theory on spectral analysis. Just as with the previous chapter on autoregressive models, the focus here is on the aspects most relevant for the research presented here instead of providing a full scale theory. However, because in the econom(etr)ic sciences (in the Netherlands), spectral analysis is not as well known as for example autoregressive models, we do try to be as complete as possible. We start of with the Fourier transform which is at the very hart of spectral analysis and work our way through to Maximum Entropy or autoregressive spectral analysis and related statistical testing procedures. This chapter also contains the results of an extensive Monte Carlo experiment to find out what estimation method and order selection criteria best to use for the (parametric) estimation of spectral densities. A central role in spectral analysis is played by the basic sine and cosine functions and the class of complex numbers. Appendices A.1 and A.2 repeat some of the basics of these mathematical concepts. Furthermore, to keep the text as easy to read as possible, proofs and derivations are moved to Appendix B as much as possible.

### 4.1 Fourier transform

Consider a time series  $\{x_t, t=0, \dots, T-1\}$  where  $T$  represents the number of observations. For the ease of argument assume  $T$  is an even number. Suppose we want to try to represent this time series as a sum of cosine functions with different frequencies, amplitudes and phases. If  $T$  cosine functions are used, such a representation looks like

$$x_t = \sum_{j=0}^{T-1} R_j \cos(\omega_j t + \phi_j) \quad (4.1.1)$$

where the parameters  $\{R_j, \omega_j$  and  $\phi_j, j = 0, \dots, T-1\}$  represent the amplitudes, frequencies and phases of the  $T$  cosine functions. An equivalent, merely re-parameterized, representation is

$$x_t = \sum_{j=0}^{T-1} A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \quad (4.1.2)$$

where

$$A_j = R_j \cos(\phi_j) \text{ and } B_j = -R_j \sin(\phi_j) \\ R_j = \sqrt{A_j^2 + B_j^2}$$

Astonishingly enough a representation such as (4.1.2) can be proven to exist for *any* particular time series. Appendix B.2 gives the proof that choosing the parameters

$$\begin{aligned}\omega_j &= 2\pi \frac{j}{T} \text{ for } j = 0, \dots, T-1 \\ A_j &= \frac{1}{T} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \\ B_j &= \frac{1}{T} \sum_{t=0}^{T-1} x_t \sin(\omega_j t)\end{aligned}\tag{4.1.3}$$

solves the equation (4.1.2) for all  $t = 0, \dots, T-1$ . This proves the remarkable result that any time series  $x_t$  can be written as a sum of cosine functions. This result leads to what we will call the *Discrete to Discrete Fourier Transform* (DDFT)

$$\text{DDFT: } A_j = \frac{1}{T} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \text{ and } B_j = \frac{1}{T} \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \quad j = 0, \dots, T-1 \tag{4.1.4}$$

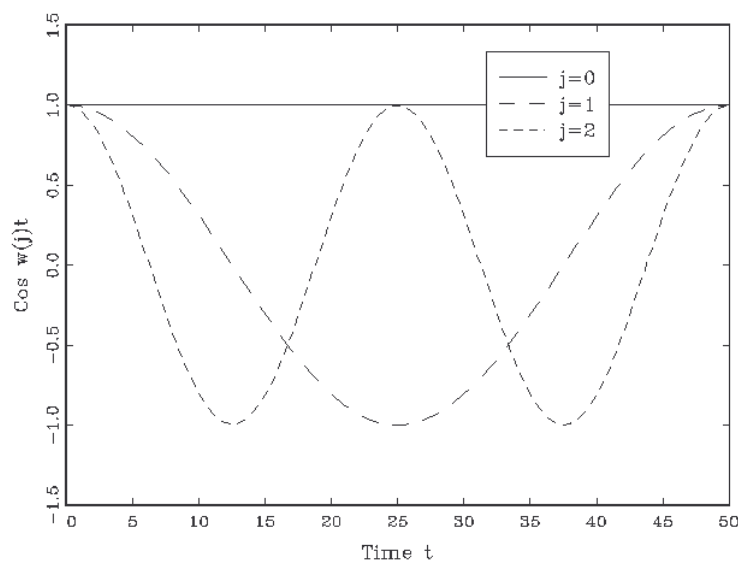
and the corresponding *Inverse Discrete to Discrete Fourier Transform* (IDDFT)

$$\text{IDDFT: } x_t = \sum_{j=0}^{T-1} A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \quad t = 0, \dots, T-1 \tag{4.1.5}$$

These are transformations from the discrete variables  $x_t$  into the discrete variables  $A_j$  and  $B_j$  and vice versa. Both are special versions of the proposition of *Fourier* which states that, given some rather weak conditions, any arbitrary single valued function, even with a finite number of discontinuities, can be written as an infinite sum of cosine functions. The series  $\{A_j, B_j, j = 0, \dots, T-1\}$  is called a *Fourier series* while the frequencies

$$\omega_j = 2\pi \frac{j}{T} \text{ for } j = 0, \dots, T-1 \tag{4.1.6}$$

are called the *Fourier frequencies*. Dividing them by  $2\pi$  indicates that the corresponding cosine functions complete  $j$  full cycles in  $T$  time periods. Figure 4.1 for example shows  $\omega_j$  for  $j=0, 1$  and  $2$  for  $T=50$ .

Figure 4.1 Cosines at Fourier frequencies  $\omega_j$  for  $j = 0, 1$  and  $2$ .

The conventional representation  $\{x_t, t=0, \dots, T-1\}$  of the time series is referred to as the representation in the *time domain*. The representations  $\{R_j, \omega_j$  and  $\phi_j, j = 0, \dots, T-1\}$  and  $\{A_j, B_j, j = 0, \dots, T-1\}$  are referred to as representations in the *frequency domain*. The DDFT therefore transforms a time series from the time domain into the frequency domain. The IDFFT does the opposite.

One of the special properties of the Fourier frequencies is that the corresponding sine and cosine functions have an orthogonal property. That is

$$\sum_{t=0}^{T-1} \cos(\omega_j t) \sin(\omega_k t) = 0 \quad \forall j, k \quad (4.1.7)$$

This special property of the Fourier frequencies has two important consequences. The first is that this zero correlation between the sines and cosines at the Fourier frequencies enables the following simple interpretation of the Fourier numbers  $A_j$  and  $B_j$  in terms of the familiar OLS criterion. If the time series  $x_t$  is regressed on the sine and cosine functions for each of the Fourier frequencies as in

$$x_t = \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) + \varepsilon_t \quad \text{for } t = 0, \dots, T-1 \quad (4.1.8)$$

by minimizing the sum of squared residuals, the estimated parameters turn out to be

$$\begin{aligned} \hat{\alpha}_j &= \begin{cases} 2A_j & \text{for } j = 0, \dots, T-1, j \neq 0 \text{ and } j \neq T/2 \\ A_j & \text{for } j = 0 \text{ and } j = T/2 \end{cases} \\ \hat{\beta}_j &= \begin{cases} 2B_j & \text{for } j = 0, \dots, T-1, j \neq 0 \text{ and } j \neq T/2 \\ B_j & \text{for } j = 0 \text{ and } j = T/2 \end{cases} \end{aligned} \quad (4.1.9)$$

The proof of this result is given in Appendix B.3.



So, the Fourier parameters  $A_j$  and  $B_j$  calculated from the DDFT give the best fitting parameters when trying to describe the time series as best as possible by the sine and cosine functions for each of the Fourier frequencies. For other than the Fourier frequencies this equivalence does not hold because these frequencies do not satisfy the orthogonal property (4.1.7). Furthermore note that given the zero correlation of the regressors, the estimated parameters are the same when the sine and cosine functions for all Fourier frequencies are estimated simultaneously as when they are estimated one by one as in (4.1.8).

For the second important consequence of (4.1.7) let us take a special look at the amplitudes  $R_j$  of the Fourier cosine functions. From the Fourier series  $A_j$  and  $B_j$  the squared amplitudes are

$$R_j^2 = A_j^2 + B_j^2 \quad (4.1.10)$$

For these squared amplitudes the following important relation holds with respect to the original time series.

$$\frac{1}{T} \sum_{t=0}^{T-1} x_t^2 = \sum_{j=0}^{T-1} A_j^2 + B_j^2 \quad (4.1.11)$$

The proof of this result is given in Appendix B.4. So, if we assume the time series  $x_t$  to have an average value of zero then this relations tells us that the DDFT actually decomposes the total variance of the time series into the squared amplitudes of the cosine functions at the Fourier frequencies. The higher the  $R_j$  for a certain  $\omega_j$ , the more this frequency contributes to the total variance of the time series.

To facilitate calculations, and only for that reason, the Fourier transforms are often used in an equivalent format using complex numbers. Appendix A.2 contains a short summary of the theory of complex numbers. The complex exponential function is defined as

$$\exp(i\omega) = \cos \omega + i \sin \omega \Leftrightarrow \begin{cases} \cos \omega = \frac{\exp(i\omega) + \exp(-i\omega)}{2} \\ \sin \omega = \frac{\exp(i\omega) - \exp(-i\omega)}{2i} \end{cases} \quad (4.1.12)$$

which for any (real) frequency  $\omega$  defines a complex number  $\exp(i\omega)$  on the unit circle in the complex plain. This directly shows the relation between the complex exponential function and the sine and cosine functions. (4.1.12) enables the DDFT and the IDFT to be defined as

$$\begin{aligned} \text{DDFT : } J_j &= \frac{1}{T} \sum_{t=0}^{T-1} x_t \exp(-i\omega_j t) = \frac{1}{T} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) - i \frac{1}{T} \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \\ &= A_j - iB_j \text{ for } j = 0, \dots, T-1 \end{aligned} \quad (4.1.13)$$

and

$$\text{IDFFT: } x_t = \sum_{j=0}^{T-1} J_j \exp(i\omega_j t) \text{ for } t = 0, \dots, T-1 \quad (4.1.14)$$

The squared amplitudes are then equal to

$$R_j^2 = |J_j|^2 = J_j J_j^* = A_j^2 + B_j^2 \text{ for } j = 0, \dots, T-1 \quad (4.1.15)$$

and the decomposition of the variance becomes

$$\frac{1}{T} \sum_{t=0}^{T-1} x_t^2 = \sum_{j=0}^{T-1} |J_j|^2 \quad (4.1.16)$$

Another important modification to the original DDFT and IDFFT is that not all of the Fourier parameters are unique. To see this note that

$$\omega_{T-j}t = 2\pi \frac{T-j}{T}t = 2\pi t - 2\pi \frac{j}{T}t = 2\pi t - \omega_j t$$

and

$$\begin{aligned} \cos(\omega_{T-j}t) &= \cos(2\pi t - \omega_j t) = \cos(-\omega_j t) = \cos(\omega_j t) \\ \sin(\omega_{T-j}t) &= \sin(2\pi t - \omega_j t) = \sin(-\omega_j t) = -\sin(\omega_j t) \end{aligned}$$

from which follows that

$$\begin{cases} A_j = A_{T-j} \\ B_j = -B_{T-j} \end{cases} \text{ and } J_j = J_{T-j}^* \text{ for } j = T/2 + 1, \dots, T-1 \quad (4.1.17)$$

So, in fact only about half of the Fourier series are uniquely determined. Therefore by limiting the Fourier transform to only the first  $T/2+1$  frequencies  $\{\omega_j, j = 0, \dots, T/2\}$ , for which hold  $0 \leq \omega_j \leq \pi$ , in the frequency domain no information about the time series is lost. The highest of these frequencies  $\omega_{T/2}$  is sometimes called the *Nyquist frequency*. This frequency has a period length of two time periods. In discrete time this is the highest attainable frequency. The fact that in discrete time it suffices to only consider the frequencies between 0 and  $\pi$  is also called the *aliasing* effect as described in Appendix A.1.

The final thing that needs to be said about the Fourier transform in this section is the generalization to the case of continuous frequencies. The Fourier frequencies  $\omega_j$  define  $T$  discrete and evenly spaced frequencies in the interval  $[0, 2\pi]$ . However it is also possible to analyze any other frequency in this interval. To see this suppose we expand the time series  $x_t$  by defining  $x_t=0$  for  $t=T, \dots, T^*-1$  where  $T^*$  is an arbitrary number larger than  $T$ .

The DDFT and IDFFT for this expanded time series  $\{x_t, t=0, \dots, T^*-1\}$  are

$$J_j = \frac{1}{T^*} \sum_{t=0}^{T^*-1} x_t \exp(-i\omega_j t) = \frac{1}{T^*} \sum_{t=0}^{T-1} x_t \exp(-i\omega_j t) \quad \text{for } j = 0, \dots, T^* - 1 \quad (4.1.18)$$

and

$$x_t = \sum_{j=0}^{T^*-1} J_j \exp(i\omega_j t) \quad \text{for } t = 0, \dots, T-1 \quad (4.1.19)$$

with

$$\omega_j = 2\pi \frac{j}{T^*} \quad \text{for } j = 0, \dots, T^* - 1 \quad (4.1.20)$$

By expanding the original time series the DDFT has been expanded to  $T^*$  instead of  $T$  frequencies in the interval  $[0, 2\pi]$ . Note that these frequencies are no longer orthogonal as in (4.1.7) so that the OLS interpretation of the Fourier transform is no longer valid. This refinement of frequencies can be continued until infinity ( $T^* = \infty$ ) which results in covering the whole continuous interval  $[0, 2\pi]$ . This gives what we will call the *Discrete to Continuous Fourier Transform* (DCFT) for obvious reasons. The DCFT and its inverse are

$$\text{DCFT: } J(\omega) = \frac{1}{T} \sum_{t=0}^{T-1} x_t \exp(-i\omega t) \quad \forall \omega \quad (4.1.21)$$

and

$$\text{IDCFT: } x_t = \frac{T}{2\pi} \int_0^{2\pi} J(\omega) \exp(i\omega t) d\omega \quad \text{for } t = 0, \dots, T-1 \quad (4.1.22)$$

Appendix B.5 contains the formal proof of (4.1.21) and (4.1.22) starting from the discrete case of the DDFT and IDFFT. Just as in the discrete case, also in the continuous case the DCFT decomposes the variance of a time series over the entire frequency range  $[0, 2\pi]$ . Appendix B.6 contains the proof of the important relation

$$\frac{1}{T} \sum_{t=0}^{T-1} x_t^2 = \frac{T}{2\pi} \int_0^{2\pi} |J(\omega)|^2 d\omega \quad (4.1.23)$$

## 4.2 Periodogram

The *periodogram* of a time series  $x_t$  is in principle defined as the function

$$P(\omega) = \frac{T}{2\pi} |J(\omega)|^2 \quad \text{for } \omega \in [0, 2\pi] \quad (4.2.1)$$

From (4.1.23) it follows that integrating this function over the interval  $[0, 2\pi]$  leads to the variance of the time series denoted as  $\sigma^2$ . That is

$$\int_0^{2\pi} P(\omega) d\omega = \frac{1}{T} \sum_{t=0}^{T-1} x_t^2 = \sigma^2 \quad (4.2.2)$$

The surface below the periodogram is therefore equal to  $\sigma^2$ . We make two modifications to the definition in (4.2.1) from a practical point of view. The first is the normalization of the periodogram by dividing it by the variance  $\sigma^2$  of the time series. The surface below this so called *normalized periodogram* is now equal to one instead of  $\sigma^2$ . This normalization makes it easier to compare the periodograms of time series with very different variances. The periodogram then “only” shows how fluctuations of various frequencies contribute to the total variance of a time series while the variance should be given as a separate number. The second modification comes from the aliasing effect as discussed in the previous section and Appendix A.1. Because of this phenomenon it is sufficient to look at the periodogram on the interval  $[0, \pi]$  instead of  $[0, 2\pi]$ . On the second half of the latter interval the periodogram is just the mirror image of the periodogram on the first half reflected in the line  $\omega = \pi$ . In order for the (normalized) periodogram to still integrate to one, its value on the interval  $[0, \pi]$  is doubled. These two modifications lead to the following modified definition of the periodogram.

$$P^*(\omega) = \frac{T}{\sigma^2 \pi} |J(\omega)|^2 \quad \text{for } \omega \in [0, \pi] \quad (4.2.3)$$

for which holds

$$\int_0^{\pi} P^*(\omega) d\omega = 1 \quad (4.2.4)$$

For theoretical purposes the original definition (4.2.1) will be used. For practical purposes it is most of the times better to present the modified definition (4.2.4). Which definition is applied will be clear from the context it is used in.

Because the periodogram is defined on a continuous domain it is no longer possible, as with the DDFT, to assign a portion of the time series variance to a single frequency. Instead, a portion of the variance can be assigned to an interval of frequencies, say  $[\omega_a, \omega_b]$ . This portion of the variance is given by

$$\int_{\omega_a}^{\omega_b} P(\omega) d\omega \quad (4.2.5)$$

An extension of this idea is to define the *cumulative periodogram* as

$$CP(\omega_a) = \int_0^{\omega_a} P(\omega) d\omega \text{ for } \omega_a \in [0, 2\pi] \quad (4.2.6)$$

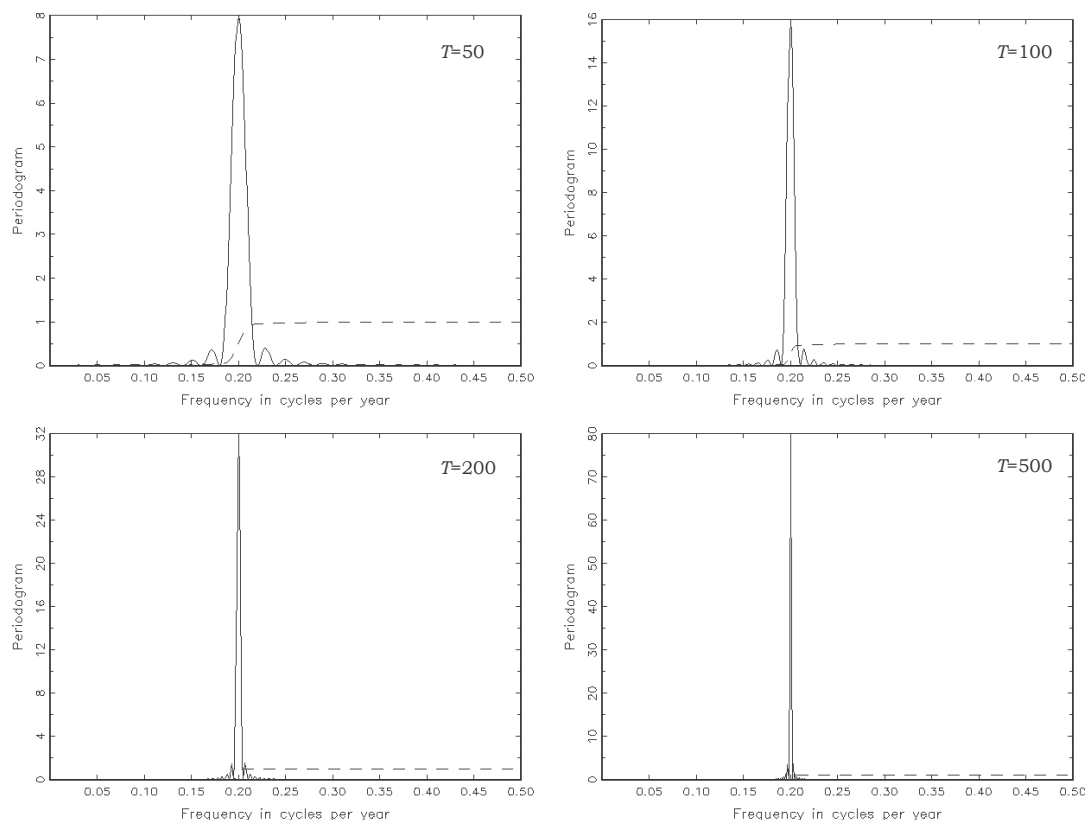
for which holds  $CP(0) = 0$  and  $CP(2\pi) = \sigma^2$ . For the cumulative periodogram the same two modifications (normalization and only the frequency interval  $[0, \pi]$ ) as for the periodogram can be made. Given the properties of the periodogram such as in (4.2.4), (4.2.5) and (4.2.6) it is easy to note the similarity between the periodogram and a probability density function (pdf). Instead of a probability mass of one, now a variance mass is distributed over some continuous interval.

As a simple example let's look at the following time series consisting of a perfect cosine function with a period length of five time periods.

$$x_t = \cos(\omega t) \text{ where } \omega = 2\pi \frac{1}{5} \text{ for } t = 0, \dots, T-1 \quad (4.2.7)$$

Figure 4.2 contains the (normalized) periodogram of this time series for sample sizes  $T=50, 100, 200$  and  $500$ . For ease of interpretation, assume that the time periods are years. The frequencies  $\omega$  on the horizontal axes are given in terms of the number of cycles per years. The lowest frequency is zero which is essentially a constant term. The highest frequency in discrete time is  $0.5$  with a period length of two years (i.e. the Nyquist frequency  $\pi$ ).  $[0, 0.5]$  is an easier representation of the relevant frequencies than the  $[0, \pi]$  representation. The dotted line in each of the pictures represent the cumulative (normalized) periodogram which always ends at a value of one for the highest frequency indicating that the entire variance of the time series is covered by the periodogram.

Figure 4.2 Periodogram of time series (4.2.7) for several sample sizes.



Given the by construction perfect periodic behavior of the time series and the interpretation of the periodogram as the distribution of the variance of the time series over the frequency interval  $[0, 0.5]$ , one would expect the periodogram only to have a peak at the frequency present in the time series,  $1/5 = 0.2$ . Looking for example at the periodogram for  $T=50$  however shows that this is not the case. The periodogram does have the largest peak around the frequency 0.2 but there is also variance mass at the surrounding frequencies. This phenomenon is known as *leakage*. The variance at a single frequency in a time series in a sense “leaks” away to surrounding frequencies in the periodogram. The dampened oscillations to the left and right starting from the dominant frequency is the typical pattern of such leakage. The peaks around the dominant peak are also called *spurious peaks* or *sidelobes*. In the next chapter on filtering techniques we will see that the leakage effect is caused by the fact that we only use a finite sample of the time series. Here this point can be illustrated by looking at the periodogram of the same time series for larger sample sizes. Figure 4.2 clearly shows that as the sample size increases the disturbing effects of leakage decrease and the periodogram gets better and better at revealing the true identity of the time series. As the peak of the periodogram at the frequency  $\omega = 0.2$  gets sharper it also gets higher because the surface below the periodogram still has to be equal to the total (normalized) variance of the time series. For a time series of infinite length the peak will eventually reach infinity because of the continuous character of the frequencies.

Because both the periodogram  $P(\omega)$  and the (sample) autocovariances  $\hat{\gamma}_k$  contain information about the dynamics of a time series it should come as no big surprise that the following relation between these two exists.

$$P(\omega) = \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \hat{\gamma}_k \exp(-i\omega k) \text{ for } \omega \in [0, 2\pi] \quad (4.2.8)$$

Assuming a zero mean for the time series the sample autocovariances are

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=|k|}^{T-1} x_t x_{t-|k|} \text{ for } -(T-1) \leq k \leq (T-1) \quad (4.2.9)$$

Appendix B.7 contains a proof of (4.2.8). So the periodogram can also be calculated based on the DCFT of the sample autocovariances. In this context, the leakage effect caused by the finite sample length presents itself through the finiteness of the (sample) autocovariance function. Autocovariances for  $k > T-1$  cannot be calculated. Therefore the autocovariance function is in a way truncated at  $k=T-1$ . Also, the other way around, the sample autocovariances can be calculated based on the IDCFT of the periodogram. That is

$$\hat{\gamma}_k = \int_0^{2\pi} P(\omega) \exp(i\omega k) d\omega \text{ for } k \in [-\infty, \infty] \quad (4.2.10)$$

### 4.3 Spectral density

It needs no argument that an empirical economic time series will never behave like a perfect periodic function as in the example at the end of the previous section. Instead, let's look at a stochastic process consisting of  $M$  cosine functions with stochastic amplitudes and phases but with fixed frequencies.

$$x_t = \sum_{j=1}^M a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \quad (4.3.1)$$

where  $a_j$  and  $b_j$  are stochastic variables with the first and second moments

$$\begin{aligned} E(a_j) &= E(b_j) = 0 \quad \forall j \\ E(a_j a_k) &= \begin{cases} \sigma_j^2 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \\ E(b_j b_k) &= \begin{cases} \sigma_j^2 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \\ E(a_j b_k) &= 0 \quad \forall j \text{ and } k \end{aligned}$$



It is easy to show that the stochastic process (4.3.1) has the moments

$$\begin{aligned}
 E(x_t) &= 0 \\
 E(x_t^2) &= \sum_{j=1}^M \sigma_j^2 \\
 E(x_t x_{t-k}) &= \sum_{j=1}^M \sigma_j^2 \cos(\omega_j k)
 \end{aligned} \tag{4.3.2}$$

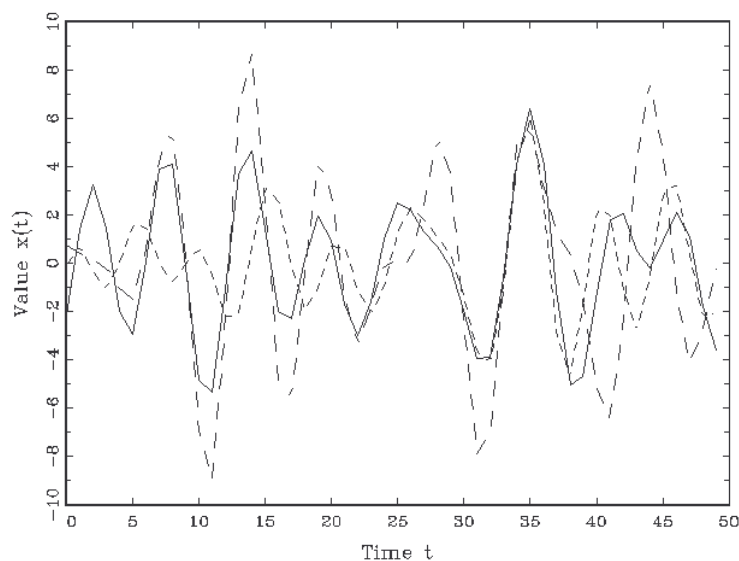
Because these moments are independent of time  $t$  the process is (covariance) stationary. Also it can be seen that the part of the total variance of the process caused by the fluctuations of frequency  $\omega_j$  is equal to  $\sigma_j^2$ . So the total variance of the process can be split up between the  $M$  frequencies. Now consider an example version of this process for  $M=6$  with the parameters from the following table and a sample size of  $T=50$ .

Table 4.1 Parameters of example version of stochastic process (4.3.1).

$j$	$\omega_j = 2\pi(5 + j - 1)/50$	Frequency = $\omega_j/2\pi$	Period = $2\pi/\omega_j$	$\sigma_j^2$
1	$2\pi 5/50$	0.10	10.0	1
2	$2\pi 6/50$	0.12	8.33	1
3	$2\pi 7/50$	0.14	7.14	1
4	$2\pi 8/50$	0.16	6.25	1
5	$2\pi 9/50$	0.18	5.55	1
6	$2\pi 10/50$	0.20	5.00	1

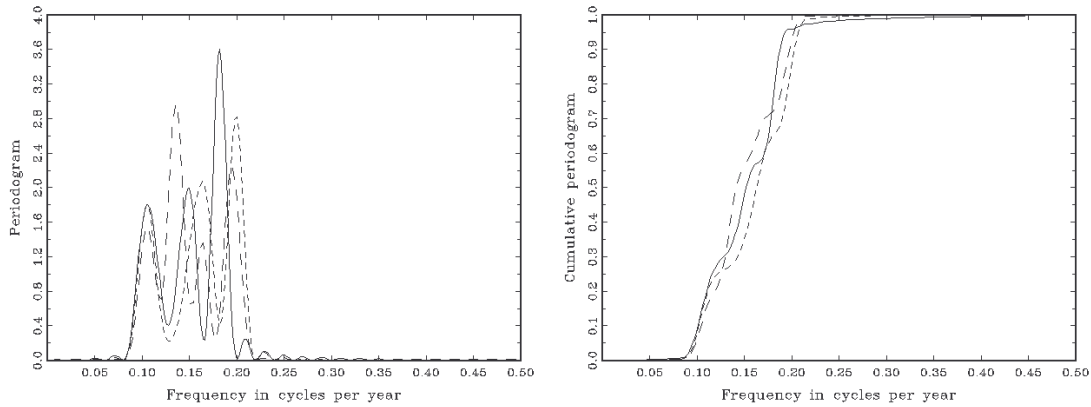
The total variance of this process is  $E(x_t^2) = 6$  to which each of the fluctuations with a period length between five and ten years contribute an equal portion of one. Three realizations of this example process are shown in Figure 4.3.

Figure 4.3 Three realizations of the stochastic process (4.3.1) from Table 4.1.



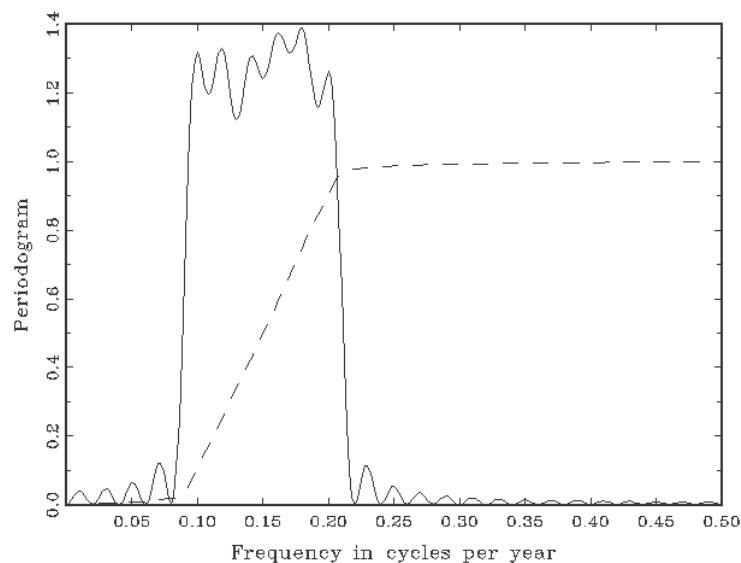
Note that these realizations of such a simple model with six stochastic sinusoids are already rather hard to distinguish from time series of real economic variables. There is nothing to prevent us from calculating the periodogram for each of the three time series, just as with the perfect cosine function at the end of the previous section. The three periodograms are shown in Figure 4.4.

Figure 4.4 Periodogram and cumulative periodogram of three realizations of the stochastic process (4.3.1) from Table 4.1 for  $T=50$ .



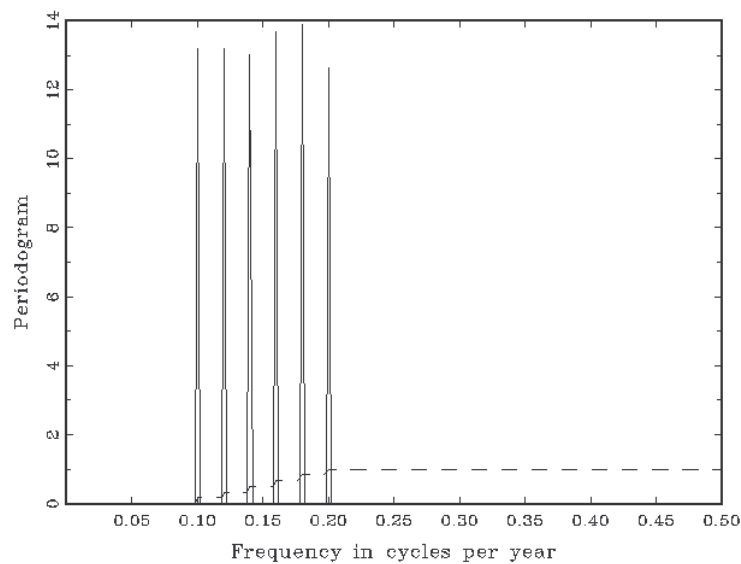
We now see that depending on the draws for the random variables  $a_j$  and  $b_j$  in each of the three realizations the periodogram has a different shape. Although the variance mass in each of the periodograms is approximately between the frequencies 0.10 and 0.20 as we would expect, the periodograms do not show that each of the six frequencies contributes an equal portion to the total variance as we know is the case. An alternative is not to look at the periodogram for each of the realizations separately but at the *average* value of these periodograms. Figure 4.5 shows the average periodogram calculated from 500 (instead of three) realizations of the process for a sample size  $T=50$ .

Figure 4.5 Average periodogram of 500 realizations of the stochastic process (4.3.1) from Table 4.1 for  $T=50$ .



We now see that the average value of the periodogram is much more in correspondence with the properties of the stochastic process. The variance seems evenly distributed over the frequency interval  $[0.10, 0.20]$ . However we also see the leakage effects. It seems as if all frequencies in and around the interval  $[0.10, 0.20]$  contribute to the variance of the process while we know that by construction this is only so for the six frequencies from Table 4.1. These leakage effects cannot be reduced by further increasing the number of realizations. Just as with the deterministic example at the end of the previous section, the leakage effect can be reduced by increasing the sample size  $T$ . Figure 4.6 shows the average periodogram calculated from 500 realizations of the stochastic process with a sample size of  $T=500$  instead of  $T=50$ . Now the average periodogram clearly shows that the variance of the process is constructed from six frequencies, each of which contributes an equal portion.

Figure 4.6 Average periodogram of 500 realizations of the stochastic process (4.3.1) from Table 4.1 for  $T=500$ .



The average periodogram calculated on a great number of realizations of sufficient length from a stochastic process in a sense describes the expected distribution of the variance of the process over periodic fluctuations with a continuous range of frequencies. This expected value of the periodogram is also known as the *spectral density* (*spectrum*) of a stochastic process. The use of the word “density” shows the analogy with a probability density function. A probability density function describes the distribution of a probability mass of one over some domain while a spectral density describes the distribution of a variance mass over the frequency domain  $[0, 2\pi]$ . The formal definition is

$$S(\omega) = \lim_{T \rightarrow \infty} E(P(\omega)) \text{ for } \omega \in [0, 2\pi] \quad (4.3.3)$$

where the periodogram  $P(\omega)$  is as defined in (4.2.1).

Similar to the definition of the cumulative periodogram, the *integrated spectral density* (*integrated spectrum*) is defined as

$$IS(\omega_a) = \int_0^{\omega_a} S(\omega) d\omega \text{ for } \omega_a \in [0, 2\pi] \quad (4.3.4)$$

In the previous example the spectrum was introduced for a special type of stationary process. The shape of this process was based on the so called *spectral representation theorem* of Cramér. Informally, this theorem states that any (covariance) stationary stochastic process  $x_t$  with expected value  $E(x_t) = \mu$  can be represented as

$$x_t = \mu + \int_0^\pi \{ a(\omega) \cos(\omega t) + b(\omega) \sin(\omega t) \} d\omega \quad (4.3.5)$$

The stochastic processes  $a(\omega)$  and  $b(\omega)$  have expected values of zero, variance  $\sigma^2(\omega)$  and are both internal (with respect to  $\omega$ ) and mutually independent. As a consequence of this theorem any stationary process has a spectrum describing the distribution of the variance over the frequency interval  $[0, 2\pi]$ . It should be clear by now that the spectrum gives very important information about the dynamics of a stochastic process.

The relation between the periodogram and the sample autocorrelations as given by (4.2.8) generalizes to the case of the spectrum and the (theoretical) autocovariances of a stochastic process. Substituting (4.2.8) into the definition (4.3.3) gives

$$\begin{aligned} S(\omega) &= \lim_{T \rightarrow \infty} E(P(\omega)) \\ &= \lim_{T \rightarrow \infty} E \left( \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \hat{\gamma}_k \exp(-i\omega k) \right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} E(\hat{\gamma}_k) \exp(-i\omega k) \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} E(\hat{\gamma}_k) \exp(-i\omega k) \end{aligned} \quad (4.3.6)$$

Assuming the process has a mean value of zero, the sample autocorvariances  $\hat{\gamma}_k$  as defined in (4.2.9) are asymptotically unbiased estimators of the true autocovariances  $\gamma_k$  as defined in (3.1.2) and it follows that

$$S(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k \exp(-i\omega k) \text{ for } \omega \in [0, 2\pi] \quad (4.3.7)$$

is an easy way to calculate a spectrum from the autocovariances of the process.

Practical calculations can be based on the equivalent expression

$$\begin{aligned} S(\omega) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k (\cos(\omega k) - i \sin(\omega k)) \\ &= \frac{1}{2\pi} \left( \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \right) \end{aligned} \quad (4.3.8)$$

which uses the symmetry property  $\gamma_k = \gamma_{-k}$ ,  $\sin(-x) = -\sin(x)$  and  $\cos(-x) = \cos(x)$ . Both the previous relations show how to calculate the spectrum from the autocorrelations. The other way around is also possible as can be seen from the following relation.

$$\gamma_k = \int_0^{2\pi} S(\omega) \exp(i\omega k) d\omega \quad \text{for } k \in [-\infty, \infty] \quad (4.3.9)$$

The spectrum  $S(\omega)$  and the autocorrelations  $\gamma_k$  contain the same information about the dynamics of a stochastic process. Neither can give information that cannot be derived from the other. The only difference is the way of presenting the information. In some applications one and in other applications the other representation may be the most appropriate. In the research presented here, the spectrum is preferred for studying fluctuations in economic time series processes. This will become clear in Part III.

Finally, note that the remarks following (4.2.2) with respect to the periodogram equally well apply to spectra. Hence in practical applications, most of the times the normalized spectrum will be used by dividing (4.3.7) by the variance  $\gamma_0 = \sigma^2$  of the process and multiplying it by two so that it can be shown on the interval  $[0, \pi]$  instead of  $[0, 2\pi]$ .

#### 4.4 Bivariate spectral parameters

The theory of spectral analysis as discussed in the previous sections is limited to the univariate case. However in order to investigate the workings of a complex multivariate process such as macroeconomics one can seldom suffice with a univariate analysis. When only individual time series are investigated no information is obtained on how the time series move together. From Chapter 2 it should be clear that how the various macroeconomic variables are interrelated may very well be one of the most important aspects of macroeconomics. In the time domain such interdependencies are reflected by the (cross) correlations between the variables. This section shows that the counterpart of these correlations in the frequency domain consists of the so called *cross-spectra* of which the *coherence* and *phase spectra* are the most useful representations.

#### 4.4.1 Cross-periodograms

Suppose that the two time series  $\{x_t, t=0, \dots, T-1\}$  and  $\{y_t, t=0, \dots, T-1\}$  have been transformed into the frequency domain by the DCFT to yield

$$\begin{aligned} J_x(\omega) &= \frac{1}{T} \sum_{t=0}^{T-1} x_t \exp(-i\omega t) \quad \forall \omega \\ J_y(\omega) &= \frac{1}{T} \sum_{t=0}^{T-1} y_t \exp(-i\omega t) \quad \forall \omega \end{aligned} \quad (4.4.1)$$

The periodograms of these time series are

$$\begin{aligned} P_x(\omega) &= \frac{T}{2\pi} |J_x(\omega)|^2 = \frac{T}{2\pi} J_x(\omega) J_x(\omega)^* \quad \text{for } \omega \in [0, 2\pi] \\ P_y(\omega) &= \frac{T}{2\pi} |J_y(\omega)|^2 = \frac{T}{2\pi} J_y(\omega) J_y(\omega)^* \quad \text{for } \omega \in [0, 2\pi] \end{aligned} \quad (4.4.2)$$

for which we know that integrating them over the interval  $[0, 2\pi]$  leads to the sample variances  $\sigma_x^2$  and  $\sigma_y^2$  of the time series. We also know that the periodograms can just as well be calculated based on a DCFT of the (univariate) sample autocovariances. Analogous to the periodogram, define the *cross-periodograms* as

$$\begin{aligned} P_{xy}(\omega) &= \frac{T}{2\pi} J_x(\omega) J_y(\omega)^* \quad \text{for } \omega \in [0, 2\pi] \\ P_{yx}(\omega) &= \frac{T}{2\pi} J_x(\omega)^* J_y(\omega) \quad \text{for } \omega \in [0, 2\pi] \end{aligned} \quad (4.4.3)$$

It is easy to see that these two cross-periodograms are complex conjugates such that

$$P_{yx}(\omega) = P_{xy}(\omega)^* \quad (4.4.4)$$

and that they therefore contain the same information. Analogous to the periodogram, it can be shown that a cross-periodogram distributes the *covariance* between the two time series instead of the variance of one of the time series over the frequency interval  $[0, 2\pi]$ . So

$$\int_0^{2\pi} P_{xy}(\omega) d\omega = \sigma_{xy}^2 \quad (4.4.5)$$

where

$$\sigma_{xy}^2 = \frac{1}{T} \sum_{t=0}^{T-1} x_t y_t \quad (4.4.6)$$

is the sample covariance between the time series  $x_t$  and  $y_t$ . The proof of (4.4.5) is given in Appendix B.8.

Besides the distribution of the covariance instead of the variance the similarity between the cross-periodogram and the conventional periodogram goes even one step further. While the periodogram can be calculated as the DCFT of the (univariate) sample autocovariances, the cross-periodogram can be calculated as the DCFT of the sample cross-covariances. That is

$$P_{xy}(\omega) = \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \hat{\gamma}_{xy,k} \exp(-i\omega k) \quad (4.4.7)$$

where

$$\hat{\gamma}_{xy,k} = \begin{cases} \frac{1}{T} \sum_{t=k}^{T-1} x_t y_{t-k} & \text{for } k = 0, \dots, T-1 \\ \frac{1}{T} \sum_{t=0}^{T-1-k} x_t y_{t-k} & \text{for } k = -T+1, \dots, -1 \end{cases} \quad (4.4.8)$$

are the sample cross-covariances. The proof of (4.4.7) is given in Appendix B.9.

#### 4.4.2 Cross-spectral density

Similar to the relation between the periodogram and the spectral density function, is defined as the *cross-spectral density function (cross-spectrum)* of a two dimensional stochastic process as the expected value of the cross-periodogram for realizations of the process of infinite length.

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} E(P_{xy}(\omega)) \text{ for } \omega \in [0, 2\pi] \quad (4.4.9)$$

Also this cross-spectrum can be shown to be equal to the DCFT of the (theoretical) cross-covariances of the process.

$$S_{xy}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{xy,k} \exp(-i\omega k) \quad (4.4.10)$$

where the cross-covariances are defined as

$$\gamma_{xy,k} = E(x_t - \mu_x)(y_{t-k} - \mu_y) \quad (4.4.11)$$

which are the off diagonal elements of the multivariate autocovariance matrices as defined in (3.1.5). Also again the inverse relation holds.

$$\gamma_{xy,k} = \int_0^{2\pi} S_{xy}(\omega) \exp(i\omega k) d\omega \text{ for } k \in [-\infty, \infty] \quad (4.4.12)$$



Note that (4.4.10) is equal to

$$S_{xy}(\omega) = \left( \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{xy,k} \cos(\omega k) \right) - i \left( \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{xy,k} \sin(\omega k) \right) \quad (4.4.13)$$

Because  $\sin(-k) = -\sin(k)$  and the autocovariances are symmetrical with respect to  $k$  (i.e.  $\gamma_{x,k} = \gamma_{x,-k}$ ) the sinus component drops out of the univariate spectra in (4.3.8). Univariate spectra are also called *auto-spectra* to distinct them from cross-spectra. Because in general cross-covariances are not symmetrical (i.e.  $\gamma_{xy,k} \neq \gamma_{yx,-k}$ ) this is not the case for the cross-spectra. Therefore a cross-spectrum will in general have an imaginary part which complicates their interpretation.

Now we have defined both the univariate (auto) spectra and the cross-spectra, these can be generalized into the spectrum of an arbitrary multivariate stochastic process. Suppose  $\bar{x}_t$  represents a  $n$  dimensional stationary stochastic process with expected value  $E(\bar{x}_t) = 0$  and the  $k$ -th order autocovariance matrices  $\Gamma_k = E(\bar{x}_t \bar{x}_{t-k}')$ . Based on the multivariate version of Cramér's theorem for this process the multivariate spectrum can be defined as

$$S(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Gamma_k \exp(-i\omega k) \quad (4.4.14)$$

The inverse transformation becomes

$$\Gamma_k = \int_0^{2\pi} S(\omega) \exp(i\omega k) d\omega \quad (4.4.15)$$

with as a special case

$$\int_0^{2\pi} S(\omega) d\omega = \Gamma_0 \quad (4.4.16)$$

The multivariate spectral density  $S(\omega)$  has now become a *spectral matrix*. Because the  $j$ -th diagonal element of  $\Gamma_k$  is the  $k$ -th order autocovariance of the  $j$ -th variable, the diagonal of the multivariate spectrum contains the auto-spectra of the  $n$  variables. We already know that these are real valued functions. The off-diagonal elements of  $\Gamma_k$  are the  $k$ -th order cross-covariances between the relevant variables. The corresponding elements in the multivariate spectrum are the cross-spectra defined in this section. In general these are complex valued functions.

### 4.4.3 Co-spectrum and quadrature spectrum

Although the definition of a cross-spectrum is clear, its interpretation is not that easy because of the imaginary part. There are two alternative representations of a cross-spectrum which enable a simpler interpretation. These representation are based on the familiar representations of complex numbers in terms of real numbers. The first is the *cartesian format* by simply splitting the cross-spectrum up into its real and its imaginary part. These are called respectively the *co-spectrum* and the *quadrature spectrum*. In formulas these are

$$C_{xy}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{xy,k} \cos(\omega k) \quad (4.4.17)$$

and

$$Q_{xy}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{xy,k} \sin(\omega k) \quad (4.4.18)$$

which can be directly seen from (4.4.13).  
Because

$$\begin{aligned} C_{xy}(\pi - \omega) &= C_{xy}(\pi + \omega) \\ Q_{xy}(\pi - \omega) &= -Q_{xy}(\pi + \omega) \end{aligned} \quad (4.4.19)$$

the co-spectrum is an even function around  $\omega=\pi$  while the quadrature spectrum is an uneven function. Just as for the univariate spectra this shows that the frequency interval  $[\pi, 2\pi]$  contains no information that is not already contained in the interval  $[0, \pi]$ . Furthermore from (4.4.5) and (B.8.2) from Appendix B.8 it follows that

$$\int_0^{2\pi} C_{xy}(\omega) d\omega = \sigma_{xy}^2 \quad (4.4.20)$$

and

$$\int_0^{2\pi} Q_{xy}(\omega) d\omega = 0 \quad (4.4.21)$$

So, the co-spectrum distributes the entire covariance while the quadrature spectrum integrates to zero. Note that this does not mean that the quadrature spectrum itself is equal to zero. Mainly because of the difficult interpretation of the quadrature spectrum we prefer not to use this representation of a cross-spectrum. The best but still difficult interpretation is that the co-spectrum represents the correlations between the “in phase” components of the two time series while the quadrature spectrum represents the correlations between the “out of phase” components. The intuition here is that periodic components of a certain frequency may have an important part in the variance of both  $x_t$  and  $y_t$  while these may contribute only little to the covariance because in each of the two processes the phase of the component can be very different. It can be shown that the quadrature spectrum looks for signs of such out of phase components.

#### 4.4.4 Coherence and phase spectrum

The second representation of a complex valued cross-spectrum is its *polar format*

$$S_{xy}(\omega) = |S_{xy}(\omega)| \exp(i \arg(S_{xy}(\omega))) \quad (4.4.22)$$

Based on this representation the *coherence spectrum* is defined as

$$CH_{xy}(\omega) = \frac{|S_{xy}(\omega)|}{\sqrt{S_x(\omega)S_y(\omega)}} \quad (4.4.23)$$

and the *phase spectrum* as

$$PH_{xy}(\omega) = \arg(S_{xy}(\omega)) \quad (4.4.24)$$

These two definitions are the most useful parameters for representing the relations between two stochastic variables in the frequency domain. To illustrate the interpretation of the coherence and phase spectrum consider the following simple bivariate stochastic process which is an extension of the example (4.3.1).

$$\begin{aligned} x_t(\omega) &= a_x(\omega) \cos(\omega t) + b_x(\omega) \sin(\omega t) \\ y_t(\omega) &= a_y(\omega) \cos(\omega t + \phi(\omega)) + b_y(\omega) \sin(\omega t + \phi(\omega)) \end{aligned} \quad (4.4.25)$$

where

$$E \begin{bmatrix} a_x(\omega) \\ a_y(\omega) \\ b_x(\omega) \\ b_y(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E \begin{bmatrix} a_x(\omega) \\ a_y(\omega) \\ b_x(\omega) \\ b_y(\omega) \end{bmatrix} \begin{bmatrix} a_x(\omega) & a_y(\omega) & b_x(\omega) & b_y(\omega) \end{bmatrix} = \begin{bmatrix} \sigma_x^2(\omega) & \sigma_{xy}^2(\omega) & 0 & 0 \\ \sigma_{xy}^2(\omega) & \sigma_y^2(\omega) & 0 & 0 \\ 0 & 0 & \sigma_x^2(\omega) & \sigma_{xy}^2(\omega) \\ 0 & 0 & \sigma_{xy}^2(\omega) & \sigma_y^2(\omega) \end{bmatrix}$$

Both  $x_t(\omega)$  and  $y_t(\omega)$  are sinusoids with a stochastic amplitude and phase but with a fixed frequency  $\omega$ . The parameter  $\phi(\omega)$  is the expected difference in phase between the two variables. The expected lead of  $y_t(\omega)$  over  $x_t(\omega)$  is  $\phi(\omega)/\omega$  time periods. The stochastic variables  $a_x(\omega)$ ,  $b_x(\omega)$ ,  $a_y(\omega)$  and  $b_y(\omega)$  cause the stochastic nature of the phase and amplitudes of the two variables. Because of the covariance  $\sigma_{xy}^2$  between  $a_x(\omega)$  and  $a_y(\omega)$  and between  $b_x(\omega)$  and  $b_y(\omega)$ , the two processes are interrelated. The Cramér representation of a stationary process consists of a continuum of frequencies. Strictly speaking it is therefore not possible to isolate one specific frequency from this range. Nevertheless we may see  $x_t(\omega)$  and  $y_t(\omega)$  as the discrete

approximations of the components with frequency  $\omega$  from the general stochastic processes

$$\begin{aligned} x_t &= \mu_x + \int_0^{2\pi} a_x(\omega) \cos(\omega t) + b_x(\omega) \sin(\omega t) d\omega \\ y_t &= \mu_y + \int_0^{2\pi} a_y(\omega) \cos(\omega t + \phi(\omega)) + b_y(\omega) \sin(\omega t + \phi(\omega)) d\omega \end{aligned} \quad (4.4.26)$$

with  $\mu_x=0$  and  $\mu_y=0$ . Appendix B.10 shows that the first and second moments of the bivariate process (4.4.25) are

$$\mu = E \begin{bmatrix} x_t(\omega) \\ y_t(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4.4.27)$$

and

$$\begin{aligned} \Gamma_k &= E \begin{bmatrix} x_t(\omega) \\ y_t(\omega) \end{bmatrix} \begin{bmatrix} x_{t-k}(\omega) & y_{t-k}(\omega) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_x^2(\omega) \cos(\omega k) & \sigma_{xy}^2(\omega) \cos(\phi(\omega) - \omega k) \\ \sigma_{xy}^2(\omega) \cos(\omega k - \phi(\omega)) & \sigma_y^2(\omega) \cos(\omega k) \end{bmatrix} \end{aligned} \quad (4.4.28)$$

with a as a special case for  $k=0$  the covariance matrix

$$\Gamma_0 = E \begin{bmatrix} x_t(\omega) \\ y_t(\omega) \end{bmatrix} \begin{bmatrix} x_t(\omega) & y_t(\omega) \end{bmatrix} = \begin{bmatrix} \sigma_x^2(\omega) & \sigma_{xy}^2(\omega) \cos(\phi(\omega)) \\ \sigma_{xy}^2(\omega) \cos(\phi(\omega)) & \sigma_y^2(\omega) \end{bmatrix} \quad (4.4.29)$$

Substituting the diagonal elements of the autocovariance matrices  $\Gamma_k$  into the definition (4.3.3) of an auto-spectrum and postponing the limit for  $T \rightarrow \infty$  yields

$$\begin{aligned} S_x(\omega) &= \lim_{T \rightarrow \infty} E(P_x(\omega)) \\ &= \lim_{T \rightarrow \infty} \left( \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \gamma_{x,k} \cos(\omega k) - i \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \gamma_{x,k} \sin(\omega k) \right) \\ &= \lim_{T \rightarrow \infty} \begin{pmatrix} \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \sigma_x^2(\omega) \cos(\omega k) \cos(\omega k) \\ -i \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \sigma_x^2(\omega) \cos(\omega k) \sin(\omega k) \end{pmatrix} \\ &= \lim_{T \rightarrow \infty} \left( \frac{T-1}{2\pi} \sigma_x^2(\omega) \right) \end{aligned} \quad (4.4.30)$$

and similar

$$S_y(\omega) = \lim_{T \rightarrow \infty} \left( \frac{T-1}{2\pi} \sigma_y^2(\omega) \right) \quad (4.4.31)$$

which uses the relations (B.2.3). The fact that these auto-spectra reach infinity when applying the limit  $T \rightarrow \infty$  stems from the fact that the processes consist of only a single frequency in a continuous domain which has to represent the entire variance. Note that multiplying by the length of a single interval  $2\pi/(T-1)$  in a discrete approximation of the integral leads to exactly the variances  $\sigma_x^2$  and  $\sigma_y^2$ .

Substituting the off-diagonal elements of the autocovariance matrices  $\Gamma_k$  into the definition (4.4.9) of a cross-spectrum and again postponing the limit for  $T \rightarrow \infty$  yields

$$\begin{aligned} S_{xy}(\omega) &= \lim_{T \rightarrow \infty} E(P_{xy}(\omega)) \\ &= \lim_{T \rightarrow \infty} \begin{pmatrix} \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \sigma_{xy}^2(\omega) \cos(\phi(\omega) - \omega k) \cos(\omega k) \\ -i \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \sigma_{xy}^2(\omega) \cos(\phi(\omega) - \omega k) \sin(\omega k) \end{pmatrix} \\ &= \lim_{T \rightarrow \infty} \left( \frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \cos(\phi(\omega)) - i \frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \sin(\phi(\omega)) \right) \end{aligned} \quad (4.4.32)$$

by first applying the doubling formulas from (A.1.6) and then again the relations (B.2.3). If we now substitute these expressions for the auto-spectra and cross-spectra into the definition (4.4.23) of the coherence spectrum we get

$$\begin{aligned} CH_{xy}(\omega) &= \frac{|S_{xy}(\omega)|}{\sqrt{S_x(\omega)S_y(\omega)}} \\ &= \lim_{T \rightarrow \infty} \frac{\sqrt{\left( \frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \cos(\phi(\omega)) \right)^2 + \left( \frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \sin(\phi(\omega)) \right)^2}}{\sqrt{\left( \frac{T-1}{2\pi} \sigma_x^2(\omega) \right) + \left( \frac{T-1}{2\pi} \sigma_y^2(\omega) \right)}} \\ &= \frac{|\sigma_{xy}^2(\omega)|}{\sigma_x(\omega)\sigma_y(\omega)} \end{aligned} \quad (4.4.33)$$

This expression shows that the coherence  $CH_{xy}(\omega)$  can be interpreted as the absolute correlation coefficient between the stochastic components disturbing the phase and amplitude at the frequency  $\omega$ . So it holds

$$0 \leq CH_{xy}(\omega) \leq 1 \quad (4.4.34)$$

A coherence close to one for some frequency  $\omega$  means that there exists a strong relation between the fluctuations of that frequency in the two time series. It is important to note that the phase difference  $\phi(\omega)$  has no effect on the coherence and its interpretation. As we shall see, this information is given by the phase spectrum. Furthermore, for example Koopmans (1974) shows that the squared coherence can also be interpreted as the portion of the variance at frequency  $\omega$  in one of the

variables that is explained by a linear regression on the component of the same frequency from the other variable. In such a regression possible phase differences are corrected in order to get the best explanation power.

Substituting the previous expressions for the auto-spectra and cross-spectra into the definition (4.4.24) of the phase spectrum because of (A.2.5) comes down to solving

$$\begin{aligned}\cos(PH_{xy}(\omega)) &= \frac{\text{real}(S_{xy}(\omega))}{|S_{xy}(\omega)|} \quad \text{and} \quad \sin(PH_{xy}(\omega)) = \frac{\text{imag}(S_{xy}(\omega))}{|S_{xy}(\omega)|} \\ \Leftrightarrow \cos(PH_{xy}(\omega)) &= \lim_{T \rightarrow \infty} \left( \frac{\frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \cos(\phi(\omega))}{\frac{T-1}{2\pi} \sigma_{xy}^2(\omega)} \right) = \cos(\phi(\omega)) \\ \text{and } \sin(PH_{xy}(\omega)) &= \lim_{T \rightarrow \infty} \left( \frac{\frac{T-1}{2\pi} \sigma_{xy}^2(\omega) \sin(\phi(\omega))}{\frac{T-1}{2\pi} \sigma_{xy}^2(\omega)} \right) = \sin(\phi(\omega))\end{aligned}\tag{4.4.35}$$

which has as the obvious solution

$$PH_{xy}(\omega) = \phi(\omega)\tag{4.4.36}$$

Similarly one can obtain

$$PH_{yx}(\omega) = -\phi(\omega)\tag{4.4.37}$$

This shows that the phase spectrum at frequency  $\omega$  gives the expected lead of the second variable on the first variable. The variable  $y_t$  leads  $x_t$  by  $\phi(\omega)$  which is the same as saying that  $x_t$  lags  $y_t$  by  $\phi(\omega)$  (i.e. a lead of  $-\phi(\omega)$ ). It is more convenient to interpret the phase spectrum as the expected lead of the *first* variable over the *second* variable and also in terms of time periods instead of radians. In practical applications we will therefore use the following slightly modified definition of the phase spectrum.

$$PH_{xy}^*(\omega) = \frac{-PH_{xy}(\omega)}{\omega}\tag{4.4.38}$$

For a final interpretation of the phase spectrum, define a third stochastic variable  $z_t$  which is the same as  $x_t$  but only with a lead of  $\alpha$ .

$$z_t(\omega) = a_x(\omega) \cos(\omega t + \alpha) + b_x(\omega) \sin(\omega t + \alpha)\tag{4.4.39}$$

Along the lines of the derivations in Appendix B.10 it follows that the covariance between  $z_t$  and  $y_t$  is

$$\begin{aligned} E(z_t(\omega)y_t(\omega)) &= \sigma_{xy}^2(\omega)(\cos(\alpha)\cos(\phi(\omega)) + \sin(\alpha)\sin(\phi(\omega))) \\ &= \sigma_{xy}^2(\omega)\cos(\alpha - \phi(\omega)) \end{aligned} \quad (4.4.40)$$

Because the cosine function has its maximum value at the frequencies  $k2\pi$  for any integer  $k$ , this covariance has its maximum value at  $\alpha^* = \phi(\omega) + k2\pi$ . This shows that the value of the phase spectrum at a certain frequency can also be interpreted as the lead that has to be applied to the first variable  $x_t$  to obtain the maximum covariance or correlation with the second variable  $y_t$ . Note that when this optimal phase shift is applied (4.4.40) becomes equal to  $\sigma_{xy}^2$ . Dividing this covariance by both  $\sigma_x$  and  $\sigma_y$  then gives exactly the coherence (4.4.33). So, the coherence spectrum gives the maximum correlation after having applied the optimal phase shift to one of the variables while the phase spectrum gives the size of this optimal phase shift. In this sense one could say that the coherence and phase spectra dissect the conventional correlations at the various frequencies into a phase shift and the maximum correlation possible after such a phase shift. Note the similarity with the conventional (business cycle) approach in which the lag is found for which the conventional correlation has its maximum value. The phase spectrum however gives more detailed information in terms of the behavior at the different frequencies. An important additional remark is that in general the value of the phase spectrum is of little use when the coherence for the relevant frequency is only small. This concludes the argumentation of why the coherence and phase spectra are the most useful representation of the cross-spectrum.

Finally note that the sample analogues of the coherence spectrum in terms of the auto-periodograms and cross-periodograms does not reveal any useful information because substituting (4.4.2) and (4.4.3) yields

$$\begin{aligned} \frac{|P_{xy}(\omega)|}{\sqrt{P_x(\omega)P_y(\omega)}} &= \frac{\sqrt{\left(\frac{T}{2\pi}J_x(\omega)J_y(\omega)^*\right)^* \left(\frac{T}{2\pi}J_x(\omega)J_y(\omega)^*\right)}}{\sqrt{\left(\frac{T}{2\pi}J_x(\omega)J_x(\omega)^*\right) \left(\frac{T}{2\pi}J_y(\omega)J_y(\omega)^*\right)}} \\ &= \sqrt{\frac{J_x(\omega)^* J_y(\omega) J_x(\omega) J_y(\omega)^*}{J_x(\omega) J_x(\omega)^* J_y(\omega) J_y(\omega)^*}} \\ &= 1 \end{aligned} \quad (4.4.41)$$

That is, the sample coherence is always equal to one for any frequency  $\omega$ . This is caused by the fact that the standard periodogram is a poor estimator of a spectral density. Section 4.6 shows why this is so and discusses how better estimates of spectral densities in general and of coherence and phase spectra in specific can be obtained.



## 4.5 Autoregressive and other models

Given the definitions of both univariate and multivariate spectral densities from the previous sections, this section gives the formula for the spectra of various well known time series models of stochastic processes. For the univariate case, the simple white noise process is discussed and extended to moving average (MA) and autoregressive (AR) processes. For the multivariate case, only the VAR model is discussed. Several examples are included of which the spectral density of the VAR(2) model defined in section 3.3.1 is the most important one. Note that the spectral densities of the following models are defined for stationary stochastic processes only.

### 4.5.1 White noise

A simple white noise process is defined as

$$x_t = \varepsilon_t \quad (4.5.1)$$

where

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ E(\varepsilon_t^2) &= \sigma^2 \\ E(\varepsilon_t \varepsilon_{t-k}) &= 0 \text{ for } k \neq 0 \end{aligned}$$

Substituting the autocovariances

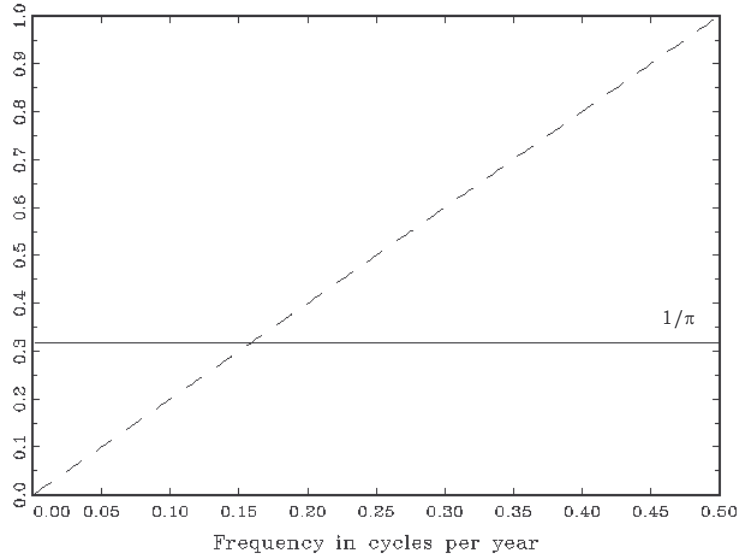
$$\gamma_0 = \sigma^2 \text{ and } \gamma_k = 0 \text{ for } k \neq 0 \quad (4.5.2)$$

into the basic spectral equation (4.3.8) gives

$$S(\omega) = \frac{1}{2\pi} \left( \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\omega k) \right) = \frac{\sigma^2}{2\pi} \quad (4.5.3)$$

as the spectral density of a white noise process. Figure 4.7 illustrates the flat format of this (normalized) spectrum. In a white noise process all frequencies are present and each of them contributes an equal portion to the total variance. It is therefore not the case, as one might have expected, that a white noise process only consists of high frequency fluctuations.

Figure 4.7 Spectrum of a white noise process (4.5.1).



#### 4.5.2 Moving Average models

A Moving Average process of order  $q$  (MA( $q$ )) is defined as

$$x_t = v + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \dots + \alpha_p \varepsilon_{t-q} \quad (4.5.4)$$

where  $\varepsilon_t$  is a white noise process as in (4.5.1). Substituting its autocovariances

$$\begin{aligned} \gamma_0 &= \sigma^2 \left( 1 + \sum_{j=1}^q \alpha_j^2 \right) \\ \gamma_k &= \sigma^2 \left( \alpha_k + \sum_{j=1}^{q-k} \alpha_{k+j} \alpha_j \right) \text{ for } 1 \leq k \leq q \\ \gamma_k &= 0 \text{ for } k > q \end{aligned} \quad (4.5.5)$$

into the basic spectral equation (4.3.8) gives

$$S(\omega) = \frac{\sigma^2}{2\pi} \left( 1 + \sum_{j=1}^q \alpha_j^2 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + 2 \sum_{k=1}^{q-1} \sum_{j=1}^{q-k} \alpha_{k+j} \alpha_j \cos(\omega k) \right) \quad (4.5.6)$$

Appendix B.11 gives the proof that this is the same as the following equivalent, though simpler, expression.

$$S(\omega) = \frac{\sigma^2}{2\pi} \left| 1 + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \right|^2 \quad (4.5.7)$$

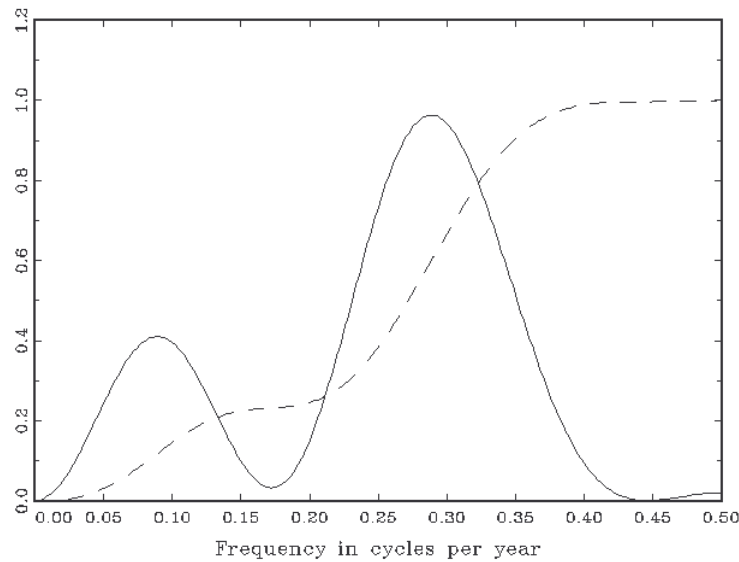
Expanding the complex exponential function gives the following formula for practical calculations.

$$\begin{aligned}
 S(\omega) &= \frac{\sigma^2}{2\pi} \left| 1 + \sum_{k=1}^q \alpha_k \cos(\omega k) - i \sum_{k=1}^q \alpha_k \sin(\omega k) \right|^2 \\
 &= \frac{\sigma^2}{2\pi} \left[ \left( 1 + \sum_{k=1}^q \alpha_k \cos(\omega k) \right)^2 + \left( \sum_{k=1}^q \alpha_k \sin(\omega k) \right)^2 \right]
 \end{aligned} \tag{4.5.8}$$

As an example consider the (normalized) spectrum shown in Figure 4.8 of the MA(6) process

$$x_t = \varepsilon_t + 0.4\varepsilon_{t-1} - 0.8\varepsilon_{t-2} + 0.2\varepsilon_{t-3} + 0.3\varepsilon_{t-4} - 0.8\varepsilon_{t-5} - 0.3\varepsilon_{t-6} \tag{4.5.9}$$

Figure 4.8 Spectrum of MA(6) process (4.5.9).



The spectrum of the process shows that a broad range of frequencies around both the frequencies  $\omega \approx 0.10$  (period length of about ten years) and  $\omega \approx 0.30$  (period length of about three years) contribute to the variance of the process. From the integrated spectrum it can be seen that the first range of frequencies contributes about 25% to the total variance and the second range of frequencies about 75%. The broadness of the peaks gives us information on how distinct the (pseudo) periodic behavior is that is present in the process. A broad peak indicates that the periodic behavior is caused by many interacting frequencies and will not be that clearly visible. A narrow peak indicates that only a limited number of frequencies cause the periodic behavior causing it to be more clearly visible. To achieve such peaked spectra rather high orders  $q$  are needed. As the order increases more complex spectra become possible. In general the spectra of Moving Average models have rather broad peaks and narrow troughs, indicating that very clear pseudo periodic behavior cannot be modeled well by Moving Average models. A class of models that is better suited for this purpose is the class of autoregressive models to which we turn next. In general to describe peaked spectra, such as in Figure 4.8, autoregressive models of only a low order are

needed. An AR(2) model for example can already describe a spectrum with a single peak.

#### 4.5.3 Autoregressive models

Both univariate and multivariate autoregressive models have been extensively discussed in the previous chapter. For the general univariate AR( $p$ ) model from that chapter

$$x_t = v + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \dots + \beta_p x_{t-p} + \varepsilon_t \quad (4.5.10)$$

the spectrum can be obtained in the same way as with the MA( $q$ ) model by first calculating the autocovariances and then substituting these into the basic spectral equation (4.3.8). These autocovariances have to be obtained from the solution (3.2.23) and (3.2.24) of the Yule-Walker equations. Koopmans (1974) shows that the resulting spectrum of an AR( $p$ ) models can also be calculated as

$$S(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{\left| 1 - \sum_{k=1}^p \beta_k \exp(-i\omega k) \right|^2} \quad (4.5.11)$$

Expanding the complex exponential function gives the following formula for practical calculations.

$$S(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{\left( 1 - \sum_{k=1}^p \beta_k \cos(\omega k) \right)^2 + \left( \sum_{k=1}^p \beta_k \sin(\omega k) \right)^2} \quad (4.5.12)$$

For the general multivariate VAR( $p$ ) model

$$x_t = v + A_1 x_{t-1} + A_2 x_{t-2} + \dots + A_p x_{t-p} + \varepsilon_t \quad (4.5.13)$$

as defined in (3.2.25) both Hamilton (1994, p. 276) and Priestley (1981, p. 689) show that the spectral matrix is equal to

$$S(\omega) = \frac{1}{2\pi} \left\{ I_n - A_1 \exp(-i\omega) - A_2 \exp(-i\omega 2) - \dots - A_p \exp(-i\omega p) \right\}^{-1} \Sigma \\ \times \left\{ I_n - A_1' \exp(i\omega) - A_2' \exp(i\omega 2) - \dots - A_p' \exp(i\omega p) \right\}^{-1} \quad (4.5.14)$$

The diagonal elements of this spectral matrix are the auto-spectra of the  $n$  variables while the off-diagonal elements are the cross-spectra between the relevant pairs of variables. Although the direct calculation of (4.5.14) is rather straightforward, in practice this does not always lead to the correct results. Depending on the properties of the VAR model, the direct calculation of (4.5.14) may lead to rather substantial numerical inaccuracies, probably caused by inverting the complex valued matrices present in the expression. These problems manifest themselves for example as auto-

spectra that do not integrate to the total variance of the relevant variable. Also, especially the coherence and phase spectra derived from the cross spectra are incorrect. The following alternative approach requires more calculations but turns out to be free of numerical errors. In this indirect approach one first uses the multivariate Yule-Walker equations from section 3.2.2 to calculate the multivariate autocovariance matrices  $\Gamma_k$  and then transforms these into the frequency domain using the expressions (4.3.8) and (4.4.10) for the calculation of respectively the auto-spectra and cross-spectra. This can be done for each element from the spectral matrix separately. Because of the required summations into infinity the autocovariance matrices  $\Gamma_k$  have to be calculated up to a sufficient high order (both positive and negative). Because for a stationary process the autocovariances will decay towards zero a stopping criterion can be that the absolute value of each element of the autocorrelation matrices  $P_k$  defined by (3.1.7) is smaller than some value, say  $\eta$ , where  $\eta$  is sufficiently small (say 0.0001).

In section 3.3 the dynamics of an example VAR(2) model of dimension  $n=2$  were discussed. There, both a deterministic simulation and the eigenvalues and eigenvectors of the model were used. Here, the indirect approach is used to calculate the spectral matrix for the same VAR(2) model. The resulting auto-spectra, coherence and phase spectrum are shown in Figure 4.9. These spectra show the familiar dynamics of the example model in a way that is very powerful and easy to understand.

First of all, the auto-spectra of the variables  $x_{1,t}$  and  $x_{2,t}$  both show a peak around the frequency  $\omega=0.22$ . These peaks are caused by the complex conjugated eigenvalues  $\lambda_1$  and  $\lambda_2$  given at the end of section 3.3. The argument of these eigenvalues is  $0.22 \cdot 2\pi$ . Their modulus, which is about 0.91, determines the sharpness of the peaks in the spectra at this frequency. The higher the modulus, the slower the dampening in a deterministic simulation and the sharper the peak in the spectra will be, indicating more distinct pseudo-periodic behavior. Furthermore, the peak in the auto-spectrum for the second variable is much higher than in the spectrum of the first variable. This shows that the pseudo-periodic behavior caused by the complex eigenvalues explains almost the total variance of the second variable while it is only about 10% of the variance of the first variable (note the value of the integrated spectrum). This is consistent with the large relative amplitude of the second element in the eigenvectors  $v_1$  and  $v_2$  (a factor of 3.78) corresponding to the complex eigenvalues.

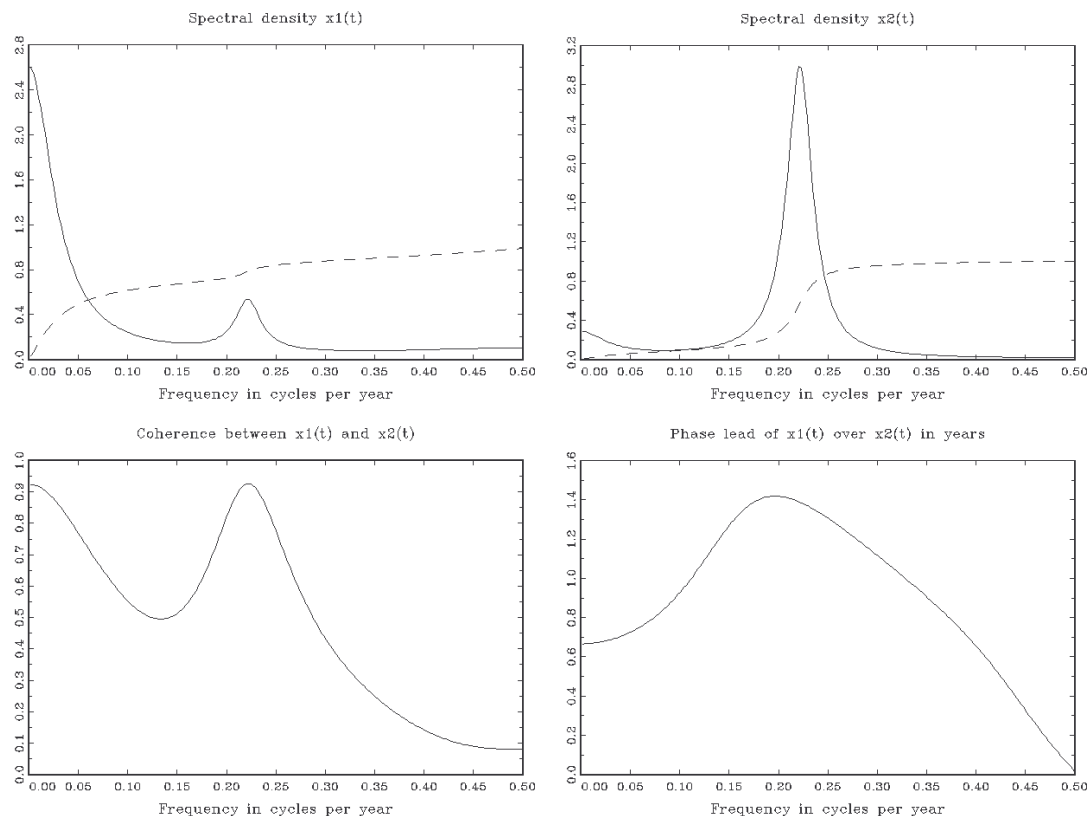
Second, the auto-spectrum of the first variable reveals that its variance is caused to a large extent by very low frequency fluctuations. These correspond to the positive real eigenvalue  $\lambda_3=0.83$  while the corresponding eigenvector  $v_3$  shows that these fluctuations are about twice as important in the first than they are in the second variable. Positive real eigenvalues turn up at the low end of the frequency range indicating very slow moving patterns. Negative real eigenvalues turn up at right hand side of the frequency range indicating alternating patterns. This type of behavior of the fourth eigenvalue  $\lambda_4=-0.37$  is only slightly visible in the auto-spectrum of the first variable (as indicated by the corresponding eigenvector).

Third, the coherence spectrum shows that, when corrected for possible differences in phase, the fluctuations at both the very low frequencies and around the frequency  $\omega=0.22$  are highly correlated. The coherence in these frequency ranges is more than 0.90. The most important information coming from the phase spectrum

is that at the frequency  $\omega=0.22$  the second variable leads the first variable with about 1.4 years. Again this is consistent with the lead / lag relations of the model that we already know from the arguments in the eigenvectors  $v_1$  and  $v_2$  and from the deterministic simulation.

Besides the eigenvalues and eigenvectors of the parameter matrices, also the covariance matrix of the error terms determines the shape of the various spectral densities of the model. If for example the elements in the parameter matrices are relatively small compared to the variance of the error terms, the auto-spectra will look more and more like that of a white noise process as shown in Figure 4.7. The more “structure” there is in the parameter matrices, the more these basic flat auto-spectra are reshaped. Furthermore, it is very likely that the correlations between the error terms will effect the coherence spectra of the model. Because of their contemporaneous character, the error terms do not directly influence the phase spectra. Finally, note that in section 20.5.2 several experiments are performed to explore the effects on the model dynamics of directly changing the eigenvalues and eigenvectors of a VAR model.

Figure 4.9 Auto-spectra, coherence and phase of VAR(2) model (3.3.2).



## 4.6 Spectral estimation techniques

Spectral analysis is a very powerful instrument for analyzing the dynamics of both univariate and multivariate stochastic processes. In previous sections the spectrum has been defined as the average or expected value of the periodogram where the averaging in principle has to take place on realizations of the process of infinite length. Because macroeconomics is a non-experimental science in general only a single and finite realization of the macroeconomic process will be available. Directly calculating the average periodogram on a large number of lengthy realizations is therefore out of the question. Other methods have to be used. Instead to estimate spectral densities. In this section we make a distinction between the conventional *non-parametric estimation techniques* on the one hand and the less well known *parametric estimation techniques* on the other. Both empirically and theoretically most of the attention has gone to the class of the non-parametric estimation techniques. When homogeneous samples of data of sufficient length are available these techniques indeed are the most robust. However if, as is the case in macroeconomics, only short samples are available a parametric technique such as autoregressive or *Maximum Entropy spectral estimation* is to be preferred.

### 4.6.1 Non-parametric estimators

The theory on non-parametric spectral estimation techniques is very extensive. The overview is therefore limited to some of the most important aspects. More complete overviews are given by Priestley (1981) and Koopmans (1974).

There are two types of non-parametric spectral estimation techniques. The first type works directly on the data. If sufficient data of a stationary time series is available, spectra can be estimated by calculating the periodogram on a number of, possibly overlapping, data segments. By averaging these periodograms an estimate of the spectrum is obtained. The averaging over multiple realizations is then replaced by averaging over different parts of the same sample. Because of the lack of long (stationary) data samples, these techniques are rarely used in macroeconomics.

The second type of non-parametric estimators does not work directly on the data but first estimates the theoretical autocovariances and then uses these to estimate the spectrum by means of its empirical counterpart, the periodogram. So

$$\hat{S}(\omega) = P(\omega) = \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \hat{\gamma}_k \exp(-i\omega k) \quad (4.6.1)$$

Note that such an estimator is in two ways different from the theoretical spectrum. First the theoretical autocovariances are replaced by the sample autocovariances as given by (4.2.9). Second the sequence of autocovariances is truncated at the maximum lag possible given the sample size. Despite its intuitive appeal, such an estimator turns out to have one rather unpleasant property. As the sample size increases to infinity, the variance of the estimator does not converge to zero. The periodogram is therefore *not a consistent estimator* of the spectrum. Because the sample autocovariances by themselves are consistent estimators, at first sight this may seem a somewhat surprising result. Realizing that a function of a consistent



estimator need not be a consistent estimator of that same function of the parameters to be estimated, already somewhat tempers this surprise. Some authors claim that the inconsistency of the periodogram stems from the fact that higher order sample autocovariances are calculated on less and less observations as the order reaches the sample size. Although it is certainly true that the quality of an estimated autocovariance decreases with the order of the autocovariance, this does not explain why the variance of the periodogram does not decrease as the sample size increases. According to Priestley (1981, p. 432) the true cause of the periodogram's inconsistency lies in the fact that the variance of every single sample autocovariance is  $O(1/T)$ . If the  $T$  sample autocovariances were independent random variables then the sum of  $T$  such estimators would have a variance that is  $T \cdot O(1/T) = O(1)$ . This would mean that the variance of the sum would not decrease as  $T$  increases to infinity. Although in reality the sample autocovariances are not independent, the argument remains the same. Simply stated, the periodogram is not a consistent estimator for the spectrum because it contains "too many" sample autocovariances. For each extra observation an extra sample autocovariance has to be estimated which introduces extra variance into the spectral estimate.

One possibility to solve the inconsistency of the periodogram is by leaving the higher order autocovariances out of the estimator (4.6.1). Although the bias will increase this will not be very severe because for a stationary process the autocovariances tend to zero as the order increases. This suggests to use a *truncated periodogram* as a spectral estimator.

$$\hat{S}_{truncated}(\omega) = \frac{1}{2\pi} \sum_{k=-M}^M \hat{\gamma}_k \exp(-i\omega k) \quad (4.6.2)$$

where  $M < T-1$  is called the *truncation point*. It can be shown that the variance of this truncated periodogram is  $O(M/T)$ . If we ensure that  $M \rightarrow \infty$  as  $T \rightarrow \infty$  then it is an asymptotically unbiased estimator. If we also ensure that  $\lim_{T \rightarrow \infty} (M/T) = 0$ , that is if  $M$  increases slow enough compared to  $T$ , then the variance of the estimator will tend to zero as  $T$  increases and a consistent spectral estimator is obtained. Finding an expression that satisfies the conditions is not that hard. As an example consider the definition  $M = T^\alpha$  for some  $0 < \alpha < 1$ . The truncated periodogram (4.6.2) is a special case of a broader class of spectral estimators called *weighted periodograms*. These type of estimators were first suggested by Grenander and Rosenblatt (1953).

$$\hat{S}_{weighted}(\omega) = \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} w(k) \hat{\gamma}_k \exp(-i\omega k) \quad (4.6.3)$$

The function  $w(k)$  attaches a weight to each sample autocovariance. For a truncated periodogram these weights are simply set at

$$w(k) = \begin{cases} 1 & \text{if } 0 \leq |k| < M-1 \\ 0 & \text{if } M-1 \leq |k| \leq T-1 \end{cases} \quad (4.6.4)$$

There is no reason not to apply other weighting functions  $w(k)$  than this special discontinuous format for the truncated periodogram. For example, functions for which the weights gradually decrease may be applied. If the function's decay is "sufficiently fast", (4.6.3) can be a consistent spectral estimator for a wide variety of weighting functions. The function  $w(k)$  is sometimes called a *lag window* because in a way it looks at the data and the sample autocovariances through a kind of time-window. These lag windows also have another representation in the frequency domain which are called *spectral windows*. This terminology was first introduced by Blackman and Tukey (1959). These spectral windows appear by substituting the IDCFT of the periodogram as given by (4.2.10) into the expression for the weighted periodogram. This gives

$$\begin{aligned}\hat{S}_{weighted}(\omega) &= \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} w(k) \left\{ \int_0^{2\pi} P(\theta) \exp(i\theta k) d\theta \right\} \exp(-i\omega k) \\ &= \int_0^{2\pi} P(\theta) \left\{ \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} w(k) \exp(-ik(\omega - \theta)) \right\} d\theta \\ &= \int_0^{2\pi} P(\theta) W(\omega - \theta) d\theta\end{aligned}\tag{4.6.5}$$

where  $W(\theta)$  is

$$W(\theta) = \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} w(k) \exp(-i\theta k)\tag{4.6.6}$$

An expression such as (4.6.5) is called a *convolution*. It writes the weighted periodogram as a weighted integral of the standard periodogram where the weighting function, the spectral window, is the Fourier transform of the lag window  $w(k)$ . In most cases  $W(\theta)$  is a function which is symmetrical and decreasing around  $\theta = 0$ . In a weighted periodogram therefore each value  $P(\omega)$  of the standard periodogram is replaced by a weighted average of this function around  $\theta = \omega$ . Applying a weighting function to the autocovariances through  $w(k)$  is therefore equivalent to *smoothing* the periodogram by replacing each value by a weighted average of the surrounding values of the periodogram. Because of the often wild fluctuations in a standard periodogram (for example the periodograms in Figure 4.4) this kind of smoothing has a certain intuitive appeal. Many different spectral windows have appeared in the literature. Some well known windows are the Barlett window, the Daniell window, the Tukey window, the Tukey-Hanning window, the Parzen window and the Tukey-Priestley window. Most of these windows have in common that they contain one or more parameters which enable the user to make a tradeoff between the bias and the variance of the spectral estimator. In general it is hard, if not impossible, to derive the exact statistical properties of these estimators. The result is that in applied work the choice of a spectral window with some specific parameter settings remains rather subjective.

### 4.6.2 The leakage effect reconsidered

Before we turn to parametric spectral estimators as an alternative for the non-parametric approach, let us take another look at the leakage effect as discussed at the end of section 4.2. Based on the experimental results of Figure 4.2, the leakage effect was introduced as the arising of disturbing side lobes around dominant peaks when calculating a periodogram on finite samples of a, theoretically, infinite stochastic process. Also it was clear that the leakage effect becomes less disturbing as the sample size increases. The weighted periodogram (4.6.5) enables a further analytical insight into the leakage effect. Let's for convenience assume we are dealing with an infinite deterministic process (for example a perfect sinusoid). Then the theoretical periodogram is given by

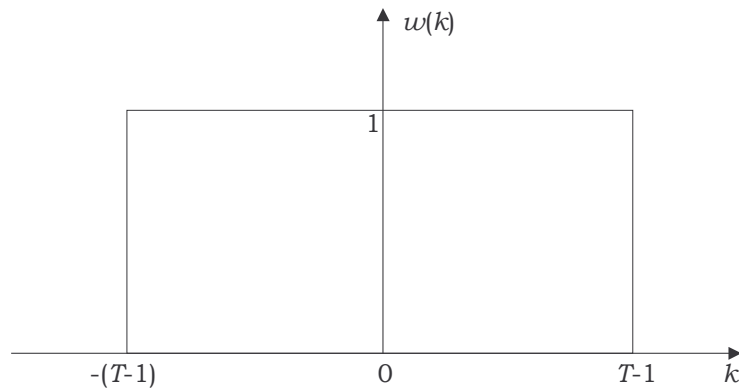
$$P_{theoretical}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{\gamma}_k \exp(-i\omega k) \quad (4.6.7)$$

In case of a perfect sinusoid this periodogram will have a single spike at the corresponding frequency. If, as in practice, we can only observe a finite sample of size  $T$ , then by calculating the periodogram on this finite sample we implicitly apply the discontinuous lag window (4.6.4). To see this note that based on (4.6.5) the empirical periodogram is

$$\begin{aligned} P_{empirical}(\omega) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} w(k) \hat{\gamma}_k \exp(-i\omega k) \\ &= \int_0^{2\pi} P_{theoretical}(\theta) W(\omega - \theta) d\theta \end{aligned} \quad (4.6.8)$$

Figure 4.10 shows the discontinuous weighting function  $w(k)$ .

Figure 4.10 Weighting function (4.6.4).

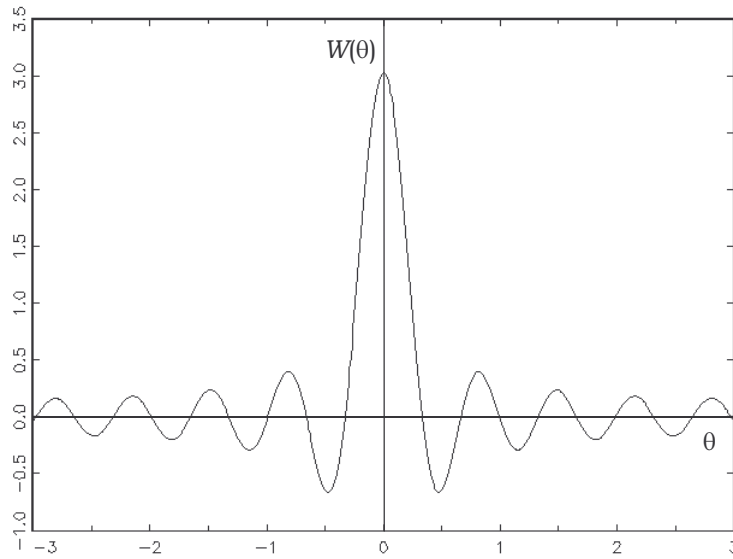


Because this lag window is symmetrical, its Fourier transform is real valued. The spectral window applied in (4.6.8) is therefore equal to

$$\begin{aligned}
 W(\theta) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} w(k) \exp(-i\theta k) \\
 &= \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \cos(\theta k) \\
 &= \frac{1}{2\pi} \left\{ \frac{\sin[(T-1/2)\theta]}{\sin(\theta/2)} \right\}
 \end{aligned} \tag{4.6.9}$$

This function is known as the *Dirichlet kernel*. For  $T=10$  it has the typical shape shown in Figure 4.11.

Figure 4.11 Dirichlet kernel (4.6.9) for  $T=10$ .



A general property of this spectral window is that at  $\theta=0$  it has a main peak (the main lobe) and adjacent, both on the left and right hand side, are a number of smaller peaks (the side lobes) of which the height decreases as they are further away from the center. This kind of behavior is typical for a Fourier transform of a function with sharp, discontinuous, edges as is the case for the weighting function shown in Figure 4.10. This behavior is also known as the *Gibb's phenomenon*. The shape of the Dirichlet kernel now shows us why the empirical periodogram calculated on a finite sample of data will always be a disturbed version of the theoretical periodogram defined on an infinite sample size. A sharp peak at some frequency in the theoretical periodogram will be spread out by the kernel onto surrounding frequencies in a dampened oscillating fashion causing the side lobes. As  $T$  increases, both the height of the main peak and the “frequency” of the side lobes increase, thereby causing the leakage effect to decrease as we already saw in Figure 4.2.

### 4.6.3 Parametric estimators

The basic approach of the traditional non-parametric spectral estimators as discussed in section 4.6.1 is that the spectrum of a stochastic process is estimated through its sample counterpart, the periodogram. By using the periodogram as an estimator in fact three types of errors are made when it is compared to the theoretical spectrum.

#### 1. *Inconsistent estimator*

The first is that the periodogram fails to be a consistent estimator in the sense that the estimator does not converge to the true spectrum as the sample size increases. We already saw that this inconsistency can be repaired by applying appropriate spectral windows.

#### 2. *Sample autocovariances*

The second error is that instead of the theoretical autocovariances, the empirical sample autocovariances are used. Although these are asymptotically unbiased estimators, especially in small samples these estimates can deviate strongly from the true values. As noted before, the number of actual observations available to estimate an autocovariance of some order decreases as the order approaches the sample size. For the highest order possible only one observation will be available. The bias in estimated high order autocovariances will therefore be larger than for low order autocovariances.

#### 3. *Finite sample*

The third, and in practice most important, error made by using the periodogram as an estimator for a spectral density is that because of the finite sample only a limited number of autocovariances are fed into the formula for the periodogram while the theoretical spectrum contains *all* (infinite) autocovariances of the process. In the previous section we saw that the disturbing leakage effect is a direct consequence of the finite sample size. Although the leakage effect can be reduced by applying spectral windows this will always come at the expense of a lower *resolution* of the estimated spectrum. By resolution is meant the extent to which a spectral estimator is able to differentiate between separate, possibly adjacent, peaks in the theoretical spectrum. A lower resolution is therefore equivalent to a larger bias in the estimate. Applying spectral windows to reduce the leakage effect comes down to averaging the periodogram over adjacent frequencies, as is illustrated by equation (4.6.8) for the weighted periodogram. It is unavoidable that this averaging of the periodogram causes adjacent peak in the spectrum to be melted together which means a possible loss of valuable information about the dynamics of the process under investigation. Especially in macroeconomics where often only limited samples of data are available, this low resolution of the conventional non-parametric estimators is very relevant. In case of small samples there will be a lot of leakage and hence a lot smoothing required which leads to a loss of information. Hence it is also very relevant with respect to the objectives of the research presented here.

Finally note that the small sample problem leads to larger problems in the frequency domain than it does in the time domain. If in the time domain for example an autocovariance of a high order cannot be estimated (well) because of a lack of data then this has no effect on the quality of the estimates of other, lower order, autocovariances. In the frequency domain however missing of a part of the

autocovariances effects the quality of the entire estimated spectrum (i.e. at all frequencies).

By estimating the spectrum through a cut off sequence of sample autocovariances, implicitly all higher order autocovariances are assumed to be zero. This also holds if a spectral window is applied although in that case additionally the sample autocovariances that are available are modified by the weighting function. In fact as a consequence of cutting off the autocovariances the spectrum of an entirely different, to some extent even arbitrary, stochastic process is estimated. Parametric spectral estimators try to circumvent this problem by first estimating the parameters of some stochastic process on the available sample. Once such a stochastic process is “known” its autocovariances can be calculated for any order up to infinity. These autocovariances can then be used to calculate the periodogram of the process as an estimate of the spectral density function. In a way such a model “extrapolates” the autocovariances observed within the sample into autocovariances for orders “outside” the sample.

Of course this approach requires first the postulation of a parameterized model of the stochastic process under investigation. Second, it requires the actual estimation of the parameters of such a model. The specification of a model can be seen as a disadvantage of the parametric approach. After all, how can we be sure that the model gives an adequate description of the true stochastic process? A partial solution to this problem is by using a model that at least can give an adequate *approximation* of the stochastic process. An obvious candidate class of models is the class of autoregressive models from the previous chapter. In principle autoregressive models can describe *any* stationary stochastic process. Even with such a general class of models in some cases it would be better to use other types of models. To adequately describe for example an MA(1) process, a very high order AR( $p$ ) model is required which may consequently produce very poor estimated model parameters and thereby also a poor spectral estimate. For non-parametric spectral estimators such a choice of model is not needed causing them, in the case of sufficiently long samples, to be the most robust approach. In the case of small samples however these techniques can produce very poor spectral estimates for reasons elaborately discussed before. Also one should not forget that also the non-parametric approach requires making a number of rather subjective choices about the spectral window and the parameter settings to use. Choosing the “right” spectral window in a sense also (just as the parametric approach) requires some knowledge of the stochastic process under investigation.

Choosing between the non-parametric and the parametric approach of spectral estimation in general requires a trade off between the various pro’s and con’s of the two approaches, given the specific problem at hand. In the specific case of small sample sizes, as is typically the case in empirical macroeconomic applications, the parametric approach is to be preferred over the non-parametric approach. The choice for this class of parametric spectral estimators is supported by a study of several spectral density estimators by Birgean and Kilian (1999) who show that the autoregressive approach produces the most accurate estimates of spectral densities. Also Berkowitz et al. (2000)<sup>29</sup> find that autoregressive spectral estimators in general provide very accurate results when compared to other types of spectral estimators.

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<sup>29</sup> Thanks go to André Lucas at the Free University of Amsterdam for pointing out this article with very similar results as those found here.



The example of using autoregressive models for parametric spectral estimation is called *autoregressive spectral analysis*. In order to obtain a spectral estimator, the following three steps have to be taken.

1. Select the order  $p$  of the model.
2. Estimate the parameters of an  $AR(p)$  model on the available data.
3. Calculate the spectrum of the estimated model.

For the first step the various order selection criteria from section 3.5 can be used. Also one can look at how well the model autocovariances fit the available sample autocovariances to find out which order in a sense best extrapolates the sample autocovariances into infinity. For the estimation of the parameters in the second step the various estimation techniques from section 3.4 can be used. The final calculation of the spectra is easily done by applying the expressions (4.5.12) and (4.5.14) for respectively univariate (AR) models and multivariate (VAR) models. Although these principles are rather straightforward, they do not give any information on *which* order selection criterion to use or on *which* estimation technique to apply. To try and answer these questions, an extensive Monte Carlo experiment was performed of which the results are presented in section 4.7. First however, a more formal justification is given of why the class of autoregressive models should be used in the first place.

#### 4.6.4 Maximum Entropy

Although Berk (1974) and Carmichael (1976) both prove in a different way that autoregressive spectral analysis leads to consistent estimators of the spectra of stochastic processes, a comment sometimes heard against using autoregressive models for estimating spectral density functions is that it lacks any theoretical justification. Such comments are however unfounded since such a justification does actually exist. Note that an infinite number of spectra can be consistent with some finite set of autocovariances. In the traditional non-parametric approach one implicitly selects the spectrum of the process which has autocovariances of zero for orders higher than the last autocovariance available (i.e. the spectrum of a truncated series of autocovariances). In the parametric approach one estimates some model that extrapolates the available autocovariances into infinity following the (stochastic) structure of the estimated model. At first sight there may seem no reason to prefer the one approach over the other. Both seem equally arbitrary. The information theoretical concept of *Maximum Entropy* however formally tells us that the best way to extrapolate the autocovariances is by an autoregressive model (i.e. by following the parametric approach).

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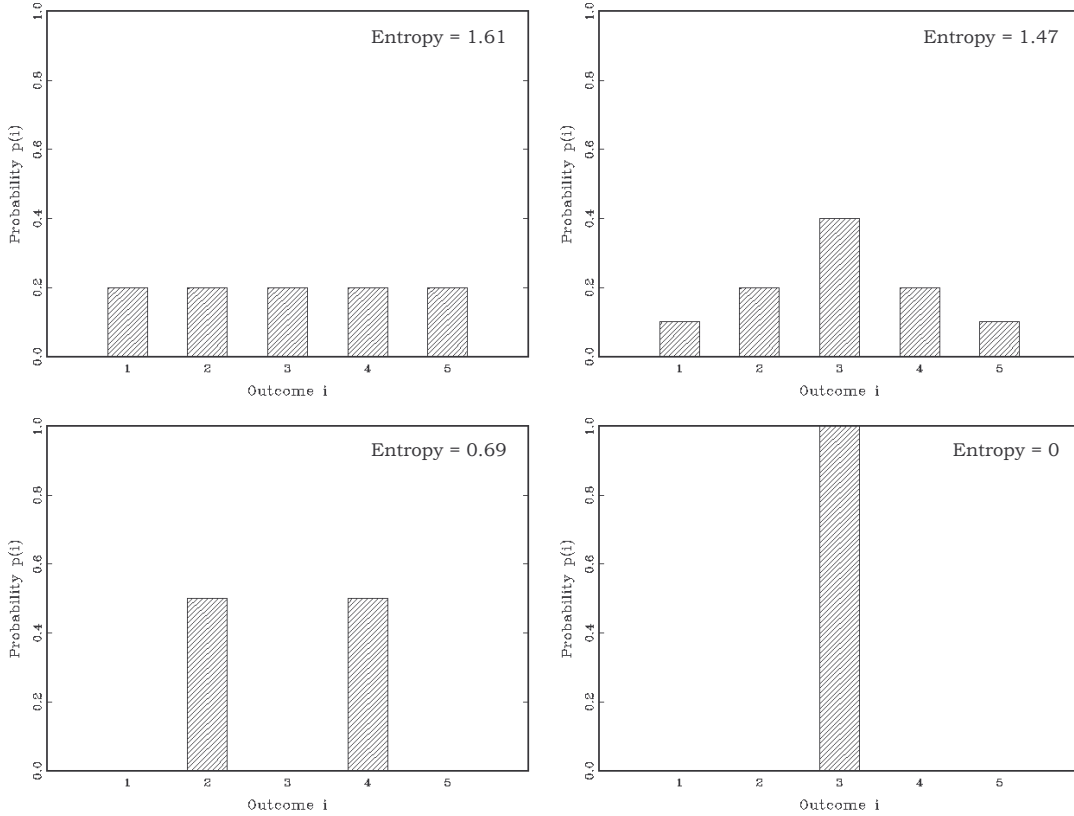
Let's first define what is meant by entropy in an intuitive way. For a more detailed discussion of the entropy concept the reader is referred to for example Akaike (1977). Shannon (1948) introduced the concept of *information* in statistics. Consider a random event  $x$  with  $m$  possible outcomes  $\{x_i, i=1, \dots, m\}$  each with a probability of  $p(x_i)$  of occurring which sum to a total probability of one. Each of the possible outcomes is said to contain a *self-information* of

$$I(x_i) = -\ln\left(\frac{1}{p(x_i)}\right) = \ln(p(x_i)) \quad (4.6.10)$$

The intuition behind this expression is as follows. Suppose that all outcomes are equally likely in a conventional probabilistic sense, such that all probabilities  $p(x_i)$  are equal to  $1/m$ . In that case one could say that up front there is no specific information available about the outcome of the random event. Now suppose the other extreme situation in which  $p(x_i)=1$  for one specific value of  $i$ , say  $j$ , and zero for all other possible outcomes  $i \neq j$ . In that case, there is very much information available about the outcome of the random event. After all, we know for sure that the outcome will be  $x_j$ . One could therefore say that the amount of information of a possible outcome is positively related to the probability of the relevant outcome. The relation (4.6.10) has this property. The *entropy* now (inversely) measures the amount of *non-information* in a probability distribution as

$$E(x) = -\sum_{i=1}^m p(x_i)I(x_i) = -\sum_{i=1}^m p(x_i)\ln(p(x_i)) \quad (4.6.11)$$

That is, the entropy of a probability distribution is the negative of the expected value of the information calculated on all possible outcomes of the random variable. The entropy is thereby a measure of “not knowing” the outcome of a random event. The higher the value of the entropy, the bigger the uncertainty about the outcome of the event. The entropy of a (discrete) probability distribution has its maximum value when all outcomes of the random event have an equal probability. As the probability distribution has a more specific format, the entropy decreases. This is illustrated in Figure 4.12 for several probability distributions for  $m=5$ .

Figure 4.12 Entropy for four probability distributions with  $m=5$  possible outcomes.


Similar to the discrete case, for a continuous random variable with a probability density function  $p(x)$  the entropy is defined as

$$E(x) = - \int_{-\infty}^{\infty} p(x) \ln(p(x)) dx \quad (4.6.12)$$

For an  $n$  dimensional multivariate Normal distribution with expectation zero and covariance matrix  $\Sigma_n$  according to Smylie et al. (1973) the entropy is

$$E(x) = \frac{1}{2} \ln(|\Sigma_n|) \quad (4.6.13)$$

As the dimension of  $n$  increases this (multivariate) measure of entropy may diverge. In that case the *entropy rate*

$$e(x) = \lim_{n \rightarrow \infty} \frac{E(x)}{n} \quad (4.6.14)$$

is used. The entropy rate is the average entropy per variable.

Smylie et al. (1973) furthermore show that the entropy rate of stationary normally distributed (time series) processes, apart from a constant term, is fully determined by the spectrum  $S(\omega)$  of the process.

$$e(x) = \frac{1}{2} \ln(\pi) + \frac{1}{2\pi} \int_0^{2\pi} \ln(S(\omega)) d\omega \quad (4.6.15)$$

Now we know the interpretation of the entropy and the dependence of the entropy on the spectral density of a stochastic process we are ready to understand the link between autoregressive spectral analysis and the entropy concept. The fundamental idea behind maximum entropy comes from Burg (1967, 1972). This idea is to select from all possible spectra that are consistent with the available information, represented by a (finite) sequence of (sample) autocovariances, the spectrum which contains the *least* additional information as the best spectral estimate. All additional information on top of the information from the available sample is not supported by the data and should therefore be minimized. This is consistent with choosing the spectrum with the *maximum* entropy from all spectra that are consistent with the observed sample of autocovariances. These basic ideas of *Maximum Entropy spectral estimation* lead to the following optimization problem

$$\max_{S(\omega)} e(x) = \int_0^{2\pi} \ln(S(\omega)) d\omega \quad (4.6.16)$$

with respect to the restrictions

$$\int_0^{2\pi} S(\omega) \exp(i\omega k) d\omega = \gamma_k \quad \text{for } k = 0, \dots, p \quad (4.6.17)$$

where  $p$  is the maximum order of the autocovariance with which consistency is required. Miraculously enough the solution to this problem turns out to be

$$S(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{\left| 1 - \sum_{k=1}^p \beta_k \exp(-i\omega k) \right|^2} \quad (4.6.18)$$

where the parameters  $\beta_1, \dots, \beta_p$  and  $\sigma^2$  are determined by the equations

$$\begin{bmatrix} \gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_p \\ \gamma_1 & -\gamma_0 & -\gamma_1 & \cdots & -\gamma_{p-1} \\ \gamma_2 & -\gamma_1 & -\gamma_0 & \cdots & -\gamma_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_p & -\gamma_{p-1} & -\gamma_{p-2} & \cdots & -\gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4.6.19)$$

The “miracle” lies in the fact that the solution (4.6.18) is exactly the spectrum of an AR( $p$ ) process as given by (4.5.11) while the parameters as determined by (4.6.19) are

exactly the solution of the Yule-Walker equations (3.2.18) and (3.2.19) assuming the first  $p$  autocovariances to be known. Proofs of this solution can be found in for example Priestley (1981), Burg (1975), Haykin and Kessler (1983), Smylie et al. (1973) and Van den Bos (1971). This optimization problem and its solution clearly show the link between Maximum Entropy and autoregressive spectral estimation. The best way to estimate a spectrum that is consistent with the observed sample information while adding the least additional information is by estimating an autoregressive model and using the spectrum of this model as the spectral estimate. Furthermore, Shannon and Weaver (1949) show that given a number of autocovariances, a Normally distributed process has the maximum entropy. So in total, the Maximum Entropy concept comes down to estimating a Normally distributed autoregressive model. It is important to note that the Maximum Entropy concept says nothing about which order  $p$  to use or how to estimate the parameters of the autoregressive model once the order is known. Maximum entropy is a concept for deriving probability models and is not concerned whatsoever with the statistics of estimating models on samples of data. Note that strictly speaking one cannot speak of Maximum Entropy spectral analysis if the process studied is not Normally distributed or if the autocovariances are replaced by sample autocovariances (as will be the case in practical applications). The next section describes the results of an extensive Monte Carlo experiment to find out which model order and which estimation procedure can best be used in autoregressive spectral estimators.

## 4.7 Monte Carlo experiment

For the autoregressive approach to spectral estimation as described in the previous section to be applied, first the order of the model has to be decided on and second the model parameters have to be estimated. For the estimation one of the techniques described in section 3.4 can be used. For determining the order of the model one of the order selection criteria from section 3.5 can be used. However, also the Maximum Entropy concept, which provides the theoretical foundations for the use of autoregressive models to estimate spectral densities, says nothing about *which* order selection criterion or *which* estimation procedure to use. In both the empirical and theoretical literature several properties of the various order selection and estimation techniques are available. For three reasons these results are not sufficient to decide on which techniques are the most appropriate for the purpose of spectral estimation in small (macroeconomic) samples. The first reason is that a lot of the theoretical properties of both estimators and order selection criteria are derived under the assumption of infinite samples and therefore only hold asymptotically or as an approximation. Therefore these results need not hold in the case of small samples. The second reason is that most of the results are not specifically obtained for the purpose of estimating spectral densities. If one of the order selection criteria is for example known to overestimate the true model order then this need not to imply that also the estimated spectrum is of poor quality. After all, a model of larger order is a special case of a model of higher order. The third reason is that many of the order selection criteria and estimation techniques are analyzed separately while for the quality of the final spectral estimate the combined effect is of crucial importance. For these three reasons, in addition to what is known in the literature, a rather extensive Monte Carlo experiment is described in this section to find out more about which order selection criterion in combination with which estimation techniques produces

the best quality spectral density estimates. The purpose of the experiment is to assess this quality also with respect to

- (a) the complexity of the spectra to be estimated
- (b) the available sample size and
- (c) the dimension of the process.

An explicit overview of what is known in the literature is not given here. Instead, some of the relevant points are mentioned throughout the text. Also some results are already given in sections 3.4 and 3.5.

#### 4.7.1 Experiment setup

The general setup of the experiment is based on a number of known (V)AR models of different dimensions and orders. For each of these models a thousand simulations of different sample sizes were made. For each of these simulations a VAR model was estimated using various combinations of order selection criteria and estimation techniques. By comparing the spectra of these estimated models with the spectra of the true model we are able to judge the quality of the estimated spectra and thereby of the order selection criteria and estimation techniques used. The fact that all of the model spectra used belong to the class of VAR models may seem a limitation of the experiment. This however need not be the case since each (second order) stationary stochastic process can be adequately described by a VAR model. If a spectrum is in reality best represented by for example a moving average (MA) model then it is assumed that this model can be approximated well by a VAR model of limited order.

##### *Models*

The order  $p$  and dimensions  $n$  of the in total eighteen VAR models that are used in the experiment are shown in Table 4.2. For models of higher dimensions, lower orders are used because higher dimensional models of lower orders are able to describe the same type of spectral densities as lower dimensional models of higher orders. For this also see the remark in section 3.3.2 about the equivalence between an  $n$  dimensional VAR( $p$ ) model and ARMA( $np$ ,  $(n-1)p$ ) models for the individual variables.

Table 4.2 Dimensions  $n$  and orders  $p$  of VAR models used in the experiment.

$n$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
1	✓	✓	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓				
3	✓	✓	✓					
4	✓	✓						
5	✓							

The actual parameters of the models are given in Appendix C. The spectral densities are not shown there but if desired these can be calculated using the formulas of the spectral densities of VAR models given in the previous chapter. In general, models were used for which the spectra look like what one would expect for macroeconomic variables. Given the literature on the various types of (business) cycles as discussed in the second chapter, this means that the spectra typically contain a number of peaks at various frequencies. Also the peaks are not that sharp because too perfect periodic behavior is not to be expected. This means that the VAR models typically have a number of complex eigenvalues with a modulus not too close to one. Besides

these complex eigenvalues also real eigenvalues are present in the models. Because the expected value of a process is irrelevant for the dynamics of the process, and hence for its spectrum, all simulated processes have an expected value of zero.

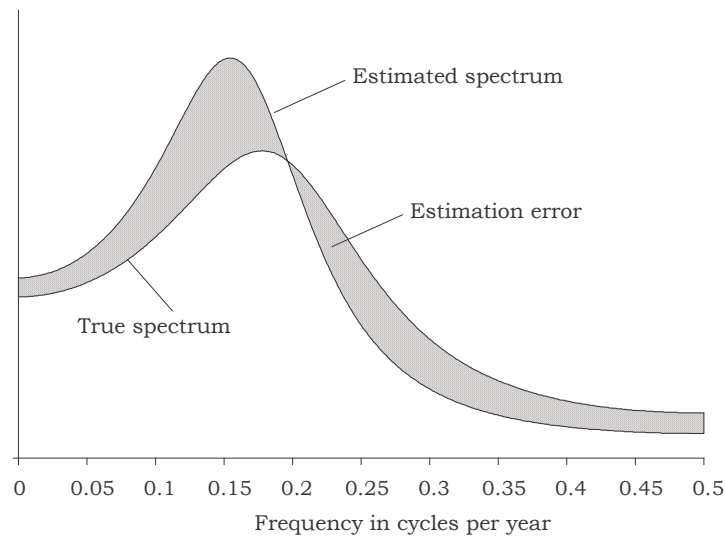
### Spectral error

To assess the quality of the spectral estimates for some estimation technique and order selection criterion, we proceeded as follows. First, for each of the thousand simulations a *relative spectral error* was calculated, as

$$S_{error} = \frac{\sum_{i=1}^n \left\{ \int_0^{\pi} |\hat{S}_i(\omega) - S_i(\omega)| d\omega \right\}}{\sum_{i=1}^n \left\{ \int_0^{\pi} S_i(\omega) d\omega \right\}} \quad (4.7.1)$$

Here  $n$  is the dimension of the process and  $S_i(\omega)$  and  $\hat{S}_i(\omega)$  are respectively the non-normalized auto-spectrum of the  $i$ -th variable from the known underlying model and its estimated counterpart. Note that since here the non-normalized spectra are used, the denominator in (4.7.1) corresponds to the sum of the model variances of the  $n$  variables. Figure 4.13 clarifies that this relative spectral error is calculated by integrating the absolute value of the difference between the estimated and the true spectral densities and dividing this by the surface below the model spectra (i.e. the sum of the variances) to obtain a relative error. So (4.7.1) in a sense measures the “distance” between the estimated and the model spectra. The smaller this distance the better the estimated spectra correspond to the true model spectra in the frequency domain. In case an estimated model turns out to be non-stationary (which can occur especially in very small samples) no estimated spectrum can be calculated. In that case the spectral error  $S_{error}$  is set at 100%.

Figure 4.13 Calculation of spectral error.



For multivariate spectra, only looking at the auto-spectra is not enough. Also the quality of the estimated cross-spectra needs to be measured. For that purpose the following *relative cross spectral error* was also calculated

$$CS_{error} = \frac{\sum_{i=1}^n \sum_{j=i+1}^n \left\{ \int_0^{\pi} |\hat{C}_{i,j}(\omega) - C_{i,j}(\omega)| d\omega + \int_0^{\pi} |\hat{Q}_{i,j}(\omega) - Q_{i,j}(\omega)| d\omega \right\}}{\sum_{i=1}^n \sum_{j=i+1}^n \left\{ \int_0^{\pi} |C_{i,j}(\omega)| d\omega + \int_0^{\pi} |Q_{i,j}(\omega)| d\omega \right\}} \quad (4.7.2)$$

Here  $C_{ij}$  and  $Q_{ij}$  represent the co-spectrum and the quadrature spectrum between variables  $i$  and  $j$ . This is an analogous definition as for the auto-spectra with the only difference that in the denominator the absolute value of the two spectra are integrated. This is so because both spectra can have positive and negative values.

Having calculated these two spectral errors for each of the thousand simulations, in the second step the 5%, 25%, 50%, 75% and 95% percentiles of the resulting error distribution were calculated and plotted in various box plots. Note that the 50% percentile is the median error. For most of the results these percentiles were calculated for an aggregated distribution constructed from the thousand simulations from all the eighteen models together. Hence the percentiles are calculated on eighteen thousand spectral estimation errors. The interpretation can therefore be as the error distribution of a general stochastic process. The effects of both the order and the dimension of the process are discussed in a separate subsection (4.7.6).

#### Order error

The previous procedure measures the error in the estimated spectra given some previously determined order  $p$ . To also assess the errors in the order selection, along the same lines for each simulation the *order error* was calculated as

$$p_{error} = \hat{p} - p \quad (4.7.3)$$

where  $\hat{p}$  and  $p$  are respectively the estimated and true model order following from some order selection criterion and estimation technique. Note that this is not a relative error as with the spectral errors and also that it is not an absolute value. The latter is because to know whether the true model order is over or underestimated is valuable information that would be lost if the absolute value of the errors would be used. Also for these order errors the percentiles were calculated and analyzed by means of box plots.



*Experimental settings*

The order selection criteria, estimation techniques and sample sizes included in the experiment are given in Table 4.3.

Table 4.3 Experimental settings of Monte Carlo experiment.

<i>Order selection</i>	<i>Estimation</i>	<i>Sample size</i>
FPE	OLS	25
AIC	Yule-Walker*	50
SC	TLS**	100
CAT	Burg**	
HQ	Optimization**	
	Top-Down***	
	Bottom-Up***	

\* Using the biased estimators of the autocovariances

\*\* Only for univariate models

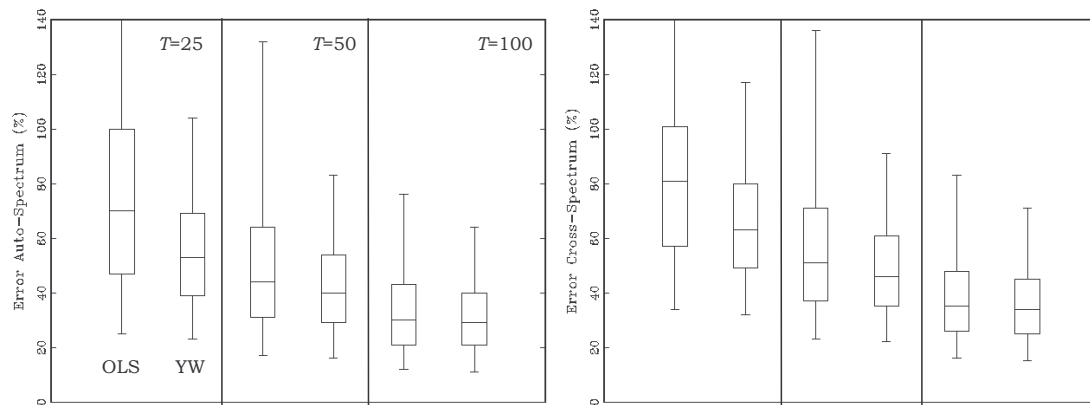
\*\*\* Only for multivariate models

The order selection criteria are all the criteria given in section 3.5. These criteria can equally well be applied to both univariate and multivariate models. Except the straightforward univariate optimization technique which will be explained in section 4.7.5, the estimation techniques are described in section 3.4. The Two sided Least Squares (TLS) and Burg approach can only be applied for univariate models. Although these estimation techniques have been analyzed, the corresponding results are not presented in the following sections. Focussing on the OLS, Yule-Walker (YW) and the various optimization techniques, which all have both univariate and multivariate versions, allows to combine the results for all univariate and multivariate models. The quality of the TLS and Burg estimates turns out to be between the results of the OLS and YW techniques. The sample size  $T$  ranges from 25 to 100 where 25 can be seen as a kind of minimum required sample size, 50 as a median sample size and 100 as a large sample size. Analyzing for example annual data for the post World War II period gives approximately  $T=50$  while a whole century of annual data would mean  $T=100$ .

#### 4.7.2 Estimation techniques

To get as pure information as possible about the effects of the estimation technique, we started by estimating the spectral densities by the OLS and YW technique while assuming that the true order of the model is known. The effects of optimization (estimation) techniques are discussed separately in section 4.7.5. So the effects of having to estimate the order by using some order selection criterion are excluded here. The box plots of the resulting distributions of the relative spectral error for the auto-spectra and the cross-spectra are given in Figure 4.14. In each box plot the lowest point represents the 5% percentile, the bottom, middle and top of the box respectively the 25%, 50% and 75% percentile and the highest point represents the 95% percentile. The box therefore contains the center 50% probability mass of the distribution while the whole vertical distance from the bottom to the top represents the central 90% probability interval.

Figure 4.14 5%, 25%, 50%, 75% and 95% percentiles of distribution of auto-spectral errors (left) and cross-spectral errors (right) assuming the true model order is known for sample sizes  $T=25$ , 50 and 100 using the OLS and YW estimation techniques.



From these results we can see that

- If the order of the autoregressive process is known, the Yule-Walker estimation technique leads to the most efficient spectral estimates, both in terms of the median spectral error and in terms of the size of the 90% probability interval. This holds for all the sample sizes analyzed and both for the auto-spectra and the cross-spectra.
- As the sample size increases the estimators seem to converge to the true spectral densities. Also the differences between the OLS and YW estimator becomes smaller.
- The relative errors in the estimates of the cross-spectra estimates are larger than those in the auto-spectra.

The fact that the Yule-Walker estimation techniques leads to better spectral estimators than the conventional OLS technique may seem rather surprising for two reasons. The first is that the OLS estimator is equal to the Maximum Likelihood estimator and can also be shown to be the best estimator within the class of linear unbiased estimators. That the OLS technique nevertheless performs worse than the YW technique can be explained in several ways. The first is to note that in the case of autoregressive models (i.e. with endogenous regressors) the results mentioned only hold asymptotically and as an approximation. The fact that as the sample size increases the difference between the OLS and YW results becomes smaller illustrates that the dominance of the YW technique, especially in small samples, need not be conflicting with the theory on OLS estimators. A second way to understand the relatively poor performance of the OLS technique is by noting that this technique does not impose a stationarity restriction on the model to be estimated. Especially in small samples, a consequence can be that a model estimated by OLS does not satisfy the stationary restriction and thereby has no spectral density. In section 3.4.4 it is explained that the (biased version of the) YW technique does impose such a restriction which guarantees that the estimated model is always stationary. Especially in small samples such a restriction can be very valuable with respect to the quality of the estimated spectra. As noted before, such a restriction can only

increase the efficiency of the estimators if it is a valid restriction. Since the stationary condition is a basic requirement for any type of spectral analysis or other modeling exercise this should always be the case. If not, first adequate variable transformations have to be applied before spectral analysis can be applied. The statement that imposing a stationarity restriction indeed is what improves the quality of the spectral estimates obtained through the YW technique can be checked by replacing the biased estimators (3.4.38) for the autocovariances by their unbiased counterparts (3.4.37). The results showed that by using the unbiased estimators the quality of the spectral estimates indeed became (much) worse. These results are not shown here.

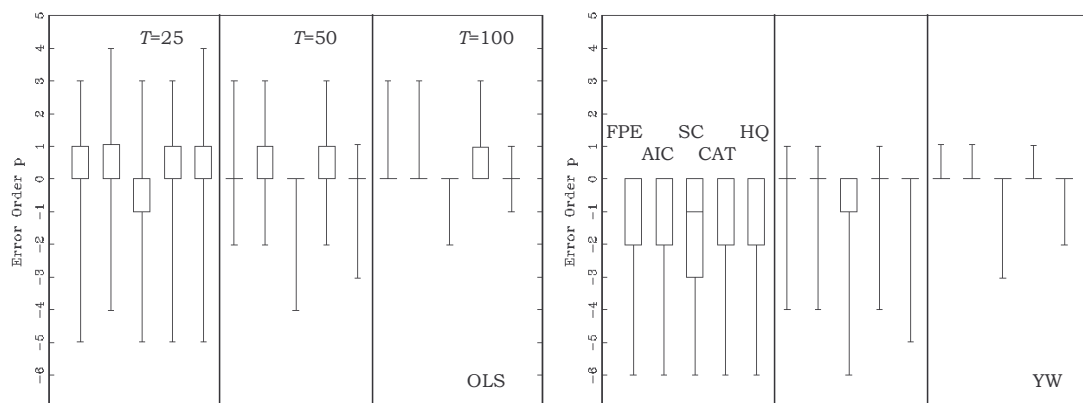
A second reason why the dominance of the YW technique may seem surprising comes from what we know from the theory on non-parametric spectral estimators. Confusion may come from the fact that the (standard) periodogram is known to be a poor spectral estimator while the previous results show that the YW technique is a good estimator while both methods are based on the same sample autocovariances. The main difference between these two methods is however that the YW method in a sense “extrapolates” the available sample autocovariances into infinity following an autoregressive scheme while the periodogram assumes autocovariances “outside” the sample to be equal to zero. If we see the YW technique as the most natural parametric extension of the non-parametric spectral estimators, which on top of that is also justified by the Maximum Entropy concept and imposes a valuable stationarity restriction, then it should no longer come as a surprise that the YW technique leads to the most efficient spectral estimates. In the concluding section 4.7.7 this result will be compared to the results of research by others on this subject.

The fact that the cross-spectra appear to be less efficiently estimated than the auto-spectra may come from the fact that estimating cross-spectra requires information of both the stochastic variables involved while an auto-spectrum only requires observations from a single variable. In the case of two variables there will be more sampling uncertainty than in the case of a single variable which may reduce the estimation efficiency.

#### 4.7.3 Order selection criteria

Of course in practical applications the order of the underlying VAR model will not be known as was assumed in the previous section. The best order to use will most of the times have to be determined by using one of the order selection criteria mentioned in Table 4.3. Given some estimation technique, the value of a criterion is calculated for an estimated model of each order until some maximum value and the order with the minimal value of the criterion is chosen as the optimal one. Note that although perhaps theoretically not completely correct, the various criteria can just as well be used by applying the OLS as well as the YW estimation technique. Therefore all ten combinations of the five order selection criteria and the two estimation techniques were applied. For all univariate models a maximum order of ten was assumed while for the multivariate models a maximum order of the actual order plus two was assumed. The box plots of the resulting distributions of the order error are given in Figure 4.15.

Figure 4.15 5%, 25%, 50%, 75% and 95% percentiles of distribution of order errors for sample sizes  $T=25$ , 50 and 100 using the OLS and YW estimation techniques combined with the various order selection criteria.



From these results we can see that

- In all but one case, the median order error is zero indicating that all order selection criteria are “unbiased”<sup>30</sup> estimates of the model order.
- Especially in small samples, using the Yule-Walker estimation technique leads to slightly more efficient (lower variance) order estimates than using the OLS technique, for each of the order selection criteria.
- Also especially in small samples however, using the Yule-Walker method results in a rather strong tendency to underestimate the model order while using the OLS method may just as well lead to underestimating or overestimating the model order.
- Regardless of the estimation technique, the SC and HQ criterion show the tendency to underestimate the model order while the other criteria FPE, AIC and CAT show the tendency to overestimate the model order.

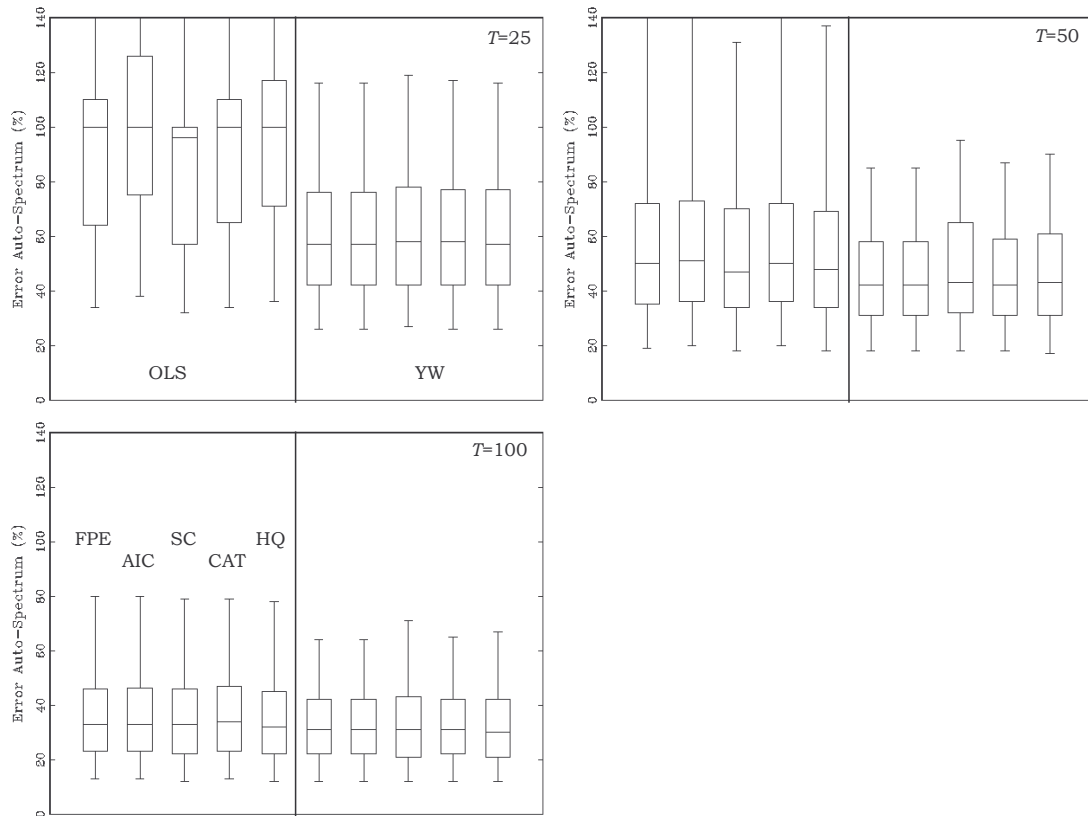
That fact that using the YW technique also leads to slightly better estimates of the model order probably comes from the fact that if the models underlying the order selection process are better estimated, also the estimated order derived from these estimates will be better. That the SC and HQ criterion underestimate the model order while the other criteria overestimate the model order is already known from the literature.

<sup>30</sup> This is not exactly true because for an (asymptotically) unbiased estimator the expected value, instead of the median, of the estimator distribution has to be equal to the true (theoretical) parameter value.

#### 4.7.4 Combinations

The estimate of the order itself as presented in the previous section is by itself not very important for the current applications. What really matters here is what effect the order coming from some order selection criterion has on the quality of the estimated spectrum. That is, it is the *combination* of an order selection criterion and an estimation technique that determines the final quality of a spectral estimate. Figure 4.16 therefore shows the box plots of the distributions of the relative spectral error for the auto-spectra if a model of the order as indicated by the order selection criteria is estimated instead of the true model order. The results for the cross-spectra are not shown since these give similar information. Also, just as in section 4.7.2, the cross-spectra are less efficiently estimated than the auto-spectra.

Figure 4.16 5%, 25%, 50%, 75% and 95% percentiles of distribution of auto-spectral errors following the model order as indicated by various order selection criteria for sample sizes  $T=25, 50$  and  $100$  using the OLS and YW estimation techniques.



In addition to the findings in the previous sections, from these results we can see that

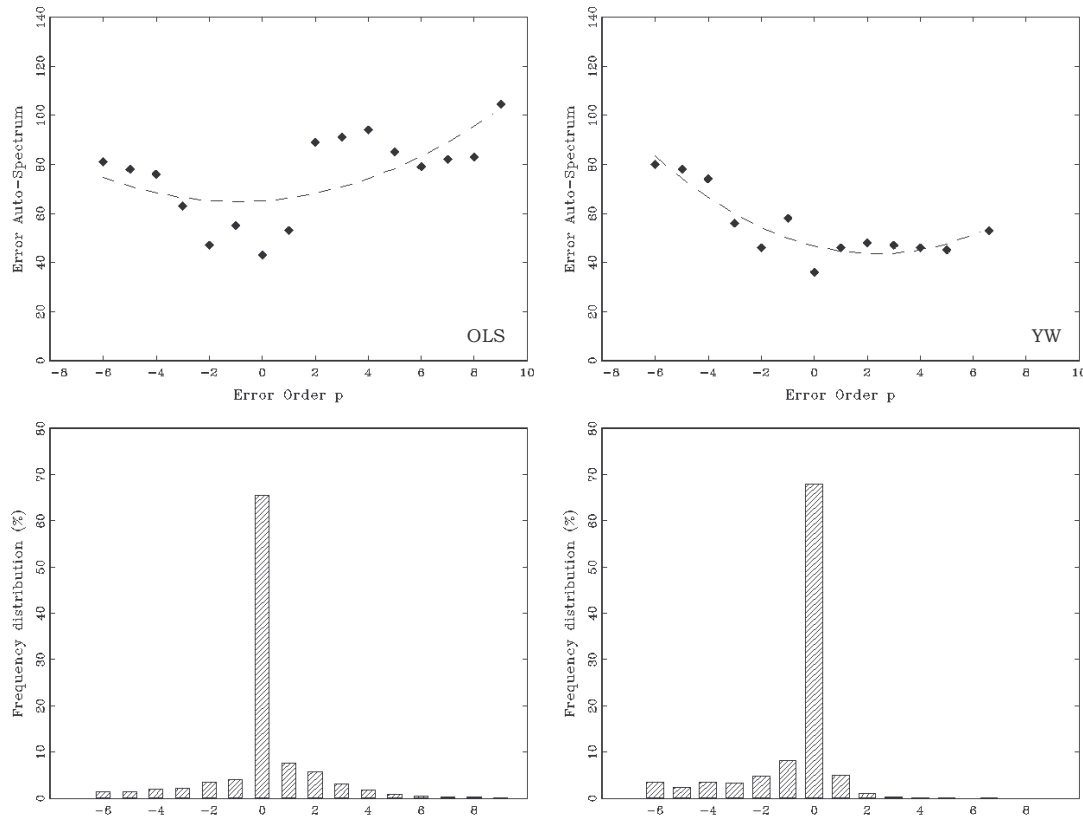
- If besides the model parameters also the model order has to be estimated, this leads to less efficient spectral estimates than is the case when the model order is assumed to be known (compare Figures 4.16 and 4.14).
- Regardless of the order selection criterion applied, the Yule-Walker estimation technique leads to the most efficient spectral estimates, especially in small samples.

- Using the SC combined with the OLS technique leads to more efficient spectral estimates than using the other criteria, especially in small samples. Using the SC combined with the YW technique however leads to slightly less efficient spectral estimates.

The first of these observations should come as no surprise. Also having to estimate the model order comes down to decreasing the information we have about the spectrum and thereby reduces the quality of the estimates. The final observation shows that choosing an order selection criterion should not be seen separately from choosing an estimation technique. In case of the OLS estimator, the SC should be preferred probably because the in general lower model order reduces the probability of (near) non-stationary models. In the OLS case, the error because of underestimating the model order is apparently compensated by the positive effects of a smaller number of parameters to be estimated. In case of the YW estimation technique, underestimating the model order however seems worse than overestimating it which renders the SC less well suited in that case.

To further explore the link between the error in the estimated order and the error in the final estimate of the spectral density take a look at Figure 4.17. In this figure the results from the thousand simulations of all eighteen models, all five order selection criteria and all three sample sizes, are combined. Since this yields 270.000 observations of order errors and spectral errors, the results were first ordered with respect to the order error before calculating the median (auto) spectral error for each possible (integer) order error. Some order errors are aggregated to ensure a minimum of one hundred observations for the calculation of the medians. The calculation of the medians also reduces the sensitivity for extreme observations. For both estimation techniques Figure 4.17 shows the frequency distributions of the order error and a scatter plot of the resulting order error versus the (median) spectral error.

Figure 4.17 Error in estimated order versus median error in estimated auto-spectrum and frequency distributions of errors in estimated orders for the OLS (left) and YW estimation techniques (right). The dashed line in the top pictures is an estimated second order polynomial.



From these additional results we can see that

- The majority (about 70%) of the order errors are zero. In the case of OLS estimation the frequency distribution of the order error is symmetrical around this median value while in the case of YW estimation the distribution is skewed to the left.
- The minimum median spectral error occurs as expected at an order error of zero. For both estimation techniques the spectral error increases as the order is under or overestimated. However in case of YW estimation, the spectral error increases less rapidly as the order is overestimated.

The median order error of zero reflects that all order selection criteria are unbiased estimators of the true order while the skewed frequency distribution for the YW estimator is consistent with its tendency to underestimate the model order as already shown by the third finding in section 4.7.3. These two results are not new. What is new is that apparently overestimating the model order is less harmful in the case of YW estimation than it is in the case of OLS estimation. That in both cases underestimating the order increases the spectral error is logical because a low order can be seen as an invalid model restriction which can never be corrected for by the subsequent estimation technique. An overestimation of the model order however can still produce good spectral estimates because the true model is merely a special version of such a high order model. A high order model however does require the

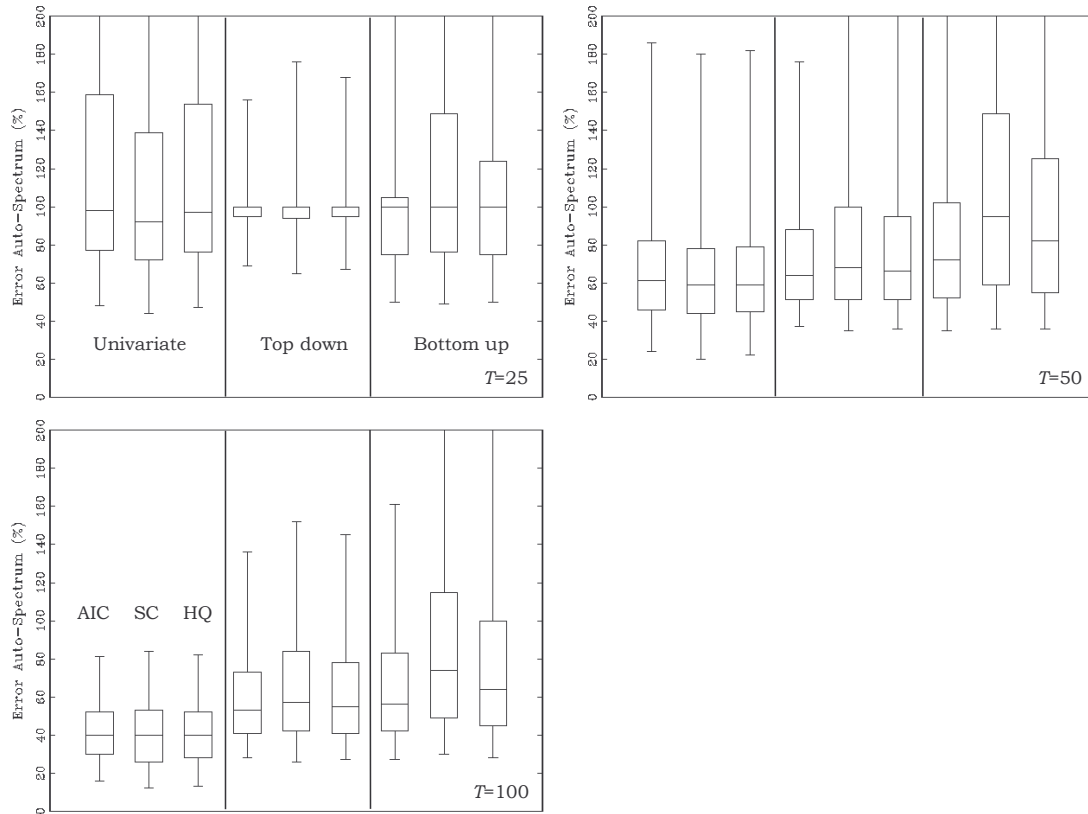


estimation of more model parameters. In the case of OLS this apparently leads to an increase in the spectral error because of the poorer quality of the estimated parameters. The quality of YW estimates decreases less rapidly, probably because of the incorporated valuable stationarity restriction, also when very high order models are estimated.

#### 4.7.5 Optimization techniques

When combining the order selection criteria and the estimation techniques as in the previous section, every time all autoregressive parameters up to the selected order were estimated. An alternative approach may be to use the order selection criteria to find, given some maximum order, the optimal set of parameter restrictions that minimizes the selected criterion. Contrary to the previous models, this may also lead to models in which lower order parameters are set to zero while one or more of the higher order parameters are still estimated. Besides the value of some order selection criterion also for example the  $t$ -values (i.e. the statistical significance) of the estimated parameters serve as a guide for deleting parameters from an autoregressive model. Here only selection criteria based optimization techniques are analyzed. For univariate models it is still possible to find the optimal set of parameters to be estimated by simply trying all parameter combinations and selecting the one that minimized the chosen criterion. For multivariate models this is practically no longer feasible because of the high computational effort involved. An alternative are the Bottom-Up and Top-Down optimization techniques as described in section 3.4.3. The results of these three optimization techniques (including the “absolute” optimization for the univariate models) are shown in Figure 4.18. Note that estimating such subset VAR models is not possible using the YW estimation technique. Therefore the following results only use (restricted) OLS estimation. Also note that the FPE and CAT criterion are not used here because for these criteria it is not clear how they should be calculated for subset VAR models.

Figure 4.18 5%, 25%, 50%, 75% and 95% percentiles of distribution of auto-spectral errors using parameter optimization techniques based on various order selection criteria for sample sizes  $T=25$ , 50 and 100 using the OLS estimation technique.



From these results we can see that

- Regardless of the order selection criterion the parameter optimization techniques lead to less efficient spectral estimates than using a order selection criterion just to find the optimal order and estimate all autoregressive parameters up to that order (compare Figures 4.18 and 4.16).
- Of the three optimization techniques the univariate method of finding the true optimum is (of course) the most efficient. It is followed by the multivariate Top-Down technique while the Bottom-Up technique produces the least efficient spectral estimates. This ordering is independent of the order selection criterion used for the optimization.
- In case of the univariate optimization technique the SC produces the most efficient estimates followed by the other two criteria while in the case of the multivariate techniques it is just the other way around.

Contrary to what one might have expected, the optimization techniques lead to less efficient spectral estimates. This holds even though the theoretical models used actually contain parameter values of zero or close to zero as can be seen in Appendix C. A possible explanation is that a possible efficiency gain because of a smaller number of parameters to be estimated is canceled if in many cases the procedures lead to invalid parameter restrictions. Just as was the case when underestimating

the model order, such restrictions can not be compensated for in the final estimation of the model.

The fact that the univariate method performs best is probably because it is guaranteed to find the model with the lowest value of the selection criterion because it simply evaluates all possibilities. The Top-Down and Bottom-Up techniques are procedures that try to find such an optimal model without the guarantee of actually finding it. The Bottom-Up technique performs worse than the Top-Down technique because by construction it more often underestimates the order, which as we know may be more harmful than overestimating the order. This is also confirmed by the effects of the order selection criteria. As we know the SC and HQ criteria have the tendency to underestimate the order while the AIC has the tendency to overestimate the order. Hence using the AIC in the multivariate optimization techniques leads to more efficient spectral estimates.

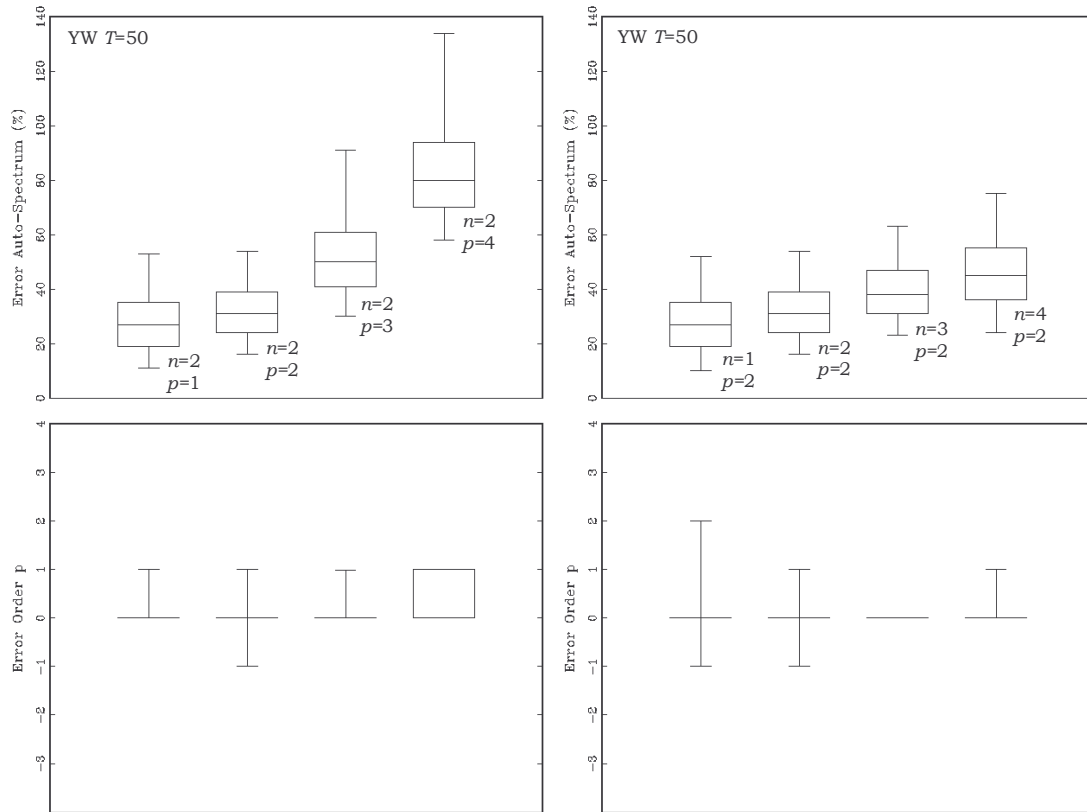
#### 4.7.6 Order and dimension

In the previous section all results were aggregated over all eighteen models of different orders and different dimensions. In order to make the effects of the order and the dimension explicit, separate percentile box plots are presented for the models of dimension  $n=2$  for order  $p=1,2,3$  and 4 on the one hand and for the models of order  $p=2$  and dimensions  $n=1,2,3$  and 4 on the other. Here only the results for the YW estimation technique and a sample size of  $T=50$  are shown while it is assumed that the model order is known and does not have to be estimated. From Figure 4.19 we can see that

- The consequence of both a higher order and a higher dimension is that the quality of the spectral estimates decreases. Furthermore, this effect is stronger when the order is increased than when the dimension is increased.
- It seems that as the order increases the quality of the order estimate decreases while as the dimension increases for a given order, the quality of the order estimate increases.

Models of higher order and higher dimension in general mean more complex spectral densities which obviously makes them more difficult to estimate. That a higher order leads to a faster increase in the error than a higher dimension is rather surprising if we realize that the four subsequent models in each of the pictures contain the same number of roots. After all, the number of roots (or eigenvalues) are equal to  $n \times p$  which are the same in both cases. The fact that the number of roots are equal implies that also the complexity of the spectral densities involved is comparable. A reason for the better estimates in case of higher dimensional models with the same number of roots could be that a higher dimensional model with the same sample size simply means that more information is available about the process and its roots. Note that the extra information should come from the dependencies between the various variables because for a single time series of course the number of sample points does not change. That increasing the dimension leads to a better estimate of the order is probably because of the same reason. However more research on this topic is required.

Figure 4.19 5%, 25%, 50%, 75% and 95% percentiles of distribution of auto-spectral and order errors for dimension  $n=2$  and orders  $p=1,2,3$  and 4 (left) and order  $p=2$  and dimensions  $n=1,2,3$  and 4 (right) assuming the model order is known for sample size  $T=50$  while using the YW estimation technique.



#### 4.7.7 Main findings and discussion

The purpose of the Monte Carlo experiment in this section was to investigate the empirical properties of the various estimation methods and order selection criteria with respect to the quality of the spectral estimates obtained from autoregressive models as indicated by the Maximum Entropy concept. The properties of the estimators in relation to the complexity of the spectra, the available sample size and the dimension of the processes estimated have been analyzed. The main findings with respect to the estimation techniques are the following.

1. The Yule-Walker estimation method leads to the most efficient spectral estimates, especially in small samples. This holds both in case the model order is known and in case the model order has to be estimated with help from an order selection criterion. The reason for this superiority of the Yule-Walker method in the case of spectral estimation lies in the fact that this method incorporates a (valid) stationarity restriction which increases the efficiency of the spectral estimator.
2. The differences between the various estimation methods become smaller as the sample size increases. That is, in infinite samples all estimation techniques yield similar results.

3. Cross-spectra are more difficult to estimate than auto-spectra which could indicate that the dependencies between different variables are more difficult to estimate than the dynamics of a single variable.

The main findings with respect to the order selection criteria are the following.

4. When combined with the Yule-Walker technique, all order selection criteria show a strong tendency to underestimate the model order, especially in small samples. Furthermore, for the Yule-Walker estimation technique, overestimating the order is much less harmful than underestimating it.
5. Although all order selection criteria are “unbiased” estimators of the model order in the sense of the median order error, the SC and HQ criterion have the tendency to underestimate the model order while the FPE, AIC and CAT criteria have the tendency to overestimate it.

The main finding with respect to the complexity of the spectra is the following.

6. For estimating spectra of a certain degree of complexity in terms of the number of roots (or eigenvalues) it seems better to use a low order model of a high dimension than a high order model of low dimension.

Based on these findings and the fact that for macroeconomic time series in general only small samples are available, we chose to apply the Yule-Walker estimation technique combined with the AIC order selection criterion in the autoregressive approach to spectral estimation. Next, a short discussion of these choices is given.

#### *Order selection criterion*

The AIC is the most well known of the three criteria that have a tendency to overestimate the model order. However, also this criterion can (strongly) underestimate the model order in small samples when combined with the Yule-Walker estimation technique. To reduce the potential dangers of underestimating the model order, the AIC should merely be used as a first indication for the model order. The following additional information should be used to select a final model order.

- (a) Practical experience has showed that not only looking at the order with the absolute minimal value of a criterion but also at the local minimums can be very useful in the order selection process.
- (b) Also look at what the other order selection criteria indicate as the appropriate model order.
- (c) Also estimate models of higher orders until the “core” information in the estimated spectral densities does not change anymore and basically only additional “noise” is introduced into the spectral estimate.
- (d) Check for which order the sample autocorrelations are “best” fitted by the model autocorrelations. Note that an  $AR(p)$  model estimated by the YW technique implies that the first  $p$  sample autocorrelations are exactly fitted while higher order autocorrelations are estimated by “extrapolation” of the AR polynomial. Optimal is then the order that performs best at this extrapolation.

Anticipating on the empirical applications of the autoregressive spectral estimation technique combined with this order selection procedure, it is worth mentioning here that this approach in general leads to typical model orders somewhere between six and eight, in case of annual data and univariate models. Although these type of orders may seem rather high compared to conventional applications of autoregressive models, the choice of such high orders is confirmed by Berkowitz et al. (2000). Their simulation results show that in the case of monthly observations using fifteen autoregressive lags is likely to provide a good approximation for a wide range of univariate processes which is considerably larger than commonly assumed. Furthermore, just as Kilian (2001), they warn against the potential dangers of underestimating the appropriate model order, just as found in the Monte Carlo experiments described in the previous sections.

### *Estimation method*

The preference for the Yule-Walker estimation method conflicts with most of the earlier research on this topic. The very thing is that often research has indicated that the Yule-Walker technique leads to the *worst* instead of the *best* spectral estimates. Fougere et al. (1976) and Kay and Marple (1979) for example show that the (univariate) Yule-Walker method can lead to *spectral line splitting*. This phenomenon means that in an estimated spectrum for some frequencies there are two peaks while the real spectrum only has one peak at that frequency. It is often accompanied by many spurious peaks in the spectrum. Spectral line splitting for the Yule-Walker method is thought to come from the poor quality of high order autocovariance estimates because of the small number of available sample points. These estimators are very sensitive to outliers in the data. For the OLS method line splitting has not been observed<sup>31</sup>. The results however show that especially in small samples the OLS method may lead to non-stationary models for which a spectral density cannot be calculated in the first place. One may wonder what is worse, spectral line splitting or no spectrum at all. Also, Marple (1987, p. 416) indicates that line splitting has not been observed in multivariate applications of the Yule-Walker method. Furthermore Kay and Marple (1979) show that the occurrence of line splitting can be strongly reduced by using the biased estimators of the sample autocovariances instead of the unbiased estimators. Note that the experiments in this section already used the biased estimators which impose a stationarity restriction on the model. Experiments, not reported here, with the unbiased version of the estimators confirmed this finding of Kay and Marple.

Nuttal (1976) and Lang and McClellan (1980) show that in general methods that do *not* guarantee stationarity by imposing restrictions lead to better spectral estimates than methods that do impose such restrictions. Assuming that such a stationarity restriction is a *valid* restriction, so that the process under investigation indeed is a stationary process, it is hard to believe that such a statement is true in general. After all, general estimation theory tells us that imposing a valid restriction can only lead to more efficient estimates.

Nuttal (1976), Shon and Mehrotra (1984) and Marple (1987) all find in their Monte Carlo experiments that the Burg and TLS estimation methods lead to the most efficient spectral estimates (so better than the OLS and YW techniques). That in our case the Yule-Walker method performs even better can have various causes. Besides

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<sup>31</sup> Note that section 19.2.1 reports results of spectral line splitting actually also occurring in the case of using OLS estimates.



the stationarity effect of applying the biased instead of the unbiased autocovariance estimates, it may be because other types of spectral densities are estimated (several not too sharp peaks which are not too close together, consistent with what one might expect for macroeconomic variables). It may also be because a different criterion for evaluating the quality of a spectral estimate is used (the relative spectral errors instead of for example the location of spectral peaks).

Because of the well known optimal properties (Maximum Likelihood, best linear unbiased, etc.) of the OLS technique one may also have expected that the OLS technique would provide the best spectral estimates. For autoregressive models these optimal properties only hold asymptotically and as an approximation and spectral estimation is an other type of application than estimating individual model parameters. Furthermore as the sample size increases the efficiency of the different estimation techniques (for example OLS and YW) converges. This shows that finding the YW technique to be the superior spectral estimation technique in small samples need not be inconsistent with the theory on OLS estimators. Experiments, not reported here, also confirmed another well known property of OLS spectral estimates. This property is that OLS spectral estimates have a rather strong tendency to “amplify” the periodic behavior. This means that a spectrum estimated by OLS often has sharper peaks than the true spectrum. In terms of the autocorrelations, this means that they show a slower decay than the sample autocorrelations and in terms of the eigenvalues of an autoregressive model it means that the modulus of the eigenvalues is closer to one than it should be (tending towards non-stationarity).

This section is closed with two final remarks. The first concerns a phenomenon that was encountered during the experiments with the multivariate spectral estimation but that has not been reported in the previous sections. For example Fougere (1981, 1985) calls this the *feed-across effect*. It means that a multivariate estimate of the auto-spectrum of one variable shows a spectral peak which is not present in the true spectrum of that variable but that however is present in the spectrum of another variable. Again, this “transfer” between auto-spectra occurs most often in case of small sample sizes. It probably comes from the problems of estimating not that parsimonious VAR models on limited data. Despite conclusion number six, it is therefore recommended to use univariate autoregressive models for the estimation of auto-spectra since the feed-across effect can then by definition not occur. Cross-spectra can of course only be estimated on multivariate models. However, in order to prevent disturbing effects as much as possible, one should use models of dimension two which is the lowest dimension possible for estimating multivariate relations.

The second remark is the one made before with respect to the connection between parametric Yule-Walker spectral estimation on the one hand and conventional non-parametric spectral estimation on the other. The Yule-Walker method extrapolates the sample autocovariances into infinity in the most neutral way, as indicated by the Maximum Entropy concept, while the non-parametric approach falsely assumes the autocovariances outside the sample to be zero, thereby introducing significant errors (for example leakage) into the spectral estimate. In this sense, parametric Yule-Walker spectral estimation can be seen as “merely” the most natural extension and improvement of the conventional non-parametric approach.



## 4.8 Statistical significance and confidence intervals

Two important topics of spectral analysis have remained undiscussed in the previous sections. The first is how to test whether an estimated spectrum is truly different from that of a white noise process or that its spectral shape should merely be considered a statistical coincidence. This can be compared to applying the familiar  $t$ -test in a regression model to test in a statistical sense whether some estimated parameter significantly differs from zero or not. Related to this first topic is the question of how to obtain a confidence interval around a “point” estimate of a spectral density. In this final section of this chapter it is shown how these two questions can be answered for the class of autoregressive spectral estimators using the Yule-Walker estimation technique.

### 4.8.1 Statistical significance

In the theory on non-parametric spectral analysis, tools are provided for testing whether an estimated spectrum significantly differs from the (flat) spectrum of a white noise process in a statistical sense. Priestley (1981, p. 406) for example states *“we must therefore apply a test to a periodogram peak to determine whether its value is significantly larger than that which would be likely to arise if there were no genuine periodic components in the model”*. Such a test enables us to get an indication about whether the dynamic behavior of a process as indicated by an estimated spectrum is a “true” stochastic structure or not. Specifically it can tell us whether some observed peak in an estimated spectrum indicates truly pseudo periodic behavior or not. Yet another description of such a test is testing for each frequency whether the contribution of that frequency to the total variance of the process is different (more or less) from the contribution it has in the variance of a white noise process. For such a test we need the probability distribution of the value of an estimated spectrum using some estimation technique at each frequency under the null hypothesis that the true spectrum is that of a white noise process. If an estimated spectrum falls outside for example the 95% confidence interval at some frequency then it is very unlikely that the actual contribution to the variance at that frequency is the same as it would have been in a white noise process. In that case we can speak of a statistically significant peak or trough at the relevant frequency given a confidence level of 5% (i.e. there is a 5% chance that we draw the wrong conclusion). Note that it is sufficient if the value of an estimated spectrum for just one frequency falls outside the confidence interval, for the process to significantly differ from a white noise process.

For the non-parametric spectral estimators there exist for example the (two-sided) “goodness-of-fit” tests of Grenander and Rosenblatt and of Barlett which can be used to test whether some estimated spectrum significantly differs from the spectrum of some theoretical model, possibly a white noise process. These tests, which can for example be found in Priestley (1981, section 6.2.6) unfortunately cannot be applied to the parametric spectral estimators. We therefore need to find another way to test the significance of an autoregressive spectrum estimated by the Yule-Walker technique. One possibility would be to use the familiar confidence intervals for the estimated sample autocorrelations. Box and Jenkins (1976) for example show that estimated autocorrelations on a sample of independent data of size  $T$  are approximately Normally distributed with a variance of  $1/T$ . This means that a 95% confidence interval for an estimated autocorrelation lies between  $-1.96/\sqrt{T}$  and  $1.96/\sqrt{T}$ . Under the null hypothesis of a white noise process, the

expected value of the autocorrelations is zero. Because the Yule-Walker estimates are a direct transformation of these sample autocorrelations it should be possible to construct a confidence interval for the spectral estimates of a white noise process based on these statistical properties of the estimated autocorrelations. Because it concerns a rather complex transformation it is not sufficient to only transform the limits of the autocorrelation confidence intervals (in the time domain) to obtain the spectral confidence interval (in the frequency domain). Because combinations of sample correlations from within the confidence intervals may equally well lead to extreme spectral densities, combinations of autocorrelation values on a fine grid within the confidence intervals have to be transformed into the corresponding spectral density. Although rather simple to implement, in practice such an approach turns out to be problematic. The problems are caused by the combinations of autocorrelations on the grid that lead to non-stationary or almost non-stationary models with exploding spectral densities that disturb the resulting spectral confidence intervals. Note that since such combinations of autocorrelations come from a grid and are not directly estimated on the data, the stationarity restriction of the Yule-Walker technique does not work here. Sims notes something similar in his reaction on Runkle (1987). There Runkle calculated confidence intervals for impulse response functions and variance decompositions of VAR models which just as a spectral density are very complex transformations of the original data. Sims notes that *“the unstable portion of the parameter space is likely to dominate the extreme values of the distribution of variance decompositions”* and *“it seems unreasonable to take explosive parameter settings seriously in computing the probabilities of variance decompositions”*.

An alternative approach for calculating spectral confidence intervals for a white noise process that is less sensitive to the problems of non-stationarity is the following simple Monte Carlo approach. Let us start with the null hypothesis of a Normally distributed white noise process. As we know, such a process has all autocorrelations equal to zero and a flat (normalized) spectrum equal to  $1/\pi$  at all frequencies. For each of a number of sample lengths  $T$  ten thousand simulations were generated from such a white noise process. For each of the simulations then the spectral density was estimated through autoregressive models of all relevant orders  $p$ , using the Yule-Walker technique. Note that using the Yule-Walker technique now guarantees that the estimated process is stationary and has a normal spectral density so that the problems of the previous “correlation approach” do not occur. Finally, based on the ten thousand estimated spectral densities it is easy to calculate the confidence interval for any desired level of confidence. This approach may equally well be applied for other estimation techniques such as the OLS technique with the only exception that these in general do not guarantee a stationary process to be estimated. We chose to perform these simulations for all required model orders and only for a limited number of sample sizes. If the confidence interval for another sample size is needed, it is easily obtained by interpolation between the adjacent sample sizes. It is important to note that these confidence intervals are only valid in the specific case of a spectrum that is actually estimated by means of an autoregressive model of some order using the Yule-Walker technique. Here, the confidence intervals were only calculated for univariate models. However, the procedure could equally well be applied to calculate confidence intervals for multivariate white noise processes.

Intuitively, we expect two properties of the resulting confidence intervals. The first is that given some order  $p$ , the confidence intervals should get smaller as the sample size increases. This is confirmed by the 95% confidence intervals of a white noise process estimated by an AR model of order  $p=4$  for sample sizes  $T=20, 40, 60$  and  $80$  as reported in Figure 4.20. In this figure also the theoretical normalized spectrum of a white noise process with a constant value of  $1/\pi$  is plotted. A striking property of the confidence intervals is that they are smaller for the intermediate frequencies and wider for the very low and very high frequencies. A second property to expect of the resulting confidence intervals is that given a fixed sample size  $T$ , the confidence intervals will get wider as the order increases. This is confirmed by the 95% confidence intervals shown in Figure 4.21 for  $T=50$  and  $p=2, 4, 6$  and  $8$ .

Finally, Figure 4.22 shows an actual example of an estimated spectrum, in this case using an AR model of order seven, and the corresponding 95% confidence interval of a white noise process. This picture tells us that of the three peaks visible in the estimated spectrum, the first two reach outside the confidence interval while the third lies within the confidence interval. We can therefore say that the first two peaks indicate “true” pseudo periodic behavior in the process at the indicated frequencies while this cannot be said about the third peak.

Figure 4.20 95% confidence interval of a the normalized spectrum of a white noise process estimated by an AR(4) model for sample sizes  $T=20, 40, 60$  and  $80$  using the Yule-Walker estimation technique.

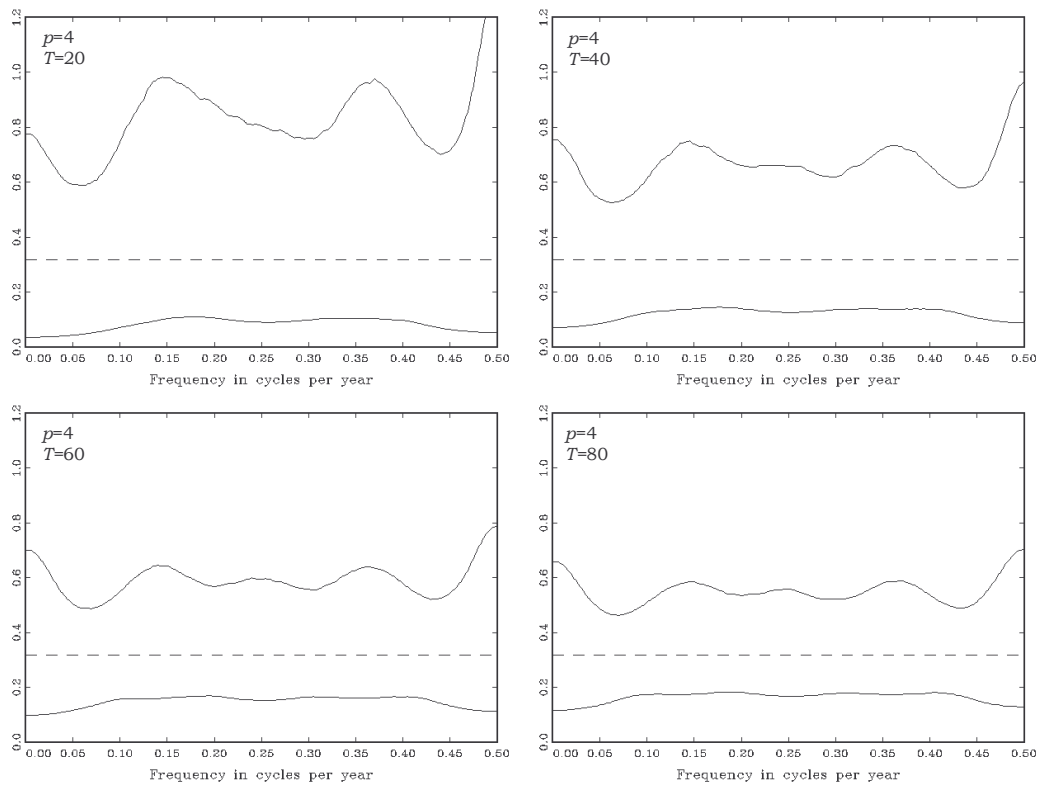


Figure 4.21 95% confidence interval of a the normalized spectrum of a white noise process estimated by an AR( $p$ ) model for  $p=2, 4, 6$  and  $8$  for sample size  $T=50$  using the Yule-Walker estimation technique.

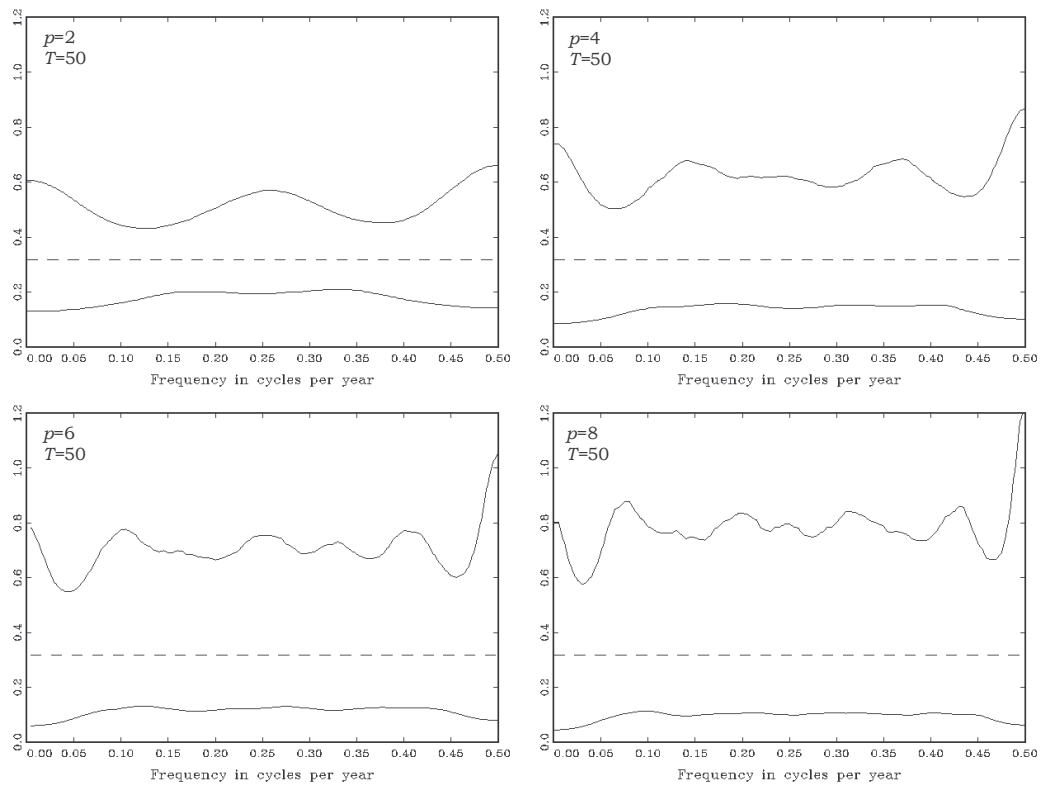
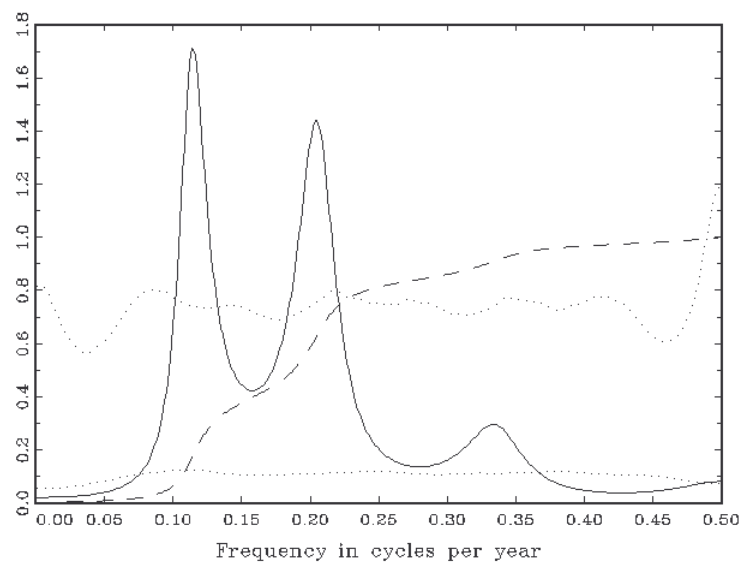


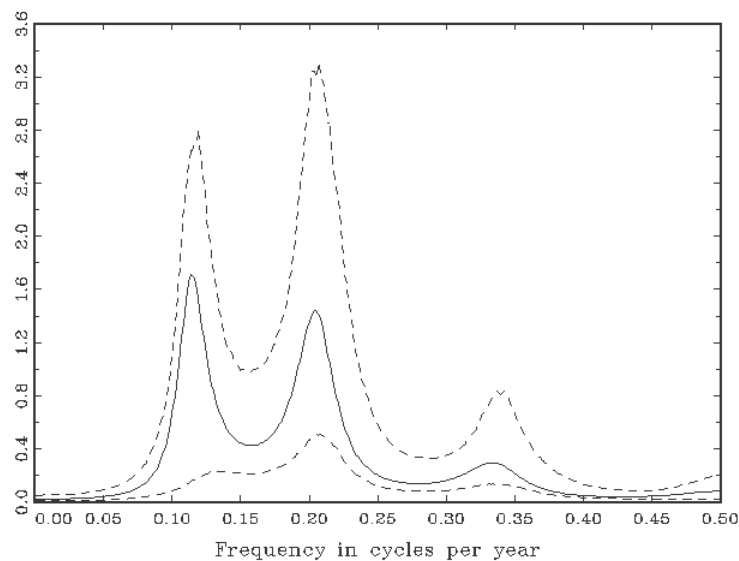
Figure 4.22 Example of an estimated spectrum and the corresponding 95% confidence interval of a white noise process



### 4.8.2 Confidence intervals

The previous subsection showed how to compare a “point” estimate of a spectral density to a confidence interval of a white noise process in order to assess the statistical significance of the spectral estimate. Using almost the same Monte Carlo principle, it is also possible to calculate a confidence interval around a “point” estimate of a spectrum. After all, any other realization of the estimated process would have led to another spectral estimate. The wider such a confidence interval, the more uncertainty there is about the actual “point” estimate. One way to calculate a confidence interval is by means of a *bootstrapping* technique. This works as follows. Once a spectral estimate is obtained we assume that this represents the true model. Under this assumption the realizations of the stochastic error terms  $\varepsilon_t$  in the model could have been very different which would have resulted in another spectral estimate while the underlying model has not changed. Now consider the empirical counterparts  $\hat{\varepsilon}_t$  as a representative sample of the error terms. Because of the assumed independence of the error terms these realizations could just as well have appeared in any different order. By keeping the first  $p$  observations of the process fixed it is now easy to construct say a thousand new realizations of the historic time serie(s) based on the original estimated model parameters and different random draws from the pool of residuals  $\hat{\varepsilon}_t$ . For each simulation, a new spectrum can be estimated without changing the model order. Each simulation will yield a different spectral estimate based on the same underlying model. From the thousand estimates it is then easy to calculate any desired confidence interval around the original “point” estimate. Figure 4.23 shows the 95% bootstrapping confidence interval for the same spectral estimate as shown in Figure 4.22. This picture for example shows that the second peak in the spectrum could very well have been higher than the first peak contrary to what the “point” estimate indicates.

Figure 4.23 Example of an estimated spectrum and its 95% confidence interval.



## 5 Linear Filters

In the previous two chapters the relevant theory of autoregressive models and the spectral analysis of time series was discussed. This final chapter of Part II discusses the relevant theory of linear filtering techniques as the last of the three central techniques mentioned in section 2.3. Filtering techniques in general can perhaps best be seen as techniques that enable a decomposition of a time series into a trend, a cycle, a seasonal and finally a “random” component as introduced by Tinbergen (see section 2.2.2). Such a decomposition enables a separate analysis and, if necessary, also modeling of different aspects of an economic time series process. In the literature a large number of such filtering techniques can be found, none of which can be shown to be the optimal one for all applications. For the specific case of detrending time series to obtain information on business cycle fluctuations for example Canova (1998), based on an extensive comparison of different techniques, concludes that all techniques subtract very different information from the same time series. Furthermore, it is well known that many of the existing filtering techniques can lead to finding so called “spurious cycles”. That is, finding cycles in time series that are not really there. This especially holds if the “true” time series process contains a stochastic trend. Examples of literature on this topic are Chan, Hayya en Ord (1977), Harvey en Jaeger (1993), Osborn (1995), Nelson and Kang (1981, 1984) and Cogley and Nason (1995), each of which discusses a different detrending technique.

These two findings, different techniques yielding different results and the potential danger of finding spurious results, necessitate a very thorough investigation of the properties of filtering techniques which then enables choosing the technique best suited for the intended application of the filter. This chapter shows that such an analysis of filtering techniques can probably best be done in the frequency domain. The discussion is limited to the class of *linear filters* which encompasses most of the common filtering techniques. Just as in the previous chapters the discussion is limited to the most relevant aspects with respect to the research presented here, instead of describing a complete theory of filtering techniques. More complete overviews of filter theory can be found in for example Rabiner en Gold (1975) and Koopmans (1974). Furthermore Gençay, Selçuk and Whitcher (2002) provide an extensive list of references. After having shown the properties of some well known filtering techniques and their main drawbacks for the current research objectives, an alternative approach is described which in our view comes closest to the definition of an “optimal” filtering technique. Finally, using this technique, some simple simulation experiments are performed to check whether the filter performs as expected.

## 5.1 Gain and Phase

Consider a simple time series  $x_t$  consisting of a perfect cosine function with frequency  $\omega$ , some amplitude  $R_x$  and some phase  $\phi_x$ .

$$x_t = R_x \cos(\omega t + \phi_x) \quad (5.1.1)$$

Now suppose that we apply a linear transformation on this time series to obtain the time series

$$y_t = \sum_{l=a}^b g_l x_{t-l} = \left( \sum_{l=a}^b g_l L^l \right) x_t = G(L) x_t \quad (5.1.2)$$

where

$$G(L) = g_a L^a + g_{a+1} L^{a+1} \dots + g_b L^b \quad (5.1.3)$$

and  $L(\cdot)$  is the *lag operator* for which holds  $L^k(x_t) = x_{t-k}$ . The linear transformation defined by  $G(L)$  is called a *linear filter*. The filter coefficients  $g_l$  as function of  $l$  is called the *Impulse Response Function* (IRF) of the filter. The IRF indicates how a single impulse

$$x_t = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.1.4)$$

turns up in  $y_{t+l}$  for  $l=a, \dots, b$ . If the impulse response of a filter is finite, it is called a *Finite Impulse Response* (FIR) *filter*. If it is not finite, the filter is called an *Infinite Impulse Response* (IIR) *filter*. Furthermore, a filter which only uses lagged values of  $x_t$  to construct a new value of  $y_t$  (that is  $a \geq 0$  and  $b > a$ ) is called a *causal filter*.

To find out how such a linear filter affects the properties of the time series  $x_t$ , first note that  $x_t$  as defined in (5.1.1) is the same as the real portion of the complex time series

$$x_t^* = R_x \exp(i(\omega t + \phi_x)) \quad (5.1.5)$$



Because the real portion of a sum of complex numbers is the same as the sum of the real portions of the complex numbers,  $y_t$  as defined in (5.1.2) is the same as the real portion of the complex time series

$$\begin{aligned}
 y_t^* &= \sum_{l=a}^b g_l x_{t-l}^* \\
 &= \sum_{l=a}^b g_l R_x \exp(i(\omega(t-l) + \phi_x)) \\
 &= R_x \exp(i(\omega t + \phi_x)) \sum_{l=a}^b g_l \exp(-i\omega l) \\
 &= x_t^* G(\exp(-i\omega))
 \end{aligned} \tag{5.1.6}$$

Note that, with the exception of a constant factor,  $G(\exp(-i\omega))$  is equal to the DCFT of  $x_t^*$  defined in (4.1.21). The function  $G(\exp(-i\omega))$  is called the *Frequency Response Function* (FRF) or the *Transfer Function* (TF) because for each frequency  $\omega$  it specifies how the amplitude and phase of the original time series  $x_t$  are effected by the linear filter. To see this, write the FRF in its polar format as

$$\begin{aligned}
 G(\exp(-i\omega)) &= \sum_{l=a}^b g_l \exp(-i\omega l) \\
 &= \sum_{l=a}^b g_l \cos(\omega l) - i \sum_{l=a}^b g_l \sin(\omega l) \\
 &= R_g(\omega) \exp(i\phi_g(\omega))
 \end{aligned} \tag{5.1.7}$$

where

$$R_g(\omega) = |G(\exp(-i\omega))| \text{ and } \phi_g(\omega) = \arg(G(\exp(-i\omega))) \tag{5.1.8}$$

are respectively the modulus and argument of the FRF at frequency  $\omega$ . Substituting this representation and (5.1.5) into (5.1.6) yields

$$\begin{aligned}
 y_t^* &= R_x \exp(i(\omega t + \phi_x)) R_g(\omega) \exp(i\phi_g(\omega)) \\
 &= R_x R_g(\omega) \exp(i(\omega t + \phi_x + \phi_g(\omega))) \\
 &= R_x R_g(\omega) (\cos(\omega t + \phi_x + \phi_g(\omega)) + i \sin(\omega t + \phi_x + \phi_g(\omega)))
 \end{aligned} \tag{5.1.9}$$

The transformed time series  $y_t$  as the real part of (5.1.9) is thus

$$y_t = R_x R_g(\omega) \cos(\omega t + \phi_x + \phi_g(\omega)) \tag{5.1.10}$$

Comparing this expression to the definition of the original time series  $x_t$  as given by (5.1.1) shows that applying the linear filter has two effects which can be characterized by the two functions

$$\text{Gain}(\omega) = |G(\exp(-i\omega))| = R_g(\omega) \quad (5.1.11)$$

and

$$\text{Phase}(\omega) = \frac{\arg(G(\exp(-i\omega)))}{2\pi} = \frac{\phi_g(\omega)}{2\pi} \quad (5.1.12)$$

The *gain* of a linear filter gives the multiplier change in the amplitude of the component of frequency  $\omega$  in a time series. The *phase* of a linear filter gives the phase shift of the component of frequency  $\omega$  in a time series, expressed as a fraction of the period length. The phase is a positive number which indicates a shift backwards in time.

If a linear filter is applied to a general time series  $x_t$  with a representation in the frequency domain, in terms of the IDFT (4.1.14), of

$$x_t = \sum_{j=0}^{T-1} J_j \exp(i\omega_j t) \quad \text{for } t = 0, \dots, T-1 \quad (5.1.13)$$

for which we already know that

$$\frac{1}{T} \sum_{t=0}^{T-1} x_t^2 = \sum_{j=0}^{T-1} |J_j|^2 \quad (5.1.14)$$

then it follows along the same lines as (5.1.6) that

$$\begin{aligned} y_t &= \sum_{l=a}^b g_l x_{t-l} \\ &= \sum_{l=a}^b g_l \sum_{j=0}^{T-1} J_j \exp(i\omega_j(t-l)) \\ &= \sum_{j=0}^{T-1} J_j R_g(\omega_j) \exp(i\omega_j t + \phi_g(\omega_j)) \end{aligned} \quad (5.1.15)$$

and thus

$$\frac{1}{T} \sum_{t=0}^{T-1} y_t^2 = \sum_{j=0}^{T-1} |J_j|^2 (R_g(\omega_j))^2 \quad (5.1.16)$$

So the *square* of the gain of a linear filter gives the multiplier change of the variance of the component with frequency  $\omega$  in a time series. The squared gain is therefore often called the *Power Transfer Function* (PTF).

This section closes with two important properties of linear filters. The first is that if  $y_t$  is a filtered time series after having applied a linear filter with some FRF  $G(\exp(-i\omega))$  to the time series  $x_t$  and  $z_t = x_t - y_t$  is its complement, then the FRF of the filter directly yielding the complement time series  $z_t$  is simply  $1 - G(\exp(-i\omega))$ . This is because of the linearity of the filters. To see this note that

$$z_t = x_t - y_t = (1 - G(L))x_t \quad (5.1.17)$$

For example think of a filter that smoothes a time series  $x_t$  to estimate a trend  $y_t$  and the complement filter that obtains the fluctuations  $z_t$  around this trend as the difference between  $x_t$  and  $y_t$ .

A second important property of linear filters holds with respect to the sequential application of several linear filters to the same time series. From the properties of complex numbers it easily follows that if  $Gain_1(\omega)$  and  $Gain_2(\omega)$  are the gains of two different linear filters,  $Phase_1(\omega)$  and  $Phase_2(\omega)$  are the corresponding phases and  $Gain_{1+2}(\omega)$  and  $Phase_{1+2}(\omega)$  are the gain and phase of the combined (sequential) filter then it holds that

$$Gain_{1+2}(\omega) = Gain_1(\omega)Gain_2(\omega) \quad (5.1.18)$$

and

$$Phase_{1+2}(\omega) = Phase_1(\omega) + Phase_2(\omega) \quad (5.1.19)$$

That is, the gain of a combined filter is simply the product of the gains of the individual gains while the phase of a combined filter is simply the sum of the individual phases. The simple proof is given in Appendix B.12.

## 5.2 Some well-known filters

Many of the well known filtering or detrending techniques fit into the general format of a linear filter as described in the previous section. Because of this, it is possible to analyze the properties of such filters in the frequency domain using the gain and phase representation of the relevant frequency response functions. In this section this is done for some well known (classes) of filters.

### 5.2.1 First order differencing

Coming from the Box and Jenkins time series methodology (also see section 2.2.6 on Classical Time Series Analysis) probably the most often applied filtering technique consists of calculating the first order differences of a time series. The filtered time series  $y_t$  is calculated as

$$y_t = x_t - x_{t-1} \quad (5.2.1)$$

In terms of the general linear filter (5.1.2) it has filter weights  $g_0=1$  and  $g_1=-1$  and all other weights equal to zero. In terms of the lag operator  $L$  it is

$$y_t = (1 - L)x_t = G(L)x_t \quad (5.2.2)$$

which implies that the FRF is

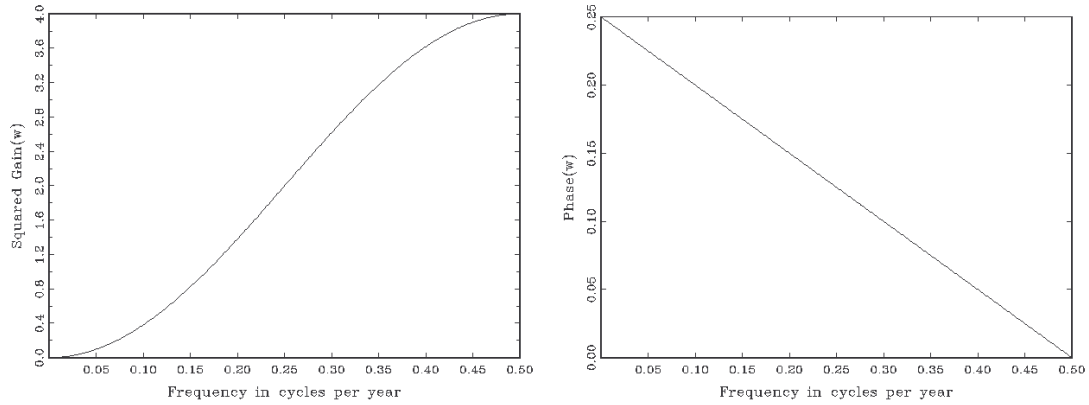
$$G(\exp(-i\omega)) = 1 - \exp(-i\omega) = 1 - \cos(\omega) + i \sin(\omega) \quad (5.2.3)$$

Its gain and phase are given below while Figure 5.1 plots the PTF (i.e. the squared gain) and the phase.

$$\text{Gain}(\omega) = |G(\exp(-i\omega))| = \sqrt{G(\exp(-i\omega))^* G(\exp(-i\omega))} = \sqrt{2 - 2 \cos(\omega)} \quad (5.2.4)$$

$$\text{Phase}(\omega) = \frac{\arg(G(\exp(-i\omega)))}{2\pi} = \frac{1}{2\pi} \arctan\left(\frac{\sin(\omega)}{1 - \cos(\omega)}\right) \quad (5.2.5)$$

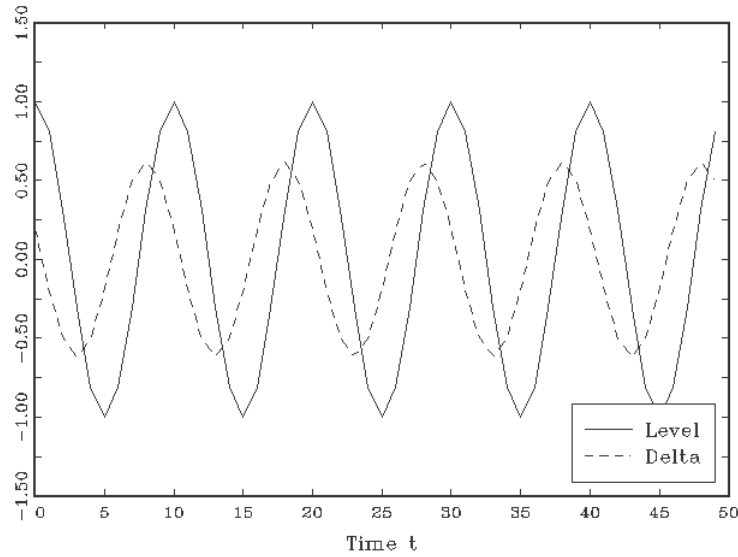
Figure 5.1 Squared gain (5.2.4) and phase (5.2.5) of the first order differencing filter.



The PTF on the left shows that the variance at frequencies below approximately 1/6 cycles per year (i.e. with a period length of more than six years) is being reduced by the first order differencing filter. This explains why the filter is often used to eliminate trending behavior (i.e. very long term and low frequency fluctuations) from empirical time series. First order differencing also (strongly) emphasizes the high frequency fluctuations. This can be seen from the value of the PTF for higher frequencies. The variance of the highest frequency fluctuations is multiplied by a factor four. Besides changing the variance at the relevant frequencies, the first order differencing filter also shifts the time series in the time domain which corresponds to phase shifts in the frequency domain. The phase on the right shows that the lowest frequencies are shifted backwards in time by approximately a quarter of the relevant period length while the phase shift for the higher frequencies decreases towards zero for the highest frequency in a linear fashion.

To illustrate these effects of the first order differencing filter let's take a look at a perfect cosine function with a period length of ten years (i.e. a frequency of 1/10 cycles per year). Figure 5.2 plots both a sample of size  $T=50$  of this original time series and of the resulting time series after calculating its first order differences.

Figure 5.2 A perfect cosine of frequency 1/10 and its first order differences.



The gain and phase effects of the first order differencing filter are clearly visible. The amplitude of the cosine function is reduced by approximately a factor 0.62 which is the value of the gain at the frequency  $\omega=2\pi/10$ . The value of the PTF at the same frequency is  $0.62^2=0.38$  as can be seen from the left hand side of Figure 5.1. Furthermore the series of first order differences leads the original cosine function by approximately two time periods which is equal to the value of the phase of 0.20 at the relevant frequency multiplied by the period length of ten years.

### 5.2.2 Symmetrical filters

The first order differencing filter from the previous section is an example of an asymmetrical filter. That is, the filter weights are not symmetrical around the weight  $g_0$ . We saw that such a filter causes phase shifts for all frequencies. The class of filters that do have symmetrical weights (i.e.  $g_l=g_{-l}$ ) has the special property that the phase is always equal to zero. This property comes from the fact that the FRF of a symmetrical linear filter is not complex valued as was the case for the first order differencing filter. A filter with  $Phase(\omega)=0$  for all  $\omega$  is also called a *zero phase filter*. Substituting the symmetry condition into the general expression for the FRF gives

$$\begin{aligned}
 G(\exp(-i\omega)) &= \sum_{l=-a}^a g_l \exp(-i\omega l) \\
 &= \sum_{l=-a}^a g_l \cos(\omega l) - i \sum_{l=-a}^a g_l \sin(\omega l) \\
 &= g_0 + 2 \sum_{l=1}^a g_l \cos(\omega l)
 \end{aligned} \tag{5.2.6}$$

which implies that for each symmetrical linear filter

$$\text{Gain}(\omega) = g_0 + 2 \sum_{l=1}^a g_l \cos(\omega l) \quad (5.2.7)$$

and

$$\text{Phase}(\omega) = 0 \quad (5.2.8)$$

As an example look at the following *Moving Average filter* of order three

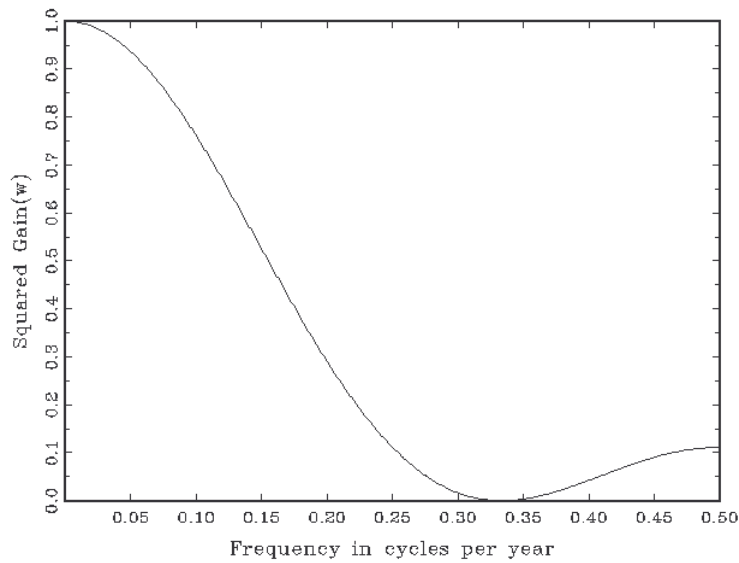
$$y_t = \frac{1}{3}(x_{t-1} + x_t + x_{t+1}) \quad (5.2.9)$$

with

$$\text{Gain}(\omega) = 1 + 2 \cos(\omega) \quad (5.2.10)$$

The PTF of this filter is given in Figure 5.3. Such a filter obviously leaves the variance at the lowest frequencies largely intact (a value of the PTF equal to one) while the higher frequencies are being suppressed (a value of the PTF smaller than one). The result of applying a moving average filter to a time series will therefore always be a smoother time series than the original time series.

Figure 5.3 Squared gain (5.2.10) of the moving average filter (5.2.9).



### 5.2.3 Exponential Smoothing

A class of (IIR) filters that just as the (FIR) moving average filters was originally designed for smoothing time series are the *Exponential Smoothing filters*. An example of such a filter, as employed by Lucas (1980) in empirical investigations, is derived from the following optimization problem

$$\min_{y_0, \dots, y_{T-1}} \sum_{t=1}^T (x_t - y_t)^2 + \lambda(y_t - y_{t-1})^2 \quad (5.2.11)$$

That is, try and fit the original time series  $x_t$  as good as possible by a time series  $y_t$  but put a penalty  $\lambda$  on the (squared) changes of  $y_t$ . The higher the value of  $\lambda$ , the smoother the time series  $y_t$  will be. The first order conditions for this problem have the following format

$$\frac{\partial}{\partial y_t} = -2(x_t - y_t) + 2\lambda(y_t - y_{t-1}) - 2\lambda(y_{t+1} - y_t) = 0 \quad (5.2.12)$$

If we assume an infinite sample, this can be written as

$$(-\lambda L^{-1} + (1 + 2\lambda) - \lambda L)y_t = (\lambda(1 - L)(1 - L^{-1}) + 1)y_t = x_t \quad (5.2.13)$$

which solves for  $y_t$  as<sup>32</sup>

$$y_t = (\lambda(1 - L)(1 - L^{-1}) + 1)^{-1} x_t = G(L)x_t \quad (5.2.14)$$

The FRF is therefore

$$\begin{aligned} G(\exp(-i\omega)) &= \frac{1}{1 + \lambda(1 - \exp(-i\omega))(1 - \exp(i\omega))} \\ &= \frac{1}{1 + 2\lambda(1 - \cos(\omega))} \end{aligned} \quad (5.2.15)$$

The gain and phase are equal to

$$Gain(\omega) = \frac{1}{1 + 2\lambda(1 - \cos(\omega))} \quad (5.2.16)$$

and

$$Phase(\omega) = 0 \quad (5.2.17)$$

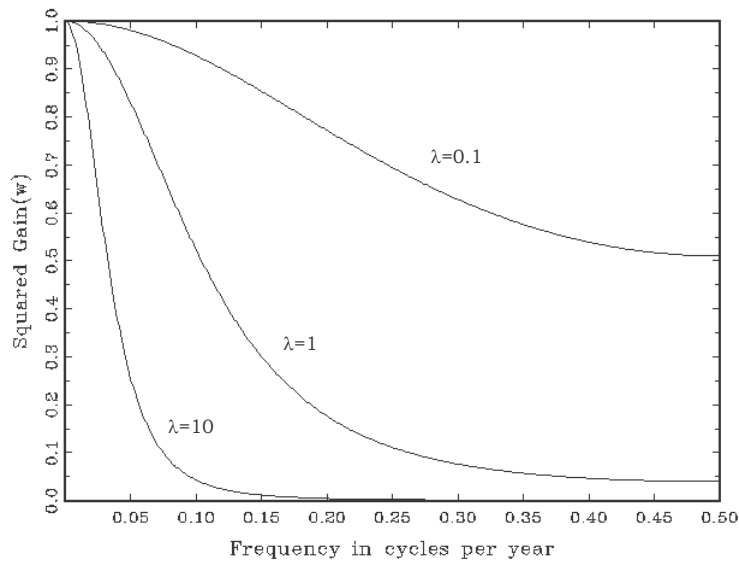
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<sup>32</sup> Note that (5.2.13) and (5.2.14) also indicate how to calculate the smoothed series in practice in a recursive format as  $y_{t+1} = (1/\lambda + 2)y_t - y_{t-1} - 1/\lambda x_t$ . It is this recursive format that makes this filter an IIR filter.



Because the phase is identical to zero, the filter weights of an exponential smoothing filter apparently are symmetrical. Figure 5.4 plots the PTF for different values of  $\lambda$ . From this we can note three things. The first is that, just as with a moving average filter, the maximum value of the PTF is equal to one which means that the filter never amplifies the variance of a component of a time series. At a maximum, it leaves the variance unchanged or otherwise it decreases the variance. The second is that as the value of  $\lambda$  decreases towards zero, less and less penalty is placed on the change of the fitted time series and therefore the fitted time series is allowed to more and more resemble the original time series. This means that the FTF and thereby also the PTF will more and more be a straight line at the value of one. As the value of  $\lambda$  increases, the penalty increases and the fitted time series will become smoother and smoother by only consisting of very low frequency fluctuations. The third and final thing to note is that the exponential smoothing filter has only a single parameter which can be used to determine the shape of the PTF.

Figure 5.4 Squared gain (5.2.16) of exponential smoothing filter for different  $\lambda$ 's.



#### 5.2.4 Hodrick-Prescott filter

The Hodrick-Prescott (HP) filter, which originally is a Whittaker-Henderson type A filter, was introduced into economics by Hodrick and Prescott (1980). Nowadays it is a very popular tool for separating trends and business cycle fluctuations in the literature on Real Business Cycles (see section 2.2.10) while it is also used for the construction of some Business Cycle Indicators (see section 2.2.4). Further readings on the pros and cons of this filter and on its background can be found in Cooley and Prescott (1995), Stadler (1994) and Kydland and Prescott (1990). The infinite sample version of the filter as given by King and Rebelo (1993) is derived from a similar type of optimization problem as the exponential smoothing filter. The only difference is that the penalty  $\lambda$  is not placed on the level of the changes of the smoothed series  $y_t$ , but it is placed on the changes of the changes of  $y_t$ . Hence

$$\min_{y_0, \dots, y_{T-1}} \sum_{t=1}^T (x_t - y_t)^2 + \lambda [(y_{t+1} - y_t) - (y_t - y_{t-1})]^2 \quad (5.2.18)$$

The first order conditions for this problem have the format

$$\begin{aligned} \frac{\partial}{\partial y_t} = & -2(x_t - y_t) + 2\lambda[(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})] \\ & - 4\lambda[(y_{t+1} - y_t) - (y_t - y_{t-1})] \\ & + 2\lambda[(y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)] = 0 \end{aligned} \quad (5.2.19)$$

Assuming an infinite sample, this can be written as

$$(\lambda L^{-2} - 4\lambda L^{-1} + (6\lambda + 1) - 4\lambda L + \lambda L^2)y_t = (\lambda(1 - L)^2(1 - L^{-1})^2 + 1)y_t = x_t \quad (5.2.20)$$

which solves for  $y_t$  as

$$y_t = (\lambda(1 - L)^2(1 - L^{-1})^2 + 1)^{-1} x_t = G(L)x_t \quad (5.2.21)$$

The FRF is therefore

$$\begin{aligned} G(\exp(-i\omega)) &= \frac{1}{1 + \lambda(1 - \exp(-i\omega))^2(1 - \exp(i\omega))^2} \\ &= \frac{1}{1 + 4\lambda(1 - \cos(\omega))^2} \end{aligned} \quad (5.2.22)$$

The gain and phase are equal to

$$Gain(\omega) = \frac{1}{1 + 4\lambda(1 - \cos(\omega))^2} \quad (5.2.23)$$

and

$$Phase(\omega) = 0 \quad (5.2.24)$$

Note that the gain of the HP filter is sort of a “squared” version of the gain (5.2.16) of the gain of the exponential smoothing filter. Both Danthine and Girardin (1989) and Krolzig (1997) show that based on the first order conditions, for a finite time series  $x_t$  of length  $T$  the filtered time series can be calculated as

$$y_t = (I + \lambda A' A)^{-1} x_t \quad (5.2.25)$$

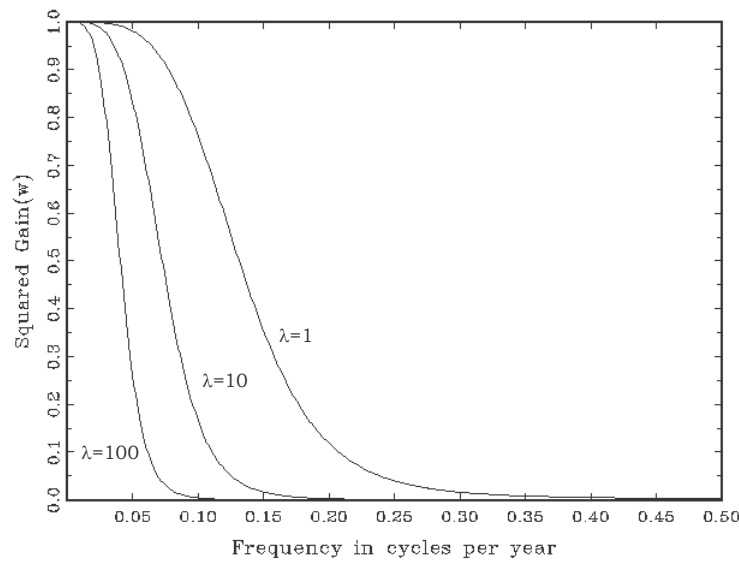
where  $A$  is the  $(T-2) \times T$  matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}$$

Figure 5.5 shows the PTF of the HP filter for several values of  $\lambda$ . The effect of the parameter  $\lambda$  on the PTF is essentially the same as for the exponential smoothing filter from the previous section. As  $\lambda$  increases, the filter passes a smaller and smaller part of the high frequency fluctuations, thereby resulting in a smoother and smoother time series. For the separation of business cycle fluctuations in practical applications, popular values for  $\lambda$  are 1600 for quarterly data, 100 for annual data of real variables (such as GDP) and 10 for annual data of nominal variables (such as a price index) which are thought to have a more volatile trend. See for example Prescott (1986) and Reiter (1995). Harvey and Jaeger (1993) argue that the value of  $\lambda=1600$  for quarterly data very specifically holds for US GDP data because it results in the same cycles as their structural model. The main difference with the exponential smoothing filter is that the FRF of a Hodrick-Prescott filter comes closer to a so called *ideal low pass filter*, or its complement, an *ideal high pass filter*. We will elaborate more on these concepts in the next section. Here we limit ourselves to the remark that in terms of the PTF this means that the PTF shows a “sharper” transition between a value of one for low frequencies and a value of zero for high frequencies which implies a more clear cut separation between high and low frequency fluctuations by the filter.

A final note concerns the phase of both the exponential smoothing and the HP filter. Although the previous analysis shows that in infinite samples these filters do not induce any phase shifts, in practical applications on finite samples of data they may do so, especially on both ends of the sample. This is shown by for example Hassler et al. (1992) and Baxter and King (1995, 1999).

Figure 5.5 Squared gain (5.2.23) of Hodrick-Prescott filter for different values of  $\lambda$ .



### 5.2.5 Inverse first order differencing

A last filter which is not really a filter in the sense that it can be used for detrending time series, is the inverse of the first order differencing filter from section 5.2.1. Hence this filter represents the integration of a time series starting from some initial level. Although it is of little use for detrending time series, it can be used for transforming a spectral density of a differenced process into that of the integrated process by using the property that the (non normalized) spectral density can be multiplied by the squared gain or PTF of the filter to give the new variance per frequency.

The gain of the inverse of a filter is simply the inverse of the gain of the original filter. To see this note that if

$$y_t = G(L)x_t \quad (5.2.26)$$

then

$$x_t = F(L)y_t = G(L)^{-1}y_t \quad (5.2.27)$$

and for the gain of the inverse filter follows as

$$\begin{aligned} \text{Gain}_F(\omega) &= |F(\exp(-i\omega))| = |G(\exp(-i\omega))^{-1}| \\ &= |G(\exp(-i\omega))|^{-1} = \text{Gain}_G(\omega)^{-1} \end{aligned} \quad (5.2.28)$$

Taking the inverse of the gain of the first order differencing filter as given by (5.2.4) therefore results in the gain of the inverse first order differencing filter.

$$\text{Gain}(\omega) = \frac{1}{\sqrt{2 - 2\cos(\omega)}} \quad (5.2.29)$$

An alternative way of determining this gain is as follows. Start with a first order differencing filter with a weight  $\alpha$

$$y_t = x_t - \alpha x_{t-1} \quad (5.2.30)$$

Integrating  $y_t$  back to  $x_t$  gives

$$\begin{aligned} x_t &= y_t + \alpha x_{t-1} \\ &= y_t + \alpha y_{t-1} + \alpha^2 x_{t-1} \\ &= \dots = \left( \sum_{i=0}^{\infty} \alpha^i L^i \right) y_t = G(L)y_t \end{aligned} \quad (5.2.31)$$

Provided that  $|\alpha| < 1$  the FRF of this filter is

$$\begin{aligned}
 G(\exp(-i\omega)) &= \sum_{i=0}^{\infty} (\alpha \exp(-i\omega))^i \\
 &= \frac{1}{1 - \alpha \exp(-i\omega)} = \frac{1}{(1 - \alpha \cos(\omega)) + i\alpha \sin(\omega)} \\
 &= \frac{1 - \alpha \cos(\omega)}{(1 - \alpha \cos(\omega))^2 + (\alpha \sin(\omega))^2} - i \frac{\alpha \sin(\omega)}{(1 - \alpha \cos(\omega))^2 + (\alpha \sin(\omega))^2}
 \end{aligned} \tag{5.2.32}$$

Its gain is then

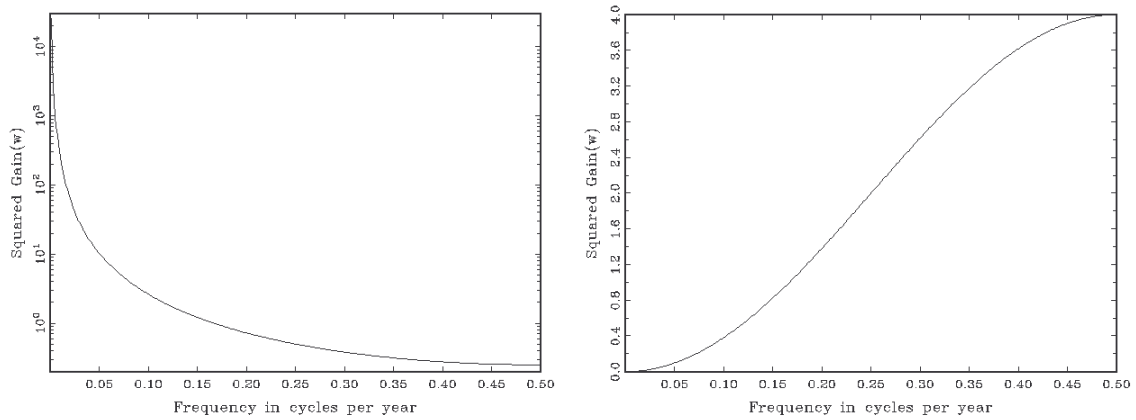
$$\begin{aligned}
 \text{Gain}(\omega) &= |G(\exp(-i\omega))| = \sqrt{G(\exp(-i\omega))^* G(\exp(-i\omega))} \\
 &= \frac{1}{\sqrt{(1 - \alpha \cos(\omega))^2 + (\alpha \sin(\omega))^2}} \\
 &= \frac{1}{\sqrt{1 - 2\alpha \cos(\omega) + \alpha^2}}
 \end{aligned} \tag{5.2.33}$$

Taking the limit for  $\alpha \rightarrow 1$  results in the gain of the inverse first order differencing filter

$$\lim_{\alpha \rightarrow 1} \text{Gain}(\omega) = \frac{1}{\sqrt{2 - 2\cos(\omega)}} \tag{5.2.34}$$

Figure 5.6 shows the PTF of the inverse first order differencing filter. Because the PTF of the differencing filter approached zero at low frequencies, the PTF of its inverse approaches infinity. This shows that integrated processes in general have a very large variance at the lowest frequencies. Hence the terminology of such processes having a *stochastic trend*.

Figure 5.6 Squared gain (5.2.4) of inverse first order differencing filter on a logarithmic scale (left) and squared gain (5.2.4) of first order differencing filter (right).



### 5.3 Frequency domain filters

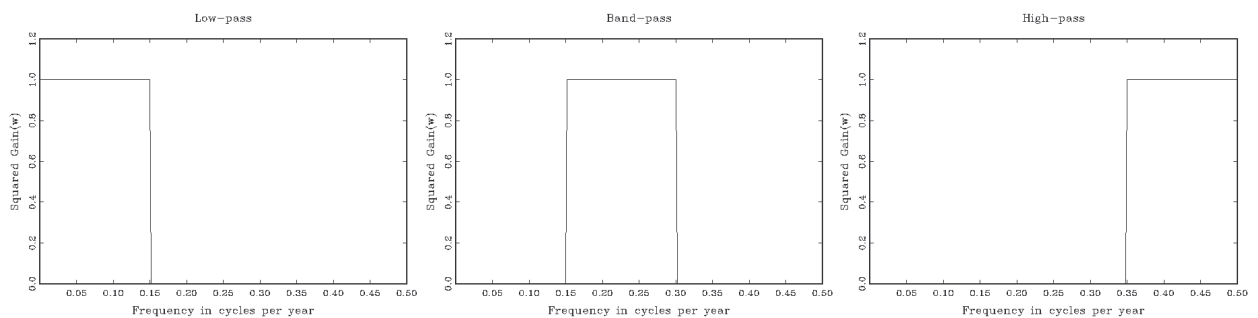
All of the filters discussed in the previous section were originally defined in the time domain in terms of their impulse response functions. In a lot of practical applications one is not aware of the properties of these filters in the frequency domain in terms of their gain and phase. Besides analyzing the frequency domain properties of a given filter, the other way around, the gain and phase of a filter also make it possible to define the desired properties of a filter and then try to design a filter that comes as close as possible to these desired properties. In general this is known as the field of *filter design*. What should such an ideal filter look like? In general we want an *ideal filter* to have a FRF with the following properties.

$$\text{Gain}(\omega) = \begin{cases} 1 & \text{for } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{for } \omega < \omega_1 \text{ or } \omega > \omega_2 \end{cases} \quad (5.3.1)$$

$$\text{Phase}(\omega) = 0 \quad (5.3.2)$$

Such a filter perfectly (that is without multiplying the variance) maintains all fluctuations of the original time series with a frequency between  $\omega_1$  and  $\omega_2$  while all fluctuations of frequencies outside this range are being suppressed completely (that is without leaving even a small portion of the fluctuations in the filtered time series). Figure 5.7 shows three types of such *band-pass filters*. The *pass-band* is the range of frequencies between  $\omega_1$  and  $\omega_2$  while its complement is also called the *stop-band*. The specific case of  $\omega_1=0$  and  $\omega_2>0$  is called a *low-pass filter* because it only passes low frequencies while the specific case of  $\omega_1>0$  and  $\omega_2=\pi$  (the highest frequency possible with half a cycle per time period) is called a *high-pass filter* for similar reasons. Besides perfectly zooming in on some frequency range, such ideal filters also need to be zero phase filters. That is, they should not induce any phase shifts and thereby preserves the possible lead / lag relations between different time series.

Figure 5.7 Gain of ideal low-pass, band-pass and high-pass filter.



In section 5.1 we saw that the FRF of a linear filter defined by the weights  $g_t$  of its IRF is obtained by calculating the DCFT of the weights of the FRF. It should therefore also be possible to first define a desired FRF and then calculate the weights of the IRF in the time domain by the IDCFT of the FRF. The first two following sub-sections show how this can be done. The third sub-section discusses the alternative of not doing the actual filtering in the time domain by applying the IRF, but doing the actual filtering in the frequency domain by directly applying the FRF.

### 5.3.1 Approximating band-pass filters

We know that the FRF of a filter can be calculated as the DCFT (4.1.21) of the weights  $g_l$  of the IRF. That is<sup>33</sup>

$$G(\omega) = G(\exp(-i\omega)) = \sum_{l=a}^b g_l \exp(-i\omega l) \quad (5.3.3)$$

This means that, the other way around, the weights of the IRF can also be calculated from the FRF by means of the IDCFT (4.1.22) as

$$g_l = \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \exp(i\omega l) d\omega \quad \text{for } l = a, \dots, b \quad (5.3.4)$$

We also know that if we impose the restriction that the filter is a zero phase filter the weights  $g_l$  have to be symmetrical around  $l=0$ , that is  $g_l = g_{-l}$ . This means that

$$\begin{aligned} g_l &= \frac{1}{2} (g_l + g_l) = \frac{1}{2} (g_l + g_{-l}) \\ &= \frac{1}{2} \left( \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \{ \cos(\omega l) + i \sin(\omega l) + \cos(\omega l) - i \sin(\omega l) \} d\omega \right) \\ &= \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \cos(\omega l) d\omega \\ &= \frac{1}{\pi} \int_0^{\pi} G(\omega) \cos(\omega l) d\omega \end{aligned} \quad (5.3.5)$$

The final equality uses the fact that  $\cos(\pi+x) = \cos(\pi-x)$ , the cosine function is symmetrical around  $x=\pi$ . So, for a zero phase band-pass filter with a pass-band between  $\omega_1$  and  $\omega_2$ , the weights of the IRF can be calculated as

$$\begin{aligned} g_l &= \frac{1}{\pi} \int_0^{\pi} G(\omega) \cos(\omega l) d\omega \\ &= \frac{1}{\pi} \int_{\omega_1}^{\omega_2} \cos(\omega l) d\omega \\ &= \begin{cases} \frac{\sin(\omega_2 l) - \sin(\omega_1 l)}{\pi l} & \text{for } l \neq 0 \\ \frac{\omega_2 - \omega_1}{\pi} & \text{for } l = 0 \end{cases} \end{aligned} \quad (5.3.6)$$

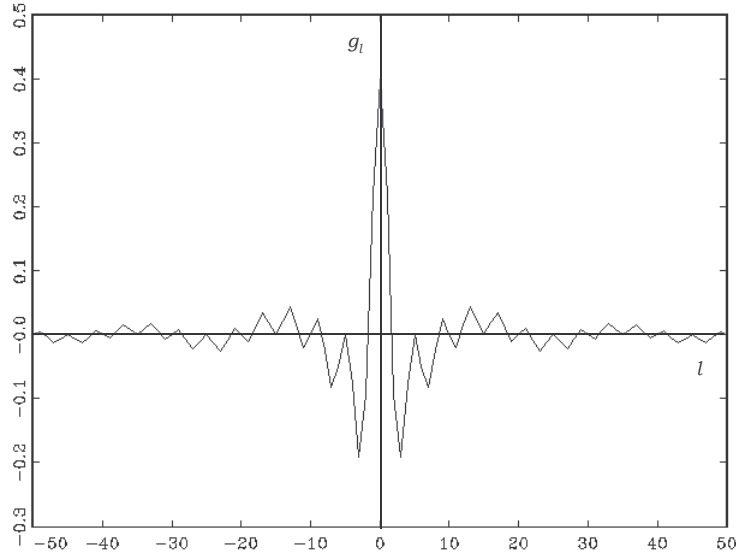
Figure 5.8 shows these weights for the case  $\omega_1 = 2\pi/20$  and  $\omega_2 = 2\pi/4$ . Such a band-pass filter cancels all fluctuations in a time series with a period length longer than twenty and shorter than four time periods (years). Despite the fact that the weights of

<sup>33</sup> Note that for the ease of notation in the following sections the short notation  $G(\omega) = G(\exp(-i\omega))$  is used.



the filter are real valued it cannot be applied directly in practice because of the infinite number of lags required. As pointed out by Koopmans (1974, p. 176-185) it is therefore not possible to apply such an ideal filter to a finite length data set, since its construction requires an infinite number of filter weights. In practice it seems that therefore we are forced to settle for an approximation of an ideal filter.

Figure 5.8 Weights (5.3.6) of IRF of zero phase band-pass filter with  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$ .



Because the amplitude of the weights tends towards zero it is logical to simply cut off the weights for all  $l > |a|$  for some value of  $a$ . To analyze the effects of such an approximation note that we are in effect applying the FRF

$$G^*(\omega) = \sum_{l=-a}^a g_l \exp(-i\omega l) = \sum_{l=-\infty}^{\infty} w_l g_l \exp(-i\omega l) \quad (5.3.7)$$

with the weights

$$w_l = \begin{cases} 1 & \text{if } |l| \leq a \\ 0 & \text{if } |l| > a \end{cases} \quad (5.3.8)$$

Expression (5.3.7) gives the ideal FRF based on an infinite IRF but with each weight  $g_l$  multiplied by a factor  $w_l$  which can be zero or one. Substituting (5.3.4) for  $g_l$  with  $a=\infty$  and  $b=\infty$  into (5.3.7) and rearranging gives the convolution

$$G^*(\omega) = \int_0^{2\pi} G(\theta) \left\{ \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} w_l \exp(-il(\omega - \theta)) \right\} d\theta = \int_0^{2\pi} G(\theta) W(\omega - \theta) d\theta \quad (5.3.9)$$

in which  $W(\cdot)$  is the Dirichlet kernel that we encountered before in section 4.6.2.

$$W(\theta) = \frac{1}{2\pi} \left\{ \frac{\sin[(a + 1/2)\theta]}{\sin(\theta/2)} \right\} \quad (5.3.10)$$

Substituting for the ideal FRF  $G(\omega)$  of a band-pass filter gives as the FRF of the approximating band-pass filter

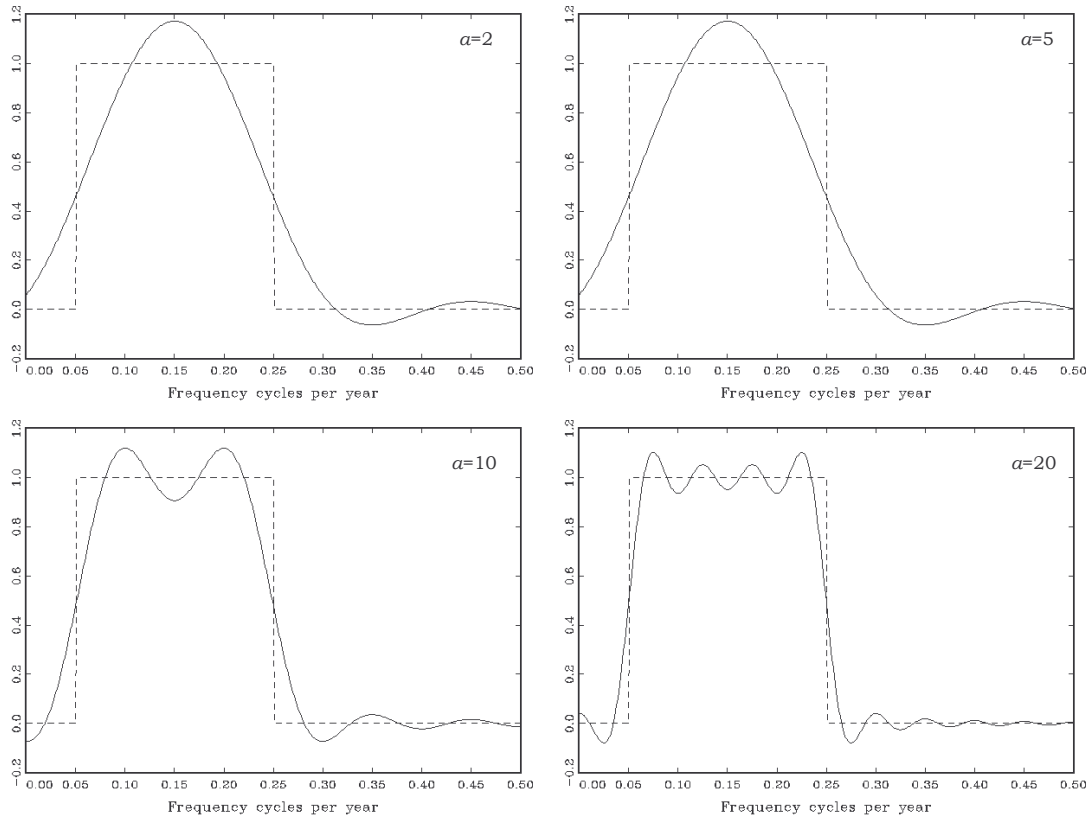
$$G^*(\omega) = \int_{\omega_1}^{\omega_2} W(\omega - \theta) d\theta \quad (5.3.11)$$

Directly calculating this integral is difficult but because we know that the weights  $w_l$  are symmetrical around  $l=0$ , (5.2.6) applies and we can write the first part of (5.3.9) as

$$\begin{aligned} G^*(\omega) &= \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} \left\{ 1 + 2 \sum_{l=1}^a \cos(l(\omega - \theta)) \right\} d\theta \\ &= \frac{\omega_2 - \omega_1}{2\pi} + \sum_{l=1}^a \left\{ \frac{1}{\pi} \int_{\omega_1}^{\omega_2} \cos(l(\omega - \theta)) d\theta \right\} \\ &= \frac{\omega_2 - \omega_1}{2\pi} + \sum_{l=1}^a \frac{\sin(l(\omega - \omega_1)) - \sin(l(\omega - \omega_2))}{\pi l} \end{aligned} \quad (5.3.12)$$

Figure 5.9 shows this expression for the approximating FRF for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  and values of  $a=2, 5, 10$  and  $20$ . In each of the four cases also the corresponding ideal FRF is plotted.

Figure 5.9 FRF of approximating band-pass filters as given by (5.3.12) for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  and different values of  $a$  including the corresponding ideal FRF.



The figure clearly shows that as more weights are included, the approximation comes closer to the ideal FRF. The over and undershoots at the edges of the pass-band are due to the Gibb's phenomenon that we also encountered in section 4.6.2 with the analysis of the leakage effect when calculating a periodogram based on a truncated series of autocovariances. Just as in that case, the shape of the Dirichlet kernel as the Fourier transform of a discontinuous function (the FRF of a band-pass filter) explains the observed behavior. Rabiner and Gold (1975) note with respect to this point that "*direct truncation of the series leads to the well known Gibb's phenomenon however, which manifests itself as a fixed percentage overshoot and ripple before and after an approximated discontinuity in the frequency response*" and "*the largest ripple . . . is about 9% of the discontinuity and its amplitude does not decrease with increasing impulse response duration. Instead the overshoot is confined to a smaller and smaller frequency range*". These properties of the approximating band-pass filters are clearly visible in Figure 5.9. Just as was the case for the non-parametric spectral estimators, the averse effects of the Gibb's phenomenon can be decreased by eliminating the discontinuities in the ideal FRF and allowing for a more smooth transition from zero to one and vice versa at the edges of the pass-band. The cost of this is off course a less ideal FRF.

To achieve such a smooth transition Schmidt (1984) proposes the function

$$f(\omega) = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi(\omega - \omega_1)}{(\omega_2 - \omega_1)}\right) \quad \text{for } \omega_1 \leq \omega \leq \omega_2 \quad (5.3.13)$$

for a transition between  $f(\omega_1)=1$  and  $f(\omega_2)=0$  and the function  $1-f(\omega)$  for a transition between  $f(\omega_1)=0$  and  $f(\omega_2)=1$ . Here the interval between  $\omega_1$  and  $\omega_2$  is called the *transition band*.

There are two more things to be said about the approximating band-pass filters. First, because of the symmetrical shape of the IRF, in finite samples further adjustments will have to be made for the first and last sections of the observations. Examples of such adjustments are given by Hassler et al. (1992), Baxter and King (1995, 1999) and Christiano and Fitzgerald (1999). Second, there is the very legitimate question whether simply truncating the weights of the infinite IRF, given the number of weights allowed, actually gives the best approximation of an ideal band-pass filter? The answer to this question is that truncating the weights indeed gives the best approximation of the ideal FRF following an OLS criterion. That is, the solution to the optimization problem

$$\min_{g_{-a}, \dots, g_a} \int_0^{2\pi} \left| G(\omega) - \sum_{l=-a}^a g_l \exp(-i\omega l) \right|^2 d\omega \quad (5.3.14)$$

where  $G(\omega)$  is the FRF of an ideal band-pass filter, is in Appendix B.13 proved to be

$$g_l^* = \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \exp(i\omega l) d\omega \quad \text{for } -a \leq l \leq a \quad (5.3.15)$$

which is independent of the value of  $\alpha$ . Because these optimal weights are identical to the relevant weights (5.3.4) of the infinite IRF, this tells us that there is in an OLS sense no better approximation of an ideal band-pass filter possible than by simply truncating the weights of its infinite IRF.

To assess the effects in the time domain of using an approximating band-pass filter instead of an ideal filter let's look at a sample of size  $T=200$  of the following stylized time series.

$$x_t = trend_t + long_t + cycle_t \quad (5.3.16)$$

with

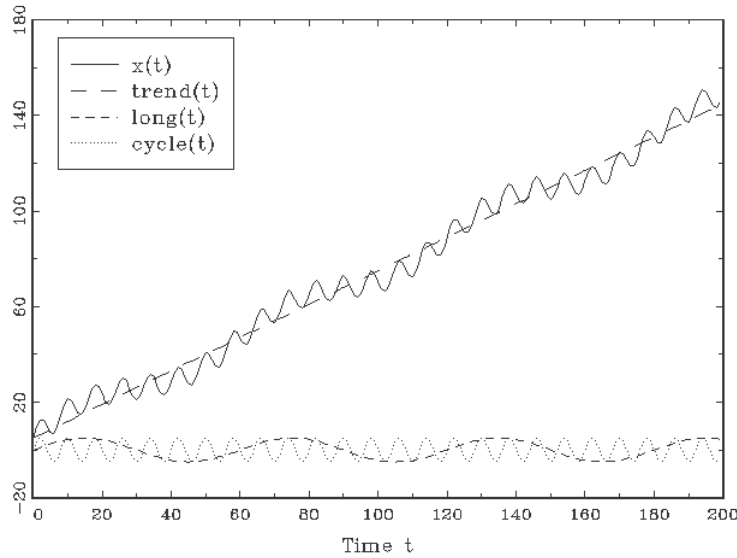
$$trend_t = 5 + 0.7 \cdot t$$

$$long_t = 5 \sin(2\pi/60 \cdot t)$$

$$cycle_t = 5 \sin(2\pi/8 \cdot t)$$

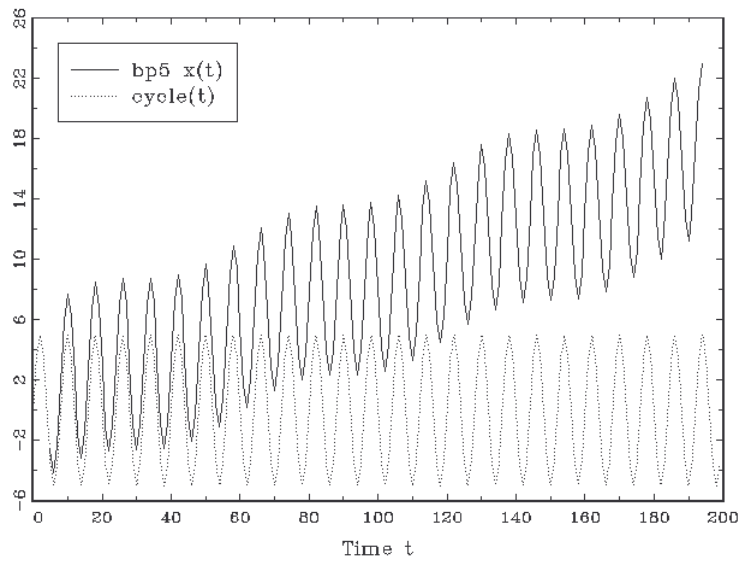
This time series consists of three components which based on the literature as discussed in section 2.2.3, may typically be present in real life time series of economic variables. It consists of a (logarithmic) linear trend, a long wave component with a period length of sixty years and a business cycle with a period length of about eight years. The time series and its components are shown in Figure 5.10.

Figure 5.10 Sample of size  $T=200$  of time series (5.3.16).



Suppose now that we would like to isolate the business cycle fluctuations of this time series by applying an approximating band-pass filter. Furthermore suppose that a-priori we define these business cycles as being fluctuations with a period length between four and twenty years. If we choose to apply the approximating band-pass filter with parameter  $\alpha=5$ , which has the FRF as displayed in the top right corner of Figure 5.9, we get the filter output shown in Figure 5.11.

Figure 5.11 Filtered time series (5.3.16) using an approximate band-pass filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  and  $a=5$  and the original cycle component.



First of all note that because  $a=5$  the IRF of the filter has eleven weights which cause the loss of five observations at the start and at the end of the sample. More important is however that the filter output is very different than the ideal output consisting of just the cyclical component. After all, the cyclical component is the only component of the time series which lies within the pass-band of the filter. All other frequencies should be suppressed by the filter. Besides the cycle however, the filtered time series also still contains a part of the trend component. A linear trend can be seen as a periodic component with a very long period length and can therefore be associated with a frequency close to  $\omega=0$ . The frequency  $\omega=0$  itself represents a constant terms in a time series which in a sense is a periodic component with an infinite period length. Although the ideal pass-band filter is exactly zero for all frequencies around  $\omega=0$ , this is not the case for the approximation for  $a=5$  as can be seen from Figure 5.9. Such a filter having a non-zero value of  $G(0)$  passes a portion of the variance at that frequency which explains the trending behavior in the filtered time series. For the same reason the approximate band-pass filter also did not succeed in fully deleting the long wave from the time series. Figure 5.9 shows that increasing the number of weights in the approximating FRF does reduce the filter error in general but the value of  $G(0)$  remains different from zero. Also this would lead to the loss of more observations at both ends of the sample. We have to conclude that an approximating band-pass filter of the form presented here is not well suited for the current filter objectives. The Baxter-King filter as discussed in the next section provides a solution to the problem of  $G(0)$  being different from zero or some other desired value.

### 5.3.2 Baxter-King filter

One specific problem of the approximating band-pass filters as discussed in the previous sub-section is that the corresponding FRF  $G(\omega)$  hasn't the desired value at the important frequency  $\omega=0$ . In the previous example the ideal value was  $G(0)=0$  for the relevant band-pass filter but it may also be  $G(0)=1$  in case of a low-pass filter. In this section it is shown how a simple adjustment of the weights of an approximating band-pass filter can ensure that the FRF has exactly the desired value at frequency  $\omega=0$ . The analysis presented next is very much based on, though not exactly the same as, the analysis presented in Baxter and King (1995, 1999). The resulting type of filters will be referred to as Baxter-King filters.

We start by repeating the relations (5.3.4) and (5.3.3) between the IRF and the FRF of a linear filter, assuming an equal number of weights  $a$  on both sides of  $g_0$ .

$$g_l = \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \exp(i\omega l) d\omega \quad \text{for } -a \leq l \leq a \quad (5.3.17)$$

and

$$G(\omega) = \sum_{l=-a}^a g_l \exp(-i\omega l) \quad (5.3.18)$$

By substituting  $\omega=0$  into (5.3.18) and wanting  $G(0)$  to have the general value of  $\alpha$  (say zero or one), we get the simple restriction

$$G(0) = \sum_{l=-a}^a g_l = \alpha \quad (5.3.19)$$

That is, if we want  $G(0)$  to have some specific value  $\alpha$ , this is the same as requiring that the weights of the IRF to sum to the same value. If we furthermore assume symmetrical weights to ensure a zero phase filter this means that the appropriate weights can be obtained by first calculating the regular weights of an approximate band-pass filter as given by (5.3.6). Next these weights can be adjusted to satisfy the restriction (5.3.19) by changing each weight by the same  $\theta$  so that

$$\sum_{l=-a}^a g_l^* = \sum_{l=-a}^a (g_l + \theta) = \alpha \quad (5.3.20)$$

The appropriate value of  $\theta$  is then

$$\theta = \frac{\alpha - \sum_{l=-a}^a g_l}{2a+1} \quad (5.3.21)$$

Of course, also other adjustments of the weights are possible to satisfy the restriction. However Baxter and King (1995, 1999) show that adding the same constant value to all weights gives the best approximation from a restricted OLS

point of view relatively to the FRF of an ideal band-pass filter. Before applying a Baxter-King band-pass filter to the example at the end of the previous sub-section, note the following special importance of the value of the FRF of a filter at the frequency  $\omega=0$ . Based on Baxter and King (1995, 1999), Appendix B.14 shows that

$$G(0) = \sum_{l=-a}^a g_l = 0 \Leftrightarrow G(L) = (1-L)H(L) \vee G(L) = (1-L^{-1})I(L) \quad (5.3.22)$$

and

$$G(0) = \sum_{l=-a}^a g_l = 0 \wedge g_l = g_{-l} \Leftrightarrow G(L) = (1-L)(1-L^{-1})J(L) \quad (5.3.23)$$

in which  $G(L)$  is the lag polynomial (5.1.3) of the linear filter having the weights  $g_l$ .  $H(L)$ ,  $I(L)$  and  $J(L)$  are remainder polynomials in the lag operator  $L$ . To understand the importance of these two relations remember that a lot of economic time series show trending behavior (also see section 2.2.6) and are thereby non-stationary. Models for such a process  $x_t$  can consist of a stationary process  $\varepsilon_t$  (not just a white noise process but any stationary process) around a *linear* or *quadratic trend* as in

$$x_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \varepsilon_t \quad (5.3.24)$$

or around a *stochastic trend* as in

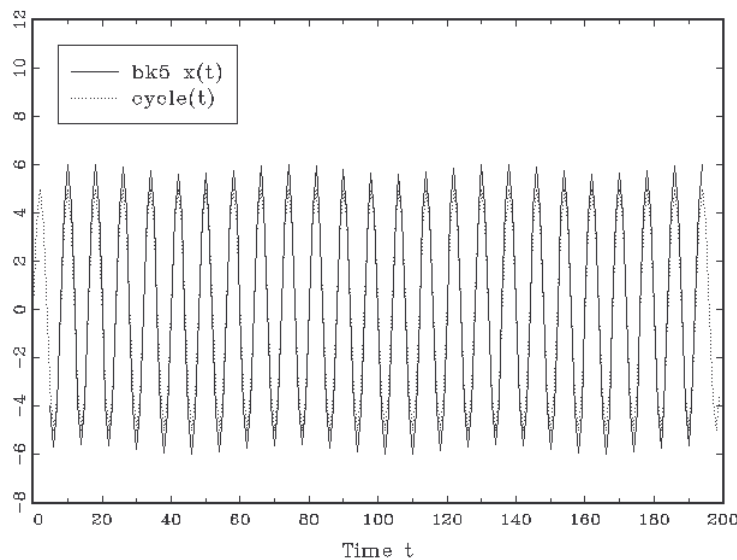
$$x_t = x_{t-1} + \varepsilon_t \quad (5.3.25)$$

In the case of (5.3.24) type of non-stationarity, the linear or quadratic trend has to be eliminated from the process while in the case of (5.3.25) type of non-stationarity it suffices to calculate the first order differences of the time series to obtain the stationary process  $\varepsilon_t$ . The importance of (5.3.22) now is that it shows that if the weights of a linear filter sum to zero, this ensures that the filter deletes both (first order) stochastic and linear trends from a time series if these are present. After all, the operation  $(1-L)$  transforms the time series into a stationary one, while the remainder polynomial only transforms this stationary process into another stationary process. Relation (5.3.23) shows that if on top of the weights summing to zero, they also are symmetrical around  $l=0$ , the filter also deletes quadratic trends and second order stochastic trends from a time series process. So the two relations (5.3.22) and (5.3.23) tell us that for a zero phase linear filter, it is very important for the FRF to have a zero value at the lowest frequency  $\omega=0$  in order for possible linear, quadratic or stochastic trends to be adequately deleted from a time series (process). Furthermore, note that, although such a filter results in a stationary time series (process), it does not guarantee that the filtered series is identical to the stationary process  $\varepsilon_t$  in the theoretical models (5.3.24) and (5.3.25). After all, in general the filter does not only apply the  $(1-L)$  type transformations, it also applies a remainder lag polynomial such as  $H(L)$ ,  $I(L)$  or  $J(L)$  on the stationary process  $\varepsilon_t$ , transforming it into another stationary process.



Let's now reconsider the example at the end of the previous sub-section by applying the same approximate band-pass filter for  $\alpha=5$  but now with the additional Baxter-King restrictions of the weights summing to zero and thereby ensuring that  $G(0)=0$ . Figure 5.11 shows the filtered time series and the original cycle component of the time series (5.3.16). From comparing this figure to the previous filter output without the Baxter-King restriction as shown in Figure 5.10 the difference is clear. Because now  $G(0)=0$  the filter has completely deleted the linear trend. As was to be expected however the filter still does not completely delete the long wave component from the time series. The reason is that the additional restriction only makes sure that the FRF passes through  $G(0)=0$  while the general shape of the approximating FRF hardly changes from what is shown in Figure 5.8. That is, the filter still only approximates the FRF of the ideal band-pass filter. However, because in this specific example the trend is the most dominant feature of the time series only making sure that this is deleted correctly already results in a filtered time series which very much looks like the ideal filter output. If other components, for example the long wave, would have been more dominant, the result could have been (much) worse. So the special importance of the zero frequency does not mean to say that applying a filter that "only" differs from the ideal filter at the non-zero frequencies cannot result in (very) poor filter results. We have to conclude that, although the Baxter-King filter provides a better approximation of the ideal band-pass filter than the filter from section 5.3.1, also this filter it is not well suited for the current filter objectives. An alternative but related approach is the Christiano-Fitzgerald filter to which we turn in the next sub-section.

Figure 5.11 Filtered time series (5.3.16) using a Baxter-King filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  and  $\alpha=5$  and the original cycle component.



### 5.3.3 Christiano-Fitzgerald filter

Besides the Baxter-King filter as described in the previous section, a filter that is also used more often is described in Christiano and Fitzgerald (1999). This filter is another special version of the approximating band-pass filters described in section 5.3.1. The Baxter-King filter adds to these basic filters the restriction of a specific value of the FRF at the frequency  $\omega=0$ . The Christiano-Fitzgerald filter aims to deal as optimal as possible with the fundamental problem of having to filter finite time series based on a time domain representation of ideal pass-band filters that in principle require time series of infinite length. This is done by minimizing the second moment of the difference between the ideally filtered time series  $y_t$  and the approximation of this time series  $\hat{y}_t$ . The filtered time series are again linear combinations of the available sample time series  $\{x_t, t=1, \dots, T\}$ <sup>34</sup>. The objective then becomes to minimize the following expression in terms of the weights of the filter.

$$\min E \left\{ (y_t - \hat{y}_t)^2 \middle| x_1, \dots, x_T \right\} \quad (5.3.26)$$

Christiano and Fitzgerald find the solution of this problem under the assumption of knowing the “true” time series representation of the series  $x_t$  that needs to be filtered. This needs to be a covariance stationary process with mean zero, either with or without a unit root. This means that possible trends have been eliminated from the time series before filtering. Note that this poses methodological problems when the objective of the filtering is exactly to remove these trends. In practice off course the true time series representation of  $x_t$  is not known and must be estimated. As Christiano and Fitzgerald claim however, for standard macro economic time series, a more straightforward approach works well. This approach uses the approximation that is optimal under the assumption that the data in  $x_t$  are generated by a pure random walk model without drift. Although this assumption is most likely false in most cases, it is nearly optimal for most types of time series encountered in economics. That is, in terms of the quality of the filter output, there is relatively little gain in knowing the precise details of the time series representation of  $x_t$ . By making this assumption, implicitly the time series  $\{x_t, t=1, \dots, T\}$  is extended backwards and forward into infinity with the value of respectively the first and last observation from the sample, which reduces the filter error. The solution to this optimization problem is as follows. To filter the time series  $x_t$  by a filter with a pass-band between the frequencies  $\omega_1=2\pi/p_l$  and  $\omega_2=2\pi/p_s$  where  $p_l$  and  $p_s$  are respectively the longest and shortest periods of oscillations to be passed with  $0 \leq p_s < p_l < \infty$ , the approximated filtered time series  $\hat{y}_t$  is calculated as

$$\hat{y}_t = \tilde{g}_{t-1}x_1 + g_{t-2}x_2 + \dots + g_1x_{t-1} + g_0x_t + g_1x_{t+1} + \dots + g_{T-1-t}x_{T-1} + \tilde{g}_{T-t}x_T \quad (5.3.27)$$

for  $t=3, 4, \dots, T-2$ .

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<sup>34</sup> Note that for convenience here we index the time series from 1 until  $T$  instead of from 0 until  $T-1$  as in the other parts of the text.

Most of the weights are equal to

$$g_l = \begin{cases} \frac{\sin(\omega_2 l) - \sin(\omega_1 l)}{\pi l} & \text{for } l > 0 \text{ and } l \neq t-1, l \neq T-t \\ \frac{\omega_2 - \omega_1}{\pi} & \text{for } l = 0 \end{cases} \quad (5.3.28)$$

while the adjusted weights at the end points are equal to

$$\begin{aligned} \tilde{g}_{T-t} &= -\frac{1}{2}g_0 - \sum_{l=1}^{T-t-1} g_l \\ \tilde{g}_{t-1} &= -(g_{t-2} + \dots + g_1 + g_0 + g_1 + \dots + g_{T-1-t} + \tilde{g}_{T-t}) \end{aligned} \quad (5.3.29)$$

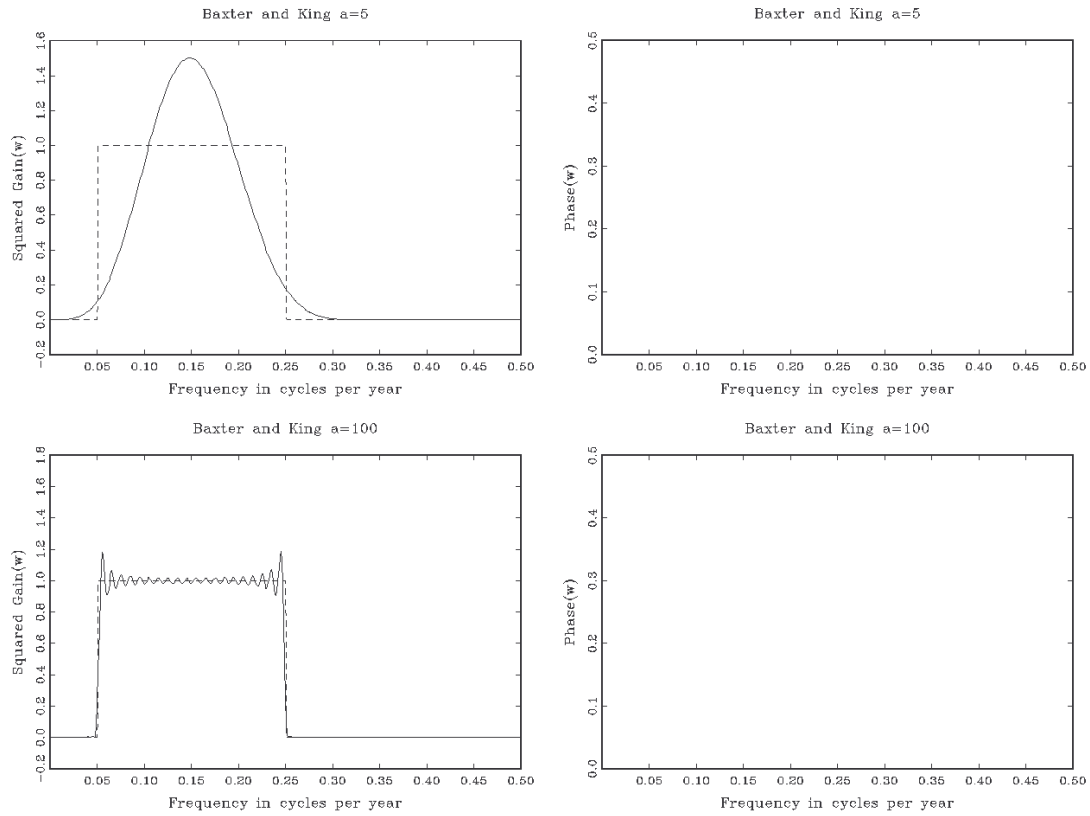
Regarding this weighting scheme of the Christiano-Fitzgerald filter the following observations can be made:

- The filter weights vary over time. For the construction of each filtered value  $\hat{y}_t$ , other filter weights are used. Christiano and Fitzgerald find that the biggest gains in terms of minimizing the distance metric (5.3.26) comes from allowing this non-stationarity in the filter weights.
- The filter weights are asymmetrical except for the weights to be applied to obtain  $\hat{y}_{(T+1)/2}$  in case  $T$  is an uneven number. That is, only exactly in the middle of the sample the filter weights are symmetrical. From section 5.2.2 we know that this implies that for most values of  $t$  in the sample, the filter will cause phase shifts. Christiano and Fitzgerald find that besides non-stationarity, asymmetry is also valuable in minimizing the distance metric, although to a lesser extent. Fortunately, they also find that the cost of these features, in terms of the quality of the filter output, is relatively minor.
- For the most part, the weights to be applied are equal to the weights of the approximate band-pass filter from (5.3.6) only know extending as far left and right as possible from the specific value of  $t$  until the near ends of the sample. At these near ends of the sample, the weights are adjusted as linear functions of the other weights.
- Although much less than for the Baxter-King filter, in the random walk case there is a loss of two observations on both sides of the sample. For higher order model assumptions about the underlying time series process, more observations will be lost.
- For most values of  $t$  in the sample, the number of weights included will be bigger than generally applied by a stationary Baxter-King or unadjusted approximate band-pass filter. From Figure 5.9 we know that this leads to a better approximation of the ideal FRF and thereby of the ideal filter output.

To illustrate these points first consider Figure 5.12. This shows the PTF (i.e. the squared gain) and the phase of the Baxter-King filter with  $\alpha=5$  for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  as applied for the results in Figure 5.11. The approximation is clearly not optimal because of the relatively low value of  $\alpha$  needed to keep the loss of observations on both sides of the sample limited. The phase is however zero for all frequencies showing that this filter causes no phase shifts. This figure also shows the PTF and phase of the same Baxter-King filter but now with much more weights by

setting  $\alpha=100$ . Including more weights clearly gives an improved fit of the ideal PTF and still causes no phase shifts. The fluctuations around the ideal PTF however remain visible because of the Gibb's phenomenon described in section 5.3.1.

Figure 5.12 PTF (squared gain) and phase of Baxter-King filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  in case of  $\alpha=5$  (top) and  $\alpha=100$  (bottom).

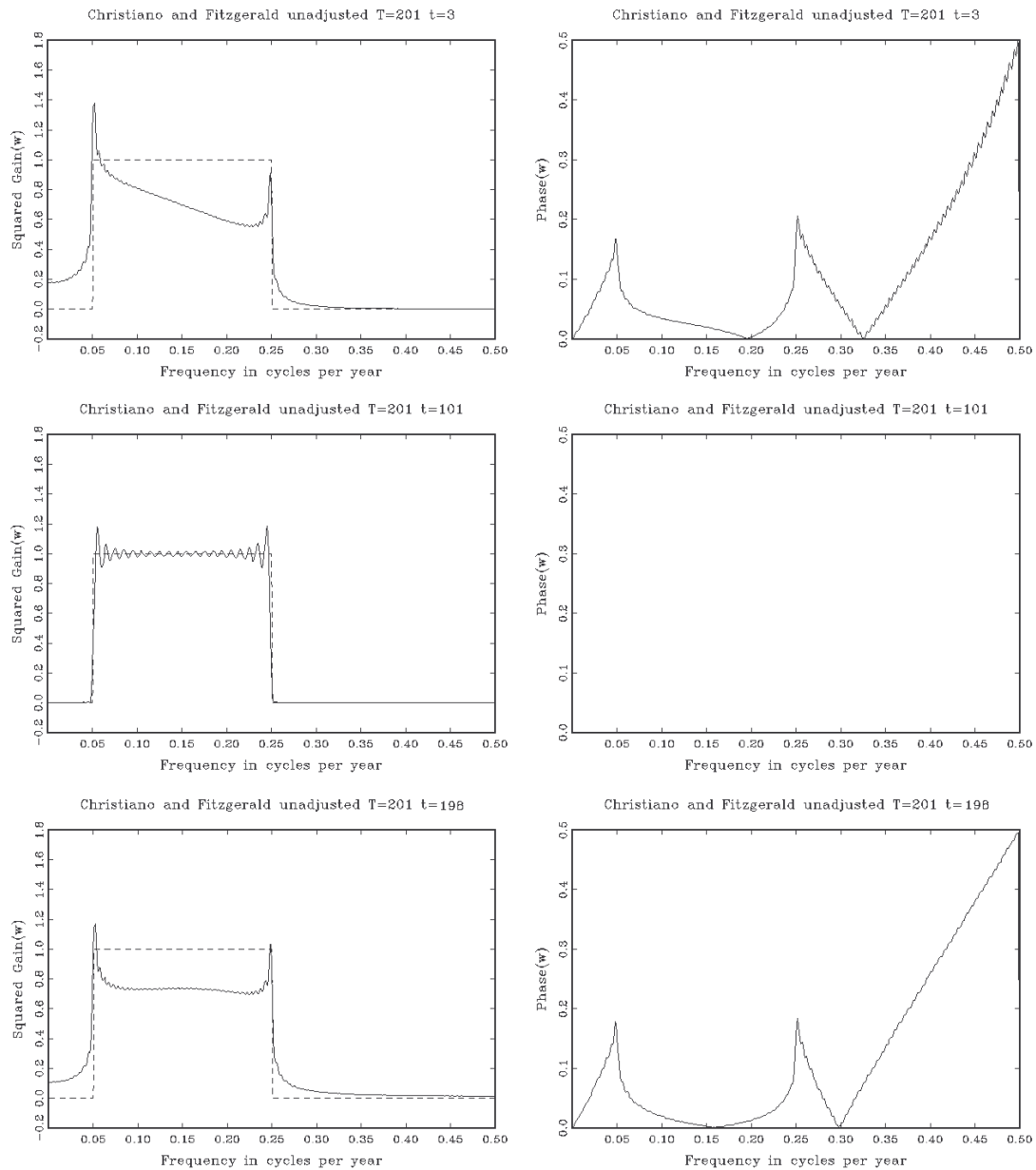


Second, consider Figure 5.13 which shows the PTF and phase of the Christiano-Fitzgerald filter but than *without* the adjustment of the end weights as in (5.3.29) but leaving them zero instead. This comes down to simply applying the weights from the approximate band-pass filter from (5.3.6) as far as possible to the beginning and end of the sample for each value of  $t$ . Because the weights vary over time (i.e. are non-stationary) it is no longer sufficient to look at only one PTF and phase. Therefore we first assumed the filter to be applied on the time series (5.3.16) with  $T=201$  to try and isolate the cycle component. We then calculated the PTF and phase three times: in the beginning of the sample ( $t=3$ ), in the middle of the sample ( $t=101$ ) and at the end of the sample ( $t=198$ ). From this figure we can observe the following:

- Exactly in the middle of the sample, the filter is identical to the Baxter-King filter with  $\alpha=100$  from Figure 5.12 and gives an adequate approximation of the ideal PTF and induces no phase shifts.
- In the beginning and at the end of the sample, the approximation of the ideal PTF is less good. The PTF of the filter is kind of skewed and also does not have the desired value of zero at the important frequency  $\omega=0$ . Furthermore, over virtually the entire pass-band, the PTF is substantially lower than the ideal PTF which implies a too low variance of the filtered series.

- In the beginning and at the end of the sample, the filter causes undesired phase shifts. This was to be expected because the weights of the filter are strongly asymmetrical in those cases. However, it seems that for the frequencies that lie in the pass-band between the frequencies  $\omega_1=0.05$  and  $\omega_2=0.25$  and hence the frequencies that are supposed to be passed by the filter, the phase shift is kept relatively small.

Figure 5.13 PTF (squared gain) and phase of Christiano-Fitzgerald filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  *without* adjusting the end weights for a sample of  $T=201$ . The top row shows the results in the beginning of the sample ( $t=3$ ), the middle row in the middle of the sample ( $t=101$ ) and the bottom row at the end of the sample ( $t=198$ ).



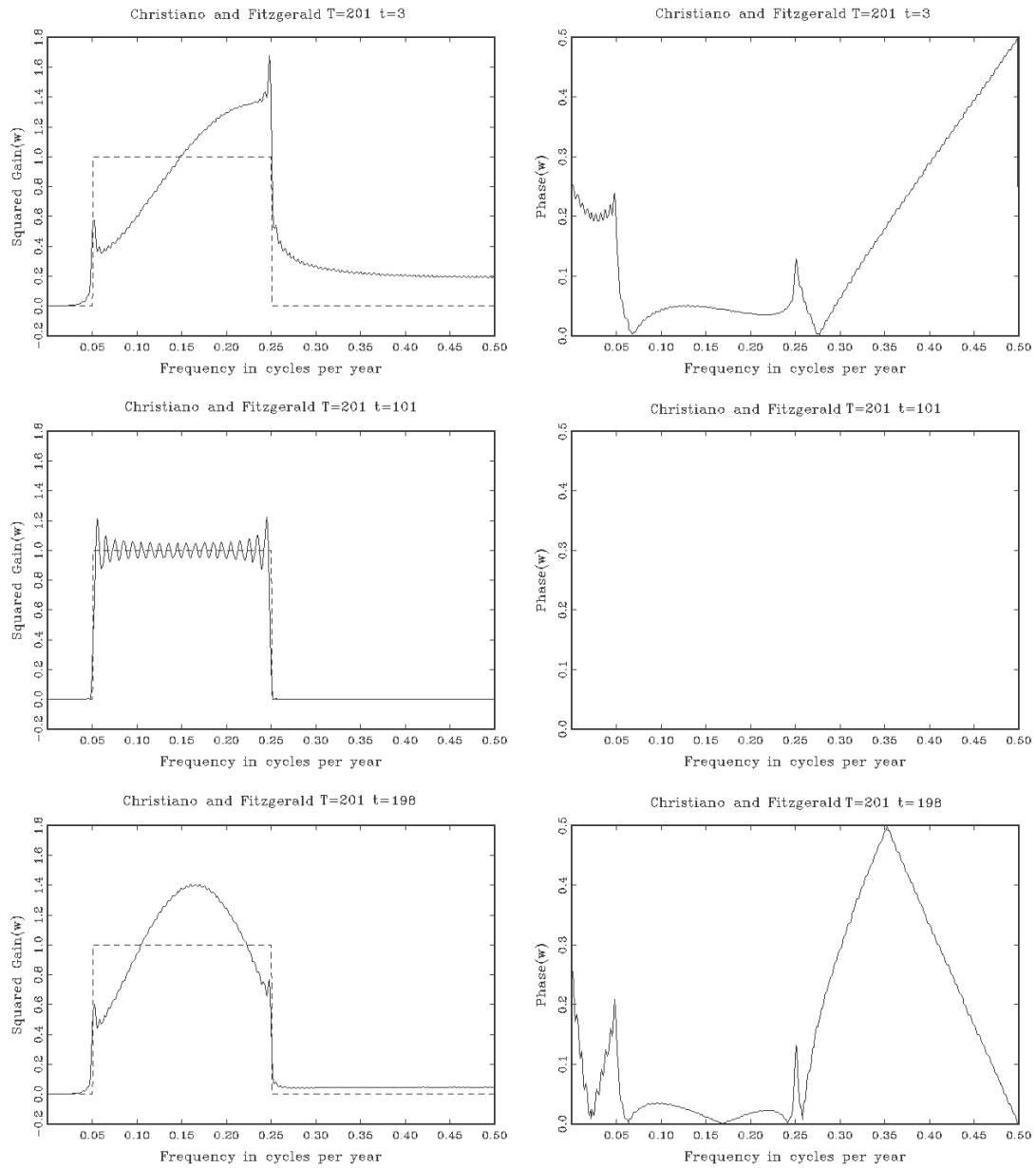
Third and finally, consider Figure 5.14 which shows the PTF and phase of the Christiano-Fitzgerald filter but than *with* the adjustment of the end weights as in (5.3.29). So, this is the full recommended Christiano-Fitzgerald filter based on the random walk assumption for the underlying time series process. In this way we can assess the impact of the end weight adjustments on the quality of the filter output when compared to the simple approach of using as much of the weights of the approximate band-pass filter as possible, given the sample size. The contents of the figure are identical to those of Figure 5.13 so that they can be directly compared. From this comparison we can observe the following:

- Exactly in the middle of the sample, the filter is no longer exactly but still virtually identical to the Baxter-King filter with  $\alpha=100$  from Figure 5.12. The volatility of the fluctuations around the ideal PTF has increases a little because of the adjustments of the end weights. In this case, the filter still induces no phase shifts.
- In the beginning and at the end of the sample, the adjustments of the end weights has not resulted in an exact replication of the ideal PTF. It is clearly still an approximation but now an optimal one in the sense of minimizing the distance metric (5.3.26) when assuming an underlying random walk time series process. The improvement can be seen from the fact that now the PTF does have the desired value of zero at the important frequency  $\omega=0$ . Furthermore, on one part of the pass-band the PTF is now lower than the ideal PTF and on another part it is higher which implies a better result with respect to the total variance of the filtered series (which was too low in the case of not adjusting the end weights).
- In the beginning and at the end of the sample, the filter still causes undesired phase shifts which have changed but not significantly improved or worsened. This is logical because the adjustment of the end weights has not changed the asymmetry in the filter weights.

Now we understand the workings of the Christiano-Fitzgerald filter and its effects in terms of PTF and phase shifts, we are ready to test the filter on the case of trying to isolate the cycle component from the time series (5.3.16). Figure 5.15 and 5.16 respectively show the results for the case *without* and *with* the adjustment of the end weights. Because for most of the sample the filtered series is virtually identical to the original cycle component, we only show the results for the first thirty observations in the sample. The last thirty observations give similar results. From these figures we can observe the following:

- Towards the ends of the sample the quality of the filter output worsens. Minor phase and volatility effects can be observed.
- Adjusting the end weights of the filter improves the quality of the filter output.
- However, in general the extent to which the filter output differs from the optimal filter output is fairly small. So, indeed, as Christiano and Fitzgerald claim, the cost of allowing for time varying weights and asymmetry is relatively minor (in this case) compared to the benefits in terms of better approximating the ideal PTF and losing less observations in the beginning and at the end of the sample.

Figure 5.14 PTF (squared gain) and phase of Christiano-Fitzgerald filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  with adjusting the end weights for a sample of  $T=201$ . The top row shows the results in the beginning of the sample ( $t=3$ ), the middle row in the middle of the sample ( $t=101$ ) and the bottom row at the end of the sample ( $t=198$ ).





To summarize, in this sub-section we have seen that the Christiano-Fitzgerald filter performs a better filtering job than the Baxter-King filter in the sense that it loses less observations at the end of the sample and gives a better approximation of the ideal PTF. The cost of non-stationarity and asymmetry in terms of causing phase shifts seem relatively minor. However, there are also still a number of drawbacks of the filter.

First of all, especially towards the ends of the sample, despite its improved approximation, the Christiano-Fitzgerald filter still does not reflect the exact FRF of an ideal pass-band filter. The consequences in the example considered here are limited but one can not be sure that it will not be different in other empirical situations.

Second, the filter requires a covariance stationary time series as input. In a lot of applications on economic time series this means that some transformation of the data needs to be made before actually filtering the series. Transformations such as detrending a series by estimating a linear trend or using the first order differences are in fact already filter exercises themselves with potential non-optimal and inconsistent properties. See for example the PTF and phase of the first order differencing filter in section 5.2.1.

A third and related drawback is that the filter is based on assumptions about the underlying time series model that generated the data that needs to be filtered. As Christiano and Fitzgerald indicate themselves, because of this, the filter does *not* closely approximate the optimal filter in *all* circumstances. For example, in case of substantial negative autocorrelations in the first order differences of the time series, the proposed filter based on the random walk assumption does not work well. From a methodological point of view, it is troubling to (implicitly) have to make assumptions about the underlying time series model in the early stages of gathering knowledge from empirical data in which filtering approaches will typically be applied. After all, from the point of the standard methodological principles, models should only be proposed and thereafter tested *after* empirical knowledge has been gathered. Also see the discussion in section 2.1.

A fourth and final drawback of the filter is that it is more difficult or even impossible to work with other desired FRF's than that of an ideal band-pass filter. If one for example wants to filter seasonal components, one needs repeated applications of the filter with different pass-bands around the seasonal frequencies. If for some reason one wants to work with smooth and slowly changing pass-bands, this is only possible with the filter after deriving a new optimal approximation, which may not be possible at all.

We have to conclude that, although the Christiano-Fitzgerald filter provides a better approximation of the ideal band-pass filter than the filters from section 5.3.1 and 5.3.2, also this filter it is not ideally suited for the current filter objectives. An alternative approach that could prevent the remaining filter errors and solve the drawbacks mentioned here, is the direct filtering of the time series in the frequency domain instead of in the time domain. We turn to this filtering approach in the next sub-section.

Figure 5.15 Filtered time series (5.3.16) using a Christiano-Fitzgerald filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  *without* adjusting the end weights for a sample of  $T=201$  together with the original cycle component. The results are shown only for the first 30 observations in the sample.

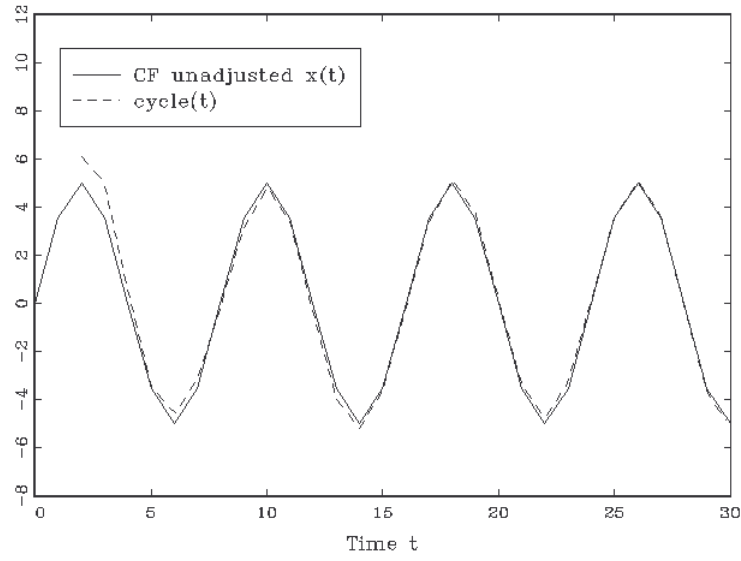
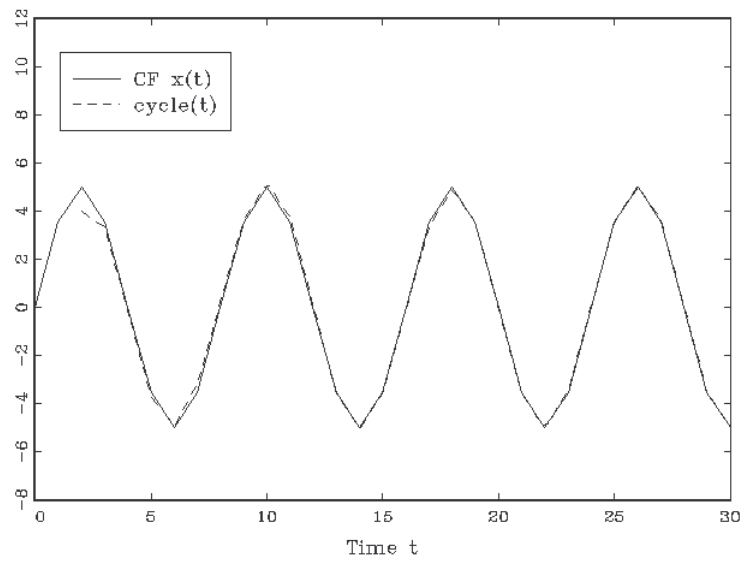


Figure 5.16 Filtered time series (5.3.16) using a Christiano-Fitzgerald filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  *with* adjusting the end weights for a sample of  $T=201$  together with the original cycle component. The results are shown only for the first 30 observations in the sample.



### 5.3.4 Direct frequency filtering

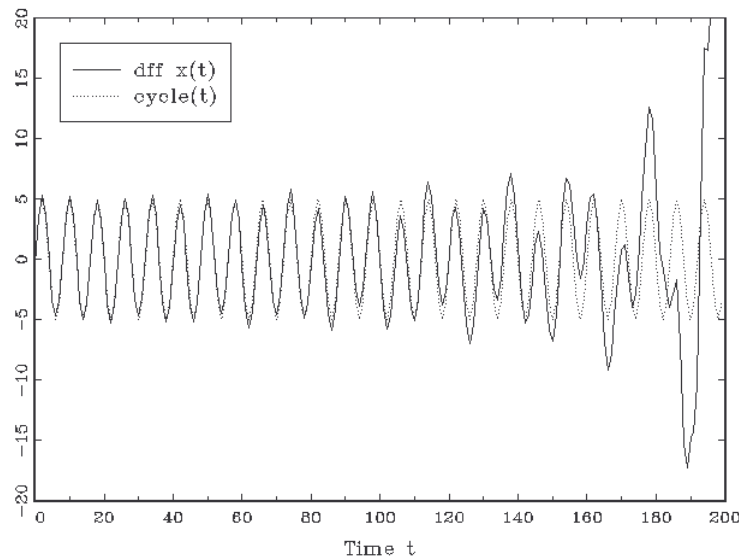
Although the approximating band-pass, Baxter-King and Christiano-Fitzgerald filters are derived from the desired properties in the frequency domain in terms of the gain and phase functions, the actual filtering is still done in the time domain using the weights of the corresponding IRF. Although using the desired properties in the frequency domain is already a great improvement over the more “ad-hoc” filters discussed in section 5.2, such filters, as we saw in the previous sub-sections, can still produce significant errors in the filter output and have a number of practical drawbacks in terms of losing observations and being based on model assumptions about the data. A simple alternative approach, which shall be called *direct frequency filtering*, consists of the following three steps.

1. Calculate the DDFT (4.1.18) of the time series to be filtered on a sufficiently large number of frequencies. That is, transform the time series into the its representation in the frequency domain.
2. Multiply the Fourier coefficients of the DDFT at each of the frequencies with the corresponding value of the FRF of the ideal filter.
3. Use the IDFFT (4.1.19) to transform the filtered series from the frequency domain back into the time domain.

In the second step, the actual filtering takes place in the frequency domain instead of in the time domain as with the other approaches. The obvious advantage of this approach is that the FRF of the ideal filter need no longer be approximated but can be applied directly without modification. Components at frequencies within the pass-band remain unaltered while components at frequencies outside the pass-band are suppressed. The consequence should be that components corresponding to frequencies outside the pass-band are no longer maintained in the filtered series and vice versa. A further advantage of such a filter would be that there is no loss of data at the start and end of the sample. In practical applications a disadvantage can be that the value at time  $t$  of a filtered time series now depends on the values of the time series in the entire sample. That is, if new data comes available, the entire previously filtered series will (slightly) change and hence the filter is not entirely stable. Note that although this does not hold for linear filters in the time domain with a finite number of weights, filters in the time domain with an infinite number of weights suffer from the same drawback. In order to reduce errors caused by the Gibb’s phenomenon when transforming the filtered series back into the time domain (step three), it is recommended that instead of using the ideal, discontinuous, FRF in step two, a slightly different FRF is used which has continuous transitions at the edges of the pass-band. For this purpose the transition function (5.3.13) can be used.

Figure 5.17 shows the results of applying such a direct frequency filter, again to try and isolate the cycle component from the time series (5.3.16). The FRF used here has transition bands between the frequencies 0.045 and 0.05 (from zero to one) and between the frequencies 0.25 and 0.28 (from one back to zero).

Figure 5.17 Filtered time series (5.3.16) using a direct frequency filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  and the original cycle component.



For about the first half of the sample, the filtered series matches nicely with the actual cycle component. However, the most striking about the filtered series are of course the large errors that occur at the end of the sample. At first sight, these errors come as a surprise. After all, we are now applying a filter with (almost) the ideal FRF so the errors due to approximating this ideal FRF that appeared with the previous two filters can no longer occur. Although indeed the errors here do not come from this approximation, they come from the very closely related leakage effect. The leakage effect was discussed in section 4.2 and in section 4.6.2. When, in the first step, transforming the time series into the frequency domain through applying the DDFT, due to the leakage effect, information about the three frequencies present in the time series (trend, long wave and cycle) is spread out over adjacent frequencies following the shape of the Dirichlet kernel as shown in Figure 4.11. Although these adjacent frequencies are not actually present in the time series, they do show up in its Fourier transform. If, in the second step, the (almost) ideal FRF is applied to this erroneous frequency domain representation, some parts of the time series will be deleted that should have been preserved and vice versa.

Note that a special version of the approach described and tested here is equal to the trigonometric regression procedure as described in Christiano and Fitzgerald (1998, 1999). This trigonometric regression procedure in principle follows the same three steps as the direct frequency filtering approach only now by restricting the frequencies involved in the DDFT and IDFFT to the standard Fourier frequencies (4.1.6). Because in this case, as we know from the results in (4.1.8) and (4.1.9), the Fourier numbers can also be obtained from the familiar OLS criterion when regressing the time series on the separate Fourier frequencies, the filtered series can be explicitly written down in terms of sine and cosine functions with (Fourier) frequencies lying within the pass-band of the filter and the original time series. The latter is in principle also possible for the direct frequency filtering approach described here. Similar to our results, Christiano and Fitzgerald (1999) find this trigonometric regression procedure not to work well.

We already know that the disturbing effects of leakage reduce as the sample size increases. Also we know that the leakage effect will be greater for greater peaks in the periodogram, that is, for components that have a greater part in the total variance of the time series. To see whether these two effects also hold for the filter errors observed here, consider the following two time series.

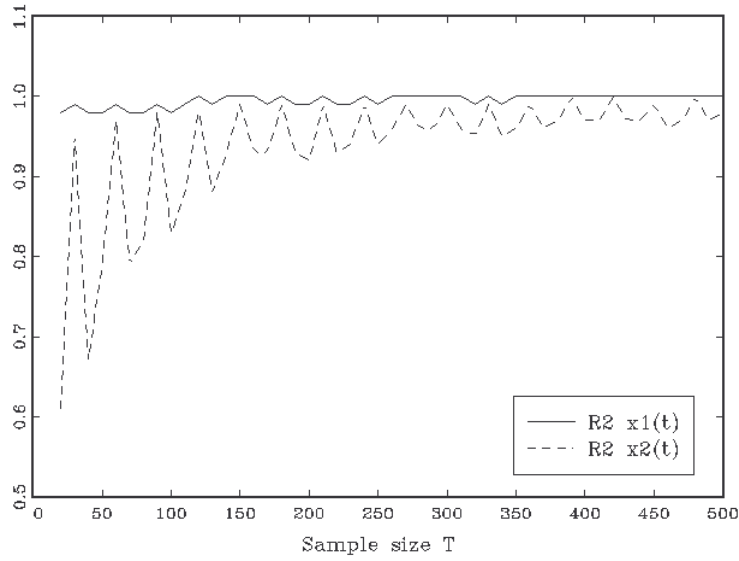
$$\begin{aligned}x_{1,t} &= long_t + cycle_t \\x_{2,t} &= long_t^* + cycle_t\end{aligned}\tag{5.3.30}$$

where

$$long_t^* = 20 \sin(2\pi/60 \cdot t)$$

and the cycle and other long wave component are the same as in (5.3.16). The first is the same time series that we analyzed before, only without the trend component. The second time series is the same with an amplitude of the long wave component that is four times as big. In the second time series therefore the long wave will be more prominently present and determine a larger part of the total variance. We applied the same direct frequency filter as before on these two time series for sample sizes ranging from  $T=20$  to  $T=500$  with steps of 10 with the objective to isolate the cycle component. For each filtering exercise we then calculated the standard  $R^2$  (3.4.8) was calculated on the filter output and the true cycle component to measure the quality of the filter output. Figure 5.18 presents the results. First of all, these results confirm that as the sample size increases, the disturbances due to the leakage effect and thereby also the filter error decreases. What exactly causes the reduction in the filter error to follow an oscillating pattern is not completely clear. The fact that the peaks in the  $R^2$  fall at sample sizes that are multiples of thirty however rises the suspicion that the leakage effect is less if the long wave, with a period length of sixty, can complete its cycles up to a half of its period length. Second, the results clearly show that the filter error is larger for  $x_{2,t}$  than for  $x_{1,t}$ . Because in the second time series the long wave component has a larger variance, this confirms that the leakage effect and thereby the filter error increases as the size of the peaks in the periodogram of the time series increase.

Figure 5.18  $R^2$  between filtered series and true cycle using a direct frequency filter for the two time series in (5.3.30) for various sample sizes.



To make the two properties of the filter error as shown in the previous experiment explicit, it is also possible to derive an upper boundary on the filter error. A similar derivation with a slightly different outcome can be found in Schmidt (1984, p. 68) and Koopmans (1974, p. 178). Suppose that the theoretical infinite time series to be filtered is  $\{x_t, t=-\infty, \dots, \infty\}$ . The theoretical output from applying a linear filter with an ideal FRF is then the time series

$$y_t = \sum_{l=-\infty}^{\infty} g_l x_{t-l} \quad \text{for } t = -\infty, \dots, \infty \quad (5.3.31)$$

In reality however, we will only have the finite time series  $\{x_t, t=0, \dots, T-1\}$  available and the output from applying the approximate (truncated) linear filter will be

$$\hat{y}_t = \sum_{l=t-T+1}^t g_l x_{t-l} \quad \text{for } t = 0, \dots, T-1 \quad (5.3.32)$$

The quadratic error for the  $t$ -th value of the filtered time series is therefore

$$|y_t - \hat{y}_t|^2 = \left| \sum_{l \in Q} g_l x_{t-l} \right|^2 \quad \text{with } Q = \{-\infty, \dots, t-T\} \cup \{t+1, \dots, \infty\} \quad (5.3.33)$$

Appendix B.15 shows that an upper boundary for this filter error is

$$|y_t - \hat{y}_t|^2 \leq M^2 \left\{ \sum_{l \in Q} |g_l| \right\}^2 \quad (5.3.34)$$

where

$$M = \max_{0 \leq \omega < 2\pi} |T \cdot J(\omega)| \quad (5.3.35)$$

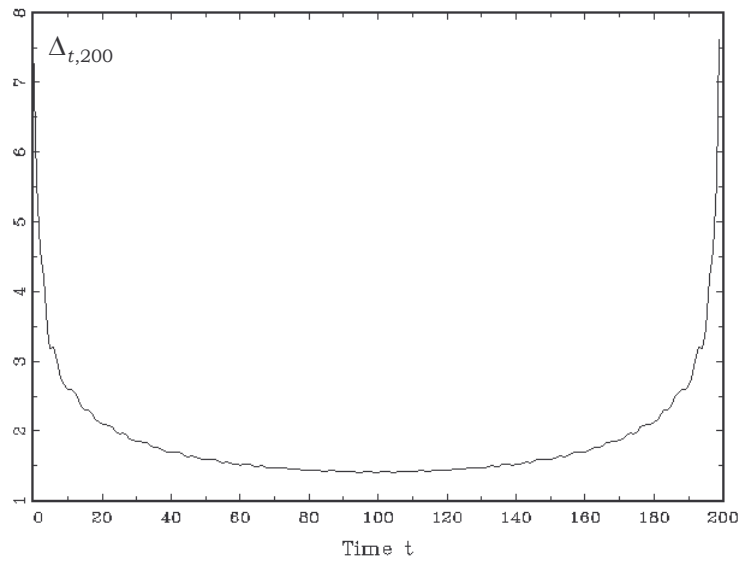
$M$  is the maximum modulus of the DCFT  $J(\omega)$  as defined in (4.1.21) multiplied by  $T$ . The  $T$  term only appears because it is included in the definitions of the DCFT and the IDCFIT for consistency with the DDFT and the IDDDFT. Note that the modulus of  $J(\omega)$ , except for a constant term, is equal to the square root of the periodogram  $P(\omega)$  given by (4.2.1). So, the upper boundary (5.3.34) for the filter error consists of two elements. The first element is defined by the maximum value of the periodogram. As this maximum value decreases, the upper boundary for the filter error of the entire sample decrease. This property of the filter error is consistent with what we know about the leakage effect. As the peaks in a periodogram are lower, the disturbing effects because of the leakage effect decrease. Because (5.3.34) gives only an upper boundary it does not guarantee that in all practical applications the filter error will indeed be smaller in case of a lower maximum value of the periodogram. The second component in (5.3.34) is the term

$$\Delta_{t,T} = \left\{ \sum_{l \in Q} |g_l| \right\}^2 = \left\{ \sum_{l=-\infty}^{t-T} |g_l| + \sum_{l=t+1}^{\infty} |g_l| \right\}^2 \quad (5.3.36)$$

which depends on the filter weights  $g_l$ , the number of the observation  $t$  and the sample size  $T$ . Because the weights of a zero phase filter are symmetrical around  $l=0$  ( $g_l=g_{-l}$ ), it holds that  $\Delta_{t,T}=\Delta_{T-1-t,T}$  which implies that, given a sample size  $T$ , the terms  $\Delta_{t,T}$  as a function of  $t$  are symmetrical around the middle of the sample. For a stable filter it holds that  $|g_l| \rightarrow 0$  as  $|l| \rightarrow \infty$ . This property implies furthermore that  $\Delta_{t,T}$  has its maximum value at the borders of the sample while it has its minimal value in the middle of the sample. These properties are illustrated in Figure 5.19 which shows the values of  $\Delta_{t,T}$  for a sample size  $T=200$  for the ideal version of the band-pass filter applied in the previous examples. The shape of the  $\Delta_{t,T}$  component of (5.3.34) implies that (the upper boundary on) the filter error is smallest in the middle of the sample and largest at the beginning and the end of the sample as we already saw in Figure 5.17.

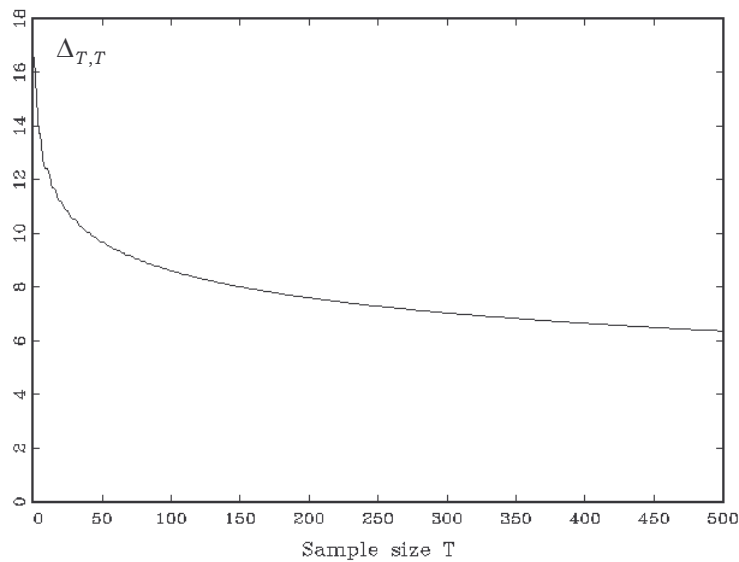


Figure 5.19  $\Delta_{t,T}$  as a function of  $t$  for sample size  $T=200$  for the ideal band-pass filter with  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$ .



To see the effect of the sample size  $T$  on the filter error note that because the maximum errors are at the boundaries of the sample it also holds that  $\Delta_{t,T} \leq \Delta_{T,T}$  for all values of  $t$ . Therefore  $\Delta_{T,T}$  as a function of  $T$  can be used to see how (the upper boundary on) the filter error changes as the sample size increases. Figure 5.20 shows this function for sample sizes  $T=1, \dots, 500$ .

Figure 5.20  $\Delta_{T,T}$  as a function of the sample size  $T$  for the ideal band-pass filter with  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$ .



We see that as the sample size increases, the upper boundary on the filter error decreases which is consistent with that the leakage effect decreases as the sample size increases. The figure also suggest that  $\Delta_{T,T}$  has some limit value. This is indeed the case and from (5.3.36) it is easily seen to be equal to

$$\lim_{T \rightarrow \infty} \Delta_{T,T} = \left\{ \sum_{l=0}^{\infty} |g_l| \right\}^2 \quad (5.3.37)$$

The conclusion of this sub-section is that although by direct frequency filtering the errors because of having to approximate the ideal FRF are avoided, it results in filter errors because of the leakage effect when transforming a time series into the frequency domain. It was shown that these filter errors are larger as the maximum value of the periodogram of the time series is larger. Also the filter errors are largest at the boundaries of the sample and they decrease as the sample size increases.

## 5.4 A zero phase frequency filter

In section 5.2 we saw that most of the conventional filtering techniques in the time domain were not originally designed to have the desired properties in the frequency domain in terms of the gain and phase of the filter. In section 5.3 we saw that also the class of approximating band-pass filters (with or without the Baxter-King adjustments) can yield significant filter errors, especially in case of small samples. Therefore, none of these techniques seems well suited for separating relatively short macroeconomic time series into the desired components such as trends, long waves and business cycles. A direct frequency filter comes rather close to the optimal properties but because of the small sample sizes on the one hand and the fact that the periodograms of macroeconomic time series will typically have a large maximum value caused by the trending behavior on the other, also this technique seems inappropriate. The leakage effect as the source of the filter errors in case of the direct frequency filters however suggests adjustments that can lead to an appropriate filtering technique.

In the first place we know from the previous section that the filter error because of the leakage effect originates from transforming a time series into the frequency domain. If we are able to (partially) avoid this transformation, then the leakage effect and thereby the filter error would decrease. Second, we know that the leakage effect decreases as the sample size increases. Although it is of course impossible to increase the number of observations of an empirical time series, we can try to decrease the maximum value of the periodogram and thereby reduce the filter error. This should be done by removing the dominant components at certain frequencies from the time series. These components should then be filtered separately while the remainder of the time series is filtered using a direct frequency filter. The more we would be able to reduce the maximum value of the periodogram without significantly changing the filter output, the bigger the reduction in the filter error would be. It turns out that such an optimal filtering technique can indeed be constructed. This section describes the workings of what will be called a *zero phase frequency filter* which can be used to filter any, possibly short sampled, time series following any desired gain function while leaving the phase properties of the time series unchanged with as little a filter error as possible. Furthermore, the filter does

not lead to any loss of observations at the beginning and end of the sample. This filtering technique is very much based on the ideas from Bloomfield (1976) and the filtering approach described in Schmidt (1984). For example Hassler et al. (1992), Baxter and King (1995, 1999) and Christiano and Fitzgerald (1999) describe different approaches to deal with the problems of filtering finite time series in the frequency domain.

#### 5.4.1 Basic algorithm

Suppose that we are able to split some infinite time series  $x_t$  that needs to be filtered into a periodic component of frequency  $\theta$  and a remainder part  $r_t$ .

$$x_t = \alpha \cos(\theta t) + \beta \sin(\theta t) + r_t \quad (5.4.1)$$

Because of the linearity of the filters analyzed, the filtered series can be written as

$$\begin{aligned} y_t &= \sum_{l=-\infty}^{\infty} g_l x_{t-l} \\ &= \sum_{l=-\infty}^{\infty} g_l (\alpha \cos(\theta(t-l)) + \beta \sin(\theta(t-l))) + \sum_{l=-\infty}^{\infty} g_l r_{t-l} \end{aligned} \quad (5.4.2)$$

Note that because  $x_t$  is assumed to be an infinite time series, the IRF can be applied with the infinite number of weights  $g_l$  that represents the ideal FRF. This shows that the filtered time series is equal to the sum of the filtered periodic component and the filtered remainder part. The crucial element of the filter is now that we *know* what the periodogram of a perfect periodic component looks like in case of an infinite sample size, that is, in the absence of the leakage effect, without actually having to transform the time series into the frequency domain. This periodogram has a single spike at the frequency  $\theta$  and is zero on all other frequencies. If we filter this component then the filter output is

$$\sum_{l=-\infty}^{\infty} g_l (\alpha \cos(\theta(t-l)) + \beta \sin(\theta(t-l))) = \begin{cases} 0 & \text{if } G(\theta) = 0 \\ \alpha \cos(\theta t) + \beta \sin(\theta t) & \text{if } G(\theta) = 1 \end{cases} \quad (5.4.3)$$

That is, if the frequency  $\theta$  lies within the pass-band of the filter, the component remains unaltered while if the frequency lies outside the pass-band, the component is set to zero by the filter. The periodic component can therefore be correctly filtered without having to make the transition into the frequency domain, thereby avoiding errors because of the leakage effect. If furthermore the parameters  $\theta$ ,  $\alpha$  and  $\beta$  of the periodic component are chosen such that the periodogram of the remainder time series  $r_t$  has a smaller maximum than the periodogram of the original time series  $x_t$ , then the error of filtering this remainder time series will be smaller than the error of filtering the original time series. In that case, the total leakage effect and thereby the total filter error will have decreased.

The question is now of course how to find the parameters of the periodic component that satisfy the condition of actually reducing the maximum value in the periodogram. Using the OLS criterion to estimate these parameters turns out to be an appropriate procedure. This is so because for the remainder time series  $r_t$ , just as for any other time series, the relation (4.1.23) holds. That is

$$\frac{1}{T} \sum_{t=0}^{T-1} r_t^2 = \frac{T}{2\pi} \int_0^{2\pi} |J_r(\omega)|^2 d\omega \quad (5.4.4)$$

from which follows

$$|J_r(\omega)|^2 d\omega \leq \frac{1}{T^2} \sum_{t=0}^{T-1} r_t^2 \quad \text{for } 0 \leq \omega \leq 2\pi \quad (5.4.5)$$

Note that we now again assumed a finite sample. Because  $d\omega$  is fixed, by reducing the sum of squares of  $r_t$ , also the periodogram of the remainder time series at all frequencies is bounded from above by a lower value. If the  $r_t$  are seen as the residuals in the equation

$$x_t = \alpha \cos(\theta t) + \beta \sin(\theta t) + r_t \quad (5.4.6)$$

then by estimating this equation by OLS, the sum of squares of  $r_t$  is reduced as far as possible. Such estimation therefore ensures that the maximum value of the periodogram of the remainder time series is as low as possible thereby accomplishing as much reduction in filter error as possible. As noted before, reducing the upper boundary on the filter error does not guarantee that in each practical application the actual filter error will be reduced. Because the equation is non-linear in the frequency parameter  $\theta$ , for the estimation of (5.4.6) standard OLS techniques (alone) cannot be applied. To see how the estimation can take place first assume that  $\theta$  is known. The objective function to minimize with respect to  $\alpha$  and  $\beta$  then becomes linear in the other parameters  $\alpha$  and  $\beta$ .

$$V(\alpha, \beta) = \sum_{t=0}^{T-1} (x_t - \alpha \cos(\theta t) - \beta \sin(\theta t))^2 \quad (5.4.7)$$

The first order conditions for minimizing this function are

$$\begin{aligned} \frac{\partial T}{\partial \alpha} &= -2 \sum_{t=0}^{T-1} (x_t - \alpha \cos(\theta t) - \beta \sin(\theta t)) \cos(\theta t) = 0 \\ \frac{\partial T}{\partial \beta} &= -2 \sum_{t=0}^{T-1} (x_t - \alpha \cos(\theta t) - \beta \sin(\theta t)) \sin(\theta t) = 0 \end{aligned} \quad (5.4.8)$$

The solution to these two (normal) equations in two unknowns is

$$\begin{aligned}\hat{\alpha}(\theta) &= \left\{ \sum_{t=0}^{T-1} x_t \cos(\theta t) \sum_{t=0}^{T-1} \sin^2(\theta t) - \sum_{t=0}^{T-1} x_t \sin(\theta t) \sum_{t=0}^{T-1} \sin(\theta t) \cos(\theta t) \right\} / \gamma \\ \hat{\beta}(\theta) &= \left\{ \sum_{t=0}^{T-1} x_t \sin(\theta t) \sum_{t=0}^{T-1} \cos^2(\theta t) - \sum_{t=0}^{T-1} x_t \cos(\theta t) \sum_{t=0}^{T-1} \sin(\theta t) \cos(\theta t) \right\} / \gamma\end{aligned}\quad (5.4.9)$$

where

$$\gamma = \sum_{t=0}^{T-1} \cos^2(\theta t) \sum_{t=0}^{T-1} \sin^2(\theta t) - \left( \sum_{t=0}^{T-1} \sin(\theta t) \cos(\theta t) \right)^2 \quad (5.4.10)$$

The notation  $\alpha(\theta)$  and  $\beta(\theta)$  explicitly shows that these optimal parameter values are conditional on the value of  $\theta$ . In practical applications, the speed of calculating the values in (5.4.9) can be increased by using the relations

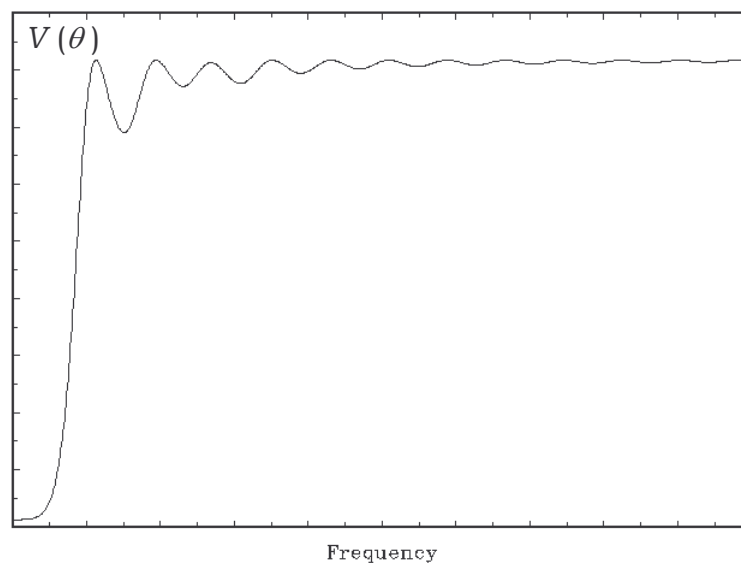
$$\begin{aligned}\sum_{t=0}^{T-1} \cos^2(\theta t) &= \frac{T}{2} + \frac{\sin(T\theta) \cos((T-1)\theta)}{2 \sin \theta} \quad \text{if } \theta \neq 0 \\ \sum_{t=0}^{T-1} \sin^2(\theta t) &= \frac{T}{2} - \frac{\sin(T\theta) \cos((T-1)\theta)}{2 \sin \theta} \quad \text{if } \theta \neq 0 \\ \sum_{t=0}^{T-1} \cos(\theta t) \sin(\theta t) &= \frac{\sin(T\theta) \sin((T-1)\theta)}{2 \sin \theta} \quad \text{if } \theta \neq 0\end{aligned}\quad (5.4.11)$$

If also the frequency parameter  $\theta$  has to be estimated this can be done by using the results (5.4.9) in order to minimize the objective function

$$V(\theta) = \sum_{t=0}^{T-1} \left( x_t - \hat{\alpha}(\theta) \cos(\theta t) - \hat{\beta}(\theta) \sin(\theta t) \right)^2 \quad (5.4.12)$$

with respect to the single parameter  $\theta$ . To solve this optimization problem one has to resort to numerical optimization procedures. Many of the optimization procedures used in practice use the first order derivative of the objective function for the minimization (so called gradient methods). In this case however these procedures are not that well suited because the objective function typically has many local minimums. Because of this property, these methods can easily get stuck in one of the local minimums instead of finding the absolute minimum of the function. Figure 5.21 shows the typical shape of  $V(\theta)$  for an arbitrary time series. The many local minimums are clearly visible.

Figure 5.21 Typical shape of the objective function (5.4.12).



Bloomfield (1976, p. 20) suggests to use the algorithm of Brent (1973) for the minimization of (5.4.12). Although it does not use derivatives of the objective function, of course also the Brent algorithm cannot guarantee to search across all local minimums and always find the absolute minimum. At the most, the Brent algorithm is more robust in this sense. The only input this algorithm needs is a starting interval  $[a, b]$  which is known to contain the optimum. On the one hand such an interval must not be too wide in order to prevent obtaining one of the local minimums as the solution. On the other hand the interval must not be too narrow because otherwise the optimal value might lie outside the interval. The only possibility is to first search a sufficiently fine grid on the interval  $[0, \pi]$  to obtain a suitable starting interval for the Brent algorithm. Such an interval runs from the first point to the left of the optimal point on the grid until the first point to the right of the optimal point on the grid. In practical applications an equally spaced grid of 500 points gives satisfactory results.

#### 5.4.2 Extensions

For the basic algorithm outlined in the previous sub-section, several extensions are required before formulating the complete filtering algorithm in the next sub-section. The first of these extensions is that besides a periodic component also a constant term  $\mu$  can be estimated and filtered separately because a constant term can be associated with the zero frequency. Because a constant term is a special case of a periodic component with a frequency  $\theta=0$  it is not clear beforehand whether the extension with a constant term leads to better filtering results or not. We will say more on this in sub-section 5.4.5 where several simulation experiments are performed to find the best parameter settings of the final filtering algorithm.

The second extension is by not estimating one single periodic component but estimating several, say  $m$ , periodic components at the same time. In the case of both a constant term and multiple periodic components the basic equation (5.4.6) extends to

$$x_t = \mu + \sum_{i=1}^m \alpha_i \cos(\theta_i t) + \beta_i \sin(\theta_i t) + r_t \quad (5.4.13)$$

The objective function to be minimized then becomes

$$V(\mu, \alpha_1, \beta_1, \theta_1, \dots, \alpha_m, \beta_m, \theta_m) = \sum_{t=0}^{T-1} \left( x_t - \mu - \sum_{i=1}^m (\alpha_i \cos(\theta_i t) - \beta_i \sin(\theta_i t)) \right)^2 \quad (5.4.14)$$

In order to minimize this even more complex and non-linear objective function Bloomfield (1976, p. 22) proposes to use a *cyclical descent* algorithm. The basic idea of such an algorithm is to optimize the parameters by means of an iterative process in which in each iteration a small subset of the parameters is optimized, keeping all other parameters unchanged. The obvious advantage of such an approach is that the sub-problems that need to be solved in each iteration are of a smaller dimension. The iterations stop if no significant reduction of the (overall) objective function can be achieved anymore. Because the previous sub-section already shows how in the case of a single periodic component the optimal values of the parameters  $\alpha$ ,  $\beta$  and  $\theta$  can be found using the Brent algorithm, obvious candidates for the parameters subsets are  $\{\mu\}$ ,  $\{\alpha_1, \beta_1, \theta_1\}$ ,  $\dots$ ,  $\{\alpha_m, \beta_m, \theta_m\}$ . To find the optimal values of the parameters  $\{\alpha_k, \beta_k, \theta_k\}$  for some value of  $k$ , given the values of all other parameters, in one of the iterations the following objective function has to be minimized.

$$\begin{aligned} V(\alpha_k, \beta_k, \theta_k) &= \sum_{t=0}^{T-1} \left( x_t - \mu - \sum_{i=1, i \neq k}^m (\alpha_i \cos(\theta_i t) - \beta_i \sin(\theta_i t)) - \alpha_k \cos(\theta_k t) - \beta_k \sin(\theta_k t) \right)^2 \\ &= \sum_{t=0}^{T-1} \left( x_t^* - \alpha_k \cos(\theta_k t) - \beta_k \sin(\theta_k t) \right)^2 \end{aligned} \quad (5.4.15)$$

where

$$x_t^* = x_t - \mu - \sum_{i=1, i \neq k}^m (\alpha_i \cos(\theta_i t) - \beta_i \sin(\theta_i t)) \quad (5.4.16)$$

The second part of (5.4.15) is equal to (5.4.7) with the only exception that the time series  $x_t$  is replaced by itself minus the constant term and the other periodic components estimated thus far. This means that the Brent algorithm from the previous sub-section can be applied without modification to solve this sub-problem. The cyclical descent algorithm always starts with all parameters equal to zero. Each subsequent iteration consists of the following two steps. The iterations are repeated until some appropriate stopping criterion is satisfied.



1. The series  $x_t$  is corrected for each of the periodic components estimated thus far and the constant term  $\mu$  is estimated on the residual time series by means of its average value.
2. For each of the periodic components  $i=1,\dots,m$ , the time series  $x_t$  is corrected for the estimated constant term  $\mu$  and all other periodic components estimated thus far (i.e.  $x_t^*$  is calculated). Then the parameters  $\alpha_i$ ,  $\beta_i$  and  $\theta_i$  are estimated on the resulting time series using the Brent algorithm.

The third extension of the basic approach from the previous sub-section is not to estimate the  $m$  periodic components just one time and then filter the remainder time series  $r_t$  using a direct frequency filter, but instead to repeat such estimation on the series  $r_t$  until the upper boundary on the filter error has been sufficiently reduced.

### 5.4.3 Final algorithm

The final algorithm of the zero phase frequency filter is a combination of the basic algorithm and the three extensions from the previous sub-sections. Filtering a time series  $\{x_t, t=0,\dots,T-1\}$  based on a FRF  $G(\omega)$  consists of the following steps.

#### Initialization:

1. Calculate  $M_0$  as the maximum of the periodogram of  $x_t$ .
2. Choose the number of periodic components  $m$  and decide whether or not a constant term  $\mu$  has to be included in (5.4.14).

#### For each $j$ -th iteration:

1. Estimate the parameters  $\{\mu_j, \alpha_{j,1}, \beta_{j,1}, \theta_{j,1}, \dots, \alpha_{j,m}, \beta_{j,m}, \theta_{j,m}\}$  of (5.4.14) using the combination of the cyclical descent and the Brent algorithm and calculate the remainder time series  $r_t$ .
2. Calculate  $M_j$  as the maximum of the periodogram of  $r_t$ .
3. Stop if the upper boundary (5.3.34) on the filter error is reduced far enough in a relative sense. This can be seen by checking whether  $M_j/M_0$  is small enough. If not, start iteration  $j+1$  using the time series  $r_t$  as input.

#### Finish:

1. Use the direct frequency filter from section 5.3.3 to calculate the filtered version of the remainder time series  $r_t$  from the last iteration and call this series  $s_t$ .
2. Calculate the total filtered time series  $y_t$  as the sum of  $s_t$  and the constant terms and estimated periodic components of all, say  $k$ , iterations multiplied by the value of the FRF at the corresponding frequencies. That is

$$y_t = s_t + \sum_{j=1}^k \left( G(0)\mu_{j,i} + \sum_{i=1}^m G(\theta_{j,i})(\alpha_{j,i} \cos(\theta_{j,i}t) + \beta_{j,i} \sin(\theta_{j,i}t)) \right) \quad (5.4.17)$$

The parameters of this zero phase frequency filter are

- (a) The number of periodic components  $m$ .
- (b) Whether or not to include a constant term  $\mu$ .
- (c) The minimal (relative) reduction of the upper boundary on the filter error ( $M_j/M_0$ ) which stops the iterations.
- (d) The minimal (relative) reduction of the objective function in the cyclical descent algorithm which stops the iterations.
- (e) The number of points on the grid before applying the Brent algorithm.
- (f) The resolution of the DDFT when applying the direct frequency filter on the final remainder time series.

Three important remarks need to be made with respect to the actual applications of this filtering algorithm. The first is that in order to reduce the filter error as much as possible, the stopping criterion (c) has to be set at a very small value. This means that the maximum value of the periodogram of the remainder time series will eventually become very small. Thereby the remainder time series will become almost zero. Note that because of this, the algorithm actually comes down to the iterative estimation of a number of periodic components and multiplying each of these components by the value of the FRF to obtain the filtered time series. Because only such a small portion of the time series is actually filtered using a direct frequency filter, it is no longer necessary to use a FRF with continuous transitions from zero to one and vice versa. The ideal FRF can be used just as well to obtain exactly the same results.

The second remark is that all filtered components of *non-overlapping* pass-bands of some time series, obtained by filtering with the zero phase frequency filter, have zero correlation in the time domain. This (theoretical) property holds for “all” filters which implement an ideal pass-band and is the continuous analogue of the orthogonal property (4.1.7) of cosine functions. This theoretical property also appears to apply to practical filter output when the filter is applied on actual time series. This property of zero correlation is very important with respect to the possible modeling of a time series process by its components (e.g. trend, long wave and business cycle). After all, if the filtered components are indeed independent, for each component a separate model can be constructed without having to model some sort of interaction between the models of the separate components. More on this modeling approach will be said in Chapter 20.

The third and last important remark about the algorithm concerns its strong analogy with Maximum Entropy spectral estimation as discussed in the previous chapter. The first analogy is that also here a kind of optimal “extrapolation” of the sample into infinity is made which reduced the disturbing effects of leakage. When estimating a spectral density, the sample autocovariances are extrapolated using an autoregressive (lag) polynomial. When filtering a time series, a periodic component of some frequency present in the time series is extrapolated using a cosine function. The second analogy is that both algorithms give no answers on the best parameter settings to use. When estimating a spectral density it is not clear which order of the AR polynomial and which estimation method to use. When filtering a time series the best setting of the parameters (a) through (f) are not known beforehand. Therefore, just as with the Maximum Entropy spectral estimation, a number of simple simulation experiments are performed to learn more about the behavior of the filtering algorithm and to find its most appropriate parameter settings for practical

applications. In the next sub-section the results of these experiments are discussed. Here, we first take a look at how the zero phase frequency filter performs when trying to filter the cycle component from the time series (5.3.16). Just as with the previous filtering methods an ideal band-pass filter with  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  is used. The following parameters were used.

- (a)  $m=1$  periodic component
- (b) A constant term  $\mu$  is included
- (c) The minimal value of  $M_j/M_0$  is 0.001%
- (d) The minimal (relative) reduction of the objective function in the cyclical descent algorithm is 0.01%
- (e) The number of points on the grid is 100
- (f) The resolution of the DDFT is  $2^{16}$  frequencies

It is not necessary to display the result of this filtering exercise here because for the eye it is identical to the theoretical cycle component. The ratio between the residual sum of squares (RSS) and the total sum of squares (TSS) is only 0.0003. That is, the  $R^2$  between the true and the filtered cycle component is 0.9997.

The many steps taken in designing the filter may have somewhat blurred the reader's overview on how the zero phase frequency filter works and more important why it works so well. To close this section it is therefore instructive to summarize the main arguments that led to the final filtering algorithm.

1. The most ideal way of filtering a time series is in the frequency domain. Only there one can exactly define the desired properties of the filter in terms of its gain and phase function.
2. When transforming short sampled (economic) time series into the frequency domain, leakage effects occur which disturb the frequency information and thereby lead to significant filter errors, especially at the boundaries of the sample.
3. The disturbing effects due to leakage decrease as the sample size increases (not possible in practical economic applications) and as the maximum of the periodogram of the series to be filtered decreases (possible).
4. Minimizing the sum of squared residuals when estimating a sinusoid on a time series is the same as minimizing an upper boundary on the filter error.
5. For a perfect periodic component the filter output is known without having to transform it into the frequency domain. Thereby the leakage effect can be avoided.

## 5.5 Simulation experiments

As stated at the end of the previous section, it is not clear beforehand which parameter settings best to use when applying the zero phase frequency filter described in the previous section. Similar to Schmidt (1984) and Metz and Stier (1992) therefore some rather simple simulation experiments are performed to further investigate the sensitivity of the filter results for the various parameters.

### *Time series*

Besides extracting a business cycle component, an application of a filtering technique will often also be to delete the trend from a time series. This is very relevant in the context of the current research objectives because many economic time series show trending behavior. The type of trend is often an exponential trend caused by some average growth or return percentage. Think for example of a GDP or stock index. In most of the following experiments therefore the following stylized time series will be used.

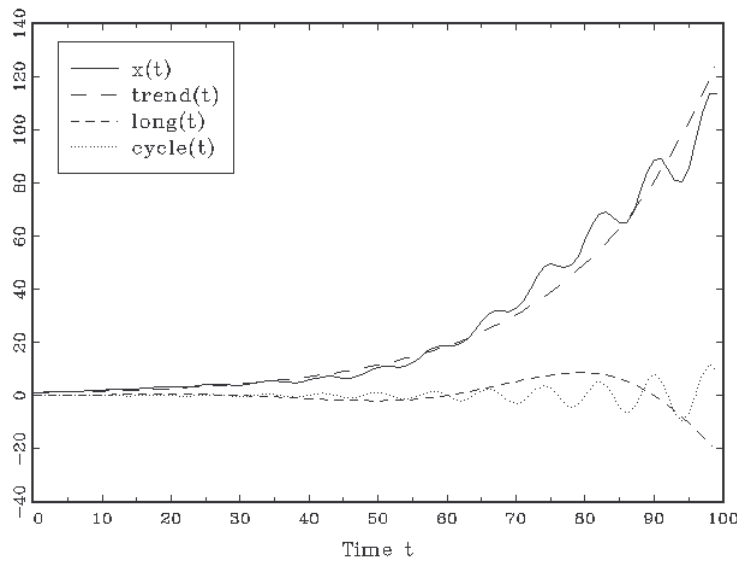
$$x_t = trend_t + long_t + cycle_t \quad (5.5.1)$$

with

$$\begin{aligned} trend_t &= 1.05^t \\ long_t &= 0.2 \cdot \sin(2\pi/60 \cdot t) \cdot trend_t \\ cycle_t &= 0.1 \cdot \sin(2\pi/8 \cdot t) \cdot trend_t \end{aligned}$$

Compared to the previous stylized time series (5.3.16), this series has an exponential trend with a growth rate of 5% instead of a linear trend. The long wave and business cycle components have the same period lengths as before but are now defined relatively to the underlying trend. The long wave for example has an amplitude of 20% of the value of the trend. Figure 5.22 shows a sample of size  $T=100$  of this time series.

Figure 5.22 Sample of size  $T=100$  of time series (5.5.1).



### Experiments

Mostly based on the time series, in the following experiments the impact of the most important parameters of the zero phase frequency filter on the filtering results are investigated. These parameters are

- (a) The number of periodic components  $m$ .
- (b) Whether or not to include a constant term  $\mu$ .
- (c) The minimal (relative) reduction of the upper boundary on the filter error ( $M_j/M_0$ ) which stops the iterations.
- (d) The minimal (relative) reduction of the objective function in the cyclical descent algorithm which stops the iterations.

The parameters (e) and (f) are in each case the same as at the end of sub-section 5.4.4. Both the quality of the filtered cycle component and the quality of the filtered trend component were tested. For the cycle component as before a band-pass filter is used for which the pass-band runs from  $\omega_1=2\pi/20$  to  $\omega_2=2\pi/4$ . A convenient short notation is  $[1/20,1/4]$  indicating that the filter passes all components with a period length between twenty and four years. A plausible frequency domain definition of a trend is that it consists of all periodic components of the time series that are not able to complete at least one full cycle within the sample length. That is, the trend is defined as all non-periodic components of the time series, given the sample size. The dependency on the sample size in this trend definition reveals the subjectivity of the trend concept in general. As an example suppose one has a sample of a hundred years of a country's GDP. In that case, business cycle fluctuations of a duration of several years are clearly not seen as a trend but rather as fluctuations around the underlying long term trends<sup>35</sup>. If however one has a sample of say two years of monthly data of the same time series, the very same business cycles will likely be seen as a trend around which for example seasonal fluctuations occur. So the same type of economic fluctuations can in a long sample be treated as fluctuations around some underlying trend while they can be treated as the trend itself in short samples of data. That is, the trend concept is sample dependent. A related concept is that of *perspective distortion* as introduced by Reijnders (1990). This states that if one looks at too short a sample, economic fluctuations with a period length longer than the sample cannot be adequately analyzed. In the case of Reijnders it concerns long waves with period lengths of somewhere around fifty years. The filter which isolates the trend following the definition of all non-periodic components has a band-pass of  $[0,1/T]$ . In case of the example time series (5.5.1) this becomes  $[0,1/100]$ . The filter error in both the trend and in the cycle component is measured by means of the RSS/TSS ratio (i.e one minus the  $R^2$ ).

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<sup>35</sup> Also see the definition of business cycles by Lucas (1977) as given in section 2.2.3.

The complete list of experiments described in the following sub-sections test

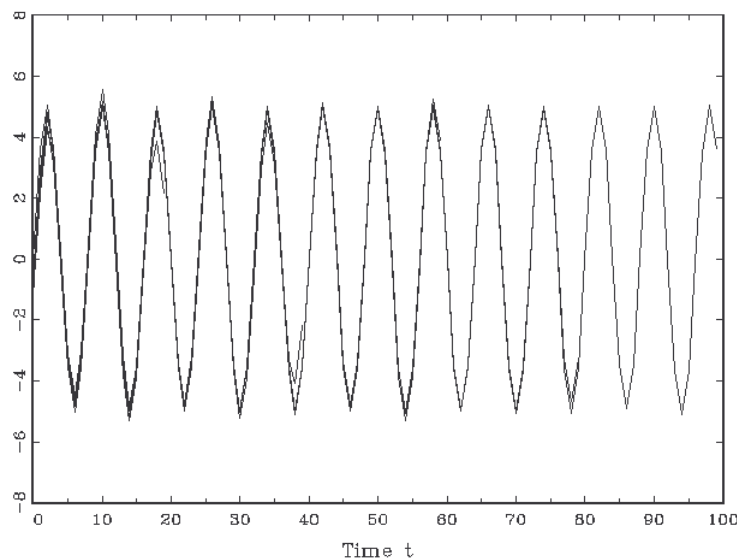
- the stability of the filter output as more data comes available (section 5.5.1).
- the impact of the stopping criterion of the overall algorithm in terms of the upper boundary on the filter error (section 5.5.2).
- the impact of the the stopping criterion of the cyclical descent algorithm (section 5.5.3).
- the impact of the number of periodic components  $m$  included (section 5.5.4).
- whether or not to include a constant term  $\mu$  (section 5.5.5).
- the effect of applying a (natural) logarithmic transformation on the time series before applying the filter (section 5.5.6).
- the effect of applying the filter on a stochastic time series (section 5.5.7).

Finally, section 5.5.8 summarizes and discusses the most important results found in these experiments.

### 5.5.1 Stability

The first of these experiments concerns the stability of the filter output as more data comes available. Although for the applications in the research presented here this is not very important, for applications such as business cycle indicators it is unpleasant if all the filter output dramatically changes when a filtering exercise is repeated on a sample which is extended by one or more extra observations. Note that perfect stability of the filter output is only possible for a causal filter (see section 5.1) which has severe drawbacks in terms of the phase shifts induced by such a filter. For a zero phase filter, perfect stability can therefore never be achieved. The objective should therefore be that the filter output is stable “enough” instead of perfectly stable. To test the stability, the filtering exercise at the end of the previous sub-section with the same parameter settings is repeated for sample sizes  $T=20, 40, 60, 80$  and  $100$ . So, the time series (5.5.1) is not used in this first experiment. The sequentially filtered cycle components are plotted in Figure 5.23.

Figure 5.23 Filtered time series (5.3.16) using the zero phase frequency filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$  for sample sizes  $T=20, 40, 60, 80$  and  $100$ .



As expected the filter output is not perfectly stable. However the changes as more data comes available are not very big. Of course a part of the non-perfect stability is also caused by the fact that the filter error becomes smaller as the sample size increases.

### 5.5.2 Stopping criterion filter error

The objective of the second simulation experiment is to find out how the stopping criterion (c) of the zero phase filter affects the filter error. Unless stated otherwise, from this experiment on, the time series (5.5.1) is used in the experiments. This stopping criterion is the minimal (relative) reduction of the upper boundary on the filter error ( $M_j/M_0$ ) which stops the iterations. Table 5.1 contains the filter errors in the trend and cycle components for different values of the stopping criterion. The other settings of the algorithm are (a)  $m=1$  frequency, (b) a constant term is included and (d) the stopping criterion for the cyclical descent algorithm is 0.0001.

Table 5.1 RSS/TSS in filtered trend and cycle components using the zero phase frequency filter for different values of the stopping criterion (c). ( $m=1$ , constant term included and stopping criterion cyclical descent  $10^{-4}$ )

<i>Stopping criterion</i>	<i>Cycle</i>	<i>Trend</i>
$10^{-2}$	0.046	0.030
$10^{-3}$	0.036	0.030
$10^{-4}$	0.013	0.030
$10^{-5}$	0.012	0.030
$10^{-6}$	0.012	0.030
$10^{-7}$	0.012	0.030

The results show that the filter error for the cycle component decreases as the stopping criterion is lowered. This is consistent with the fact that the theory tells us that as more periodic components are taken from the time series in order to be filtered separately, the maximum value of the periodogram and thereby also the upper boundary on the filter error further decreases. It also seems that without changing any of the other parameters the filter error cannot be reduced beyond the 0.012. Finally, the results show that the filter error for the exponential trend cannot be reduced at all by setting a more strict stopping criterion. Apparently the error in case of the trend already occurs in one of the first iterations of the algorithm.



### 5.5.3 Stopping criterion cyclical descent

The third experiment differs from the second only in the sense that the stopping criterion (d) of the cyclical descent is changed instead of the stopping criterion of the overall algorithm. The results are shown in Table 5.2.

Table 5.2 RSS/TSS in filtered trend and cycle components using the zero phase frequency filter for different values of the stopping criterion (d). ( $m=1$ , constant term included and stopping criterion overall algorithm  $10^{-5}$ )

<i>Stopping criterion</i>	<i>Cycle</i>	<i>Trend</i>
$10^{-3}$	0.013	0.036
$10^{-4}$	0.012	0.030
$10^{-5}$	0.010	0.026
$10^{-6}$	0.010	0.024
$10^{-7}$	0.010	0.023

These results show that also by reducing the stopping criterion of the cyclical descent algorithm the filter error can be reduced. In this case also the error in the trend component can be reduced. Lowering this stopping criterion simply means that the periodic components in each iteration are estimated using a higher degree of precision. If instead of one, several periodic components ( $m>1$ ) have to be estimated, such precision will probably become more and more important. Although of course by lowering both stopping criteria at the same time the filter errors can be decreased further, reducing the filter errors beyond the errors in the final row seems not possible without making any further modifications to the parameter settings.

### 5.5.4 Number of periodic components

The fourth experiment is designed to assess the impact on the filter errors of using multiple periodic components in each of the iterations instead of only one. The stopping criterion for the overall algorithm is  $10^{-5}$  while the stopping criterion for the cyclical descent algorithm is only  $10^{-3}$  in order to keep the computational effort within reasonable limits. Table 5.3 shows the results for  $m=1, 5, 10, 15$  and  $20$ .

Table 5.3 RSS/TSS in filtered trend and cycle components using the zero phase frequency filter for different numbers  $m$  of periodic components (a). (constant term included, stopping criterion cyclical descent  $10^{-3}$  and overall algorithm  $10^{-5}$ )

<i>m</i>	<i>Cycle</i>	<i>Trend</i>
1	0.012	0.030
5	0.002	0.092
10	0.001	0.117
15	0.001	0.128
20	0.001	0.119

By increasing the number of periodic components to be estimated in each iteration the error in the filtered cycle components reduces to almost zero. At first sight it may seem strange that estimating five periodic components at the same time leads to better filter results than estimating one periodic component in five consecutive iterations as is the case in the previous experiments. If we think purely in terms of

the fit with the time series to be filtered however, the second approach will always be less efficient than (or equally efficient as) the first approach because the second is a restricted version of the first. After all, four of the five periodic components are restricted to zero in the first approach. The second approach will therefore always lead to an equal or lower residual sum of squares, thereby further reducing (the upper boundary on) the filter error. The drawback of estimating more periodic components at the same time is that the required computational effort rapidly increases. For  $m=20$  for example  $20 \times 3 = 60$  parameters of a non-linear function have to be estimated.

While the cycle component can be estimated almost perfectly, this clearly does not hold for the trend component. Instead of decreasing, the error in the filtered trend even increases as more periodic components are included. The cause of this probably lies with the exponential character of the trend. A constant term and a linear trend can still be seen as special case of a periodic component with very low frequencies. For an exponential trend this is no longer the case. Of course the Fourier theorem tells us that any time series, also one with an exponential character, can be written as a sum of sinusoids of different frequencies. However the problem in a filtering context is that in order to describe an exponential trend, several periodic components with different frequencies are needed. Sometimes these periodic components amplify each other while at other times the periodic components dampen each other to reach as good a fit as possible. The information about the trend is thereby distributed across these frequencies. Some of the frequencies may very well lie outside the pass-band  $[0, 1/T]$  defined for the trend. During the filtering process the information about the trend is thereby falsely torn apart, despite the fact that including more periodic components means a better fit. Note the strong parallels of this effect with how leakage disturbs the workings of a direct frequency filter in the case of small sample sizes as discussed in section 5.3.3.

Filtering an exponential trend thus remains problematic for now. Another property that is often encountered in economic time series is that of a non-stable variance. Provided the appropriate filtering parameters are used this does not seem to pose a problem. After all, the cycle component in (5.5.1) has an increasing variance because it is defined proportional to the value of the trend and the results in Table 5.3 showed that the cycle component can be filtered almost perfectly. This is no surprise if one realizes that a periodic component with an increasing variance is well described by two or more interfering periodic components of frequencies close together. At some times these components are completely out of phase and dampen each other causing a low volatility while at other times they are in phase and cause a high volatility.

### 5.5.5 Constant term

The fifth experiment investigates the influence of including a constant term or not when estimating some number of periodic components in each of the iterations. The previous experiment is repeated with exactly the same parameter settings, only this time excluding the constant term. The results are shown in Table 5.4 which can be compared to Table 5.3 which does include a constant term.

Table 5.4 RSS/TSS in filtered trend and cycle components using the zero phase frequency filter for different numbers  $m$  of periodic components (a) when no constant term is included. (stopping criterion cyclical descent  $10^{-3}$  and overall algorithm  $10^{-5}$ )

$m$	Cycle	Trend
1	0.023	0.030
5	0.003	0.024
10	0.008	0.021
15	0.001	0.023
20	0.000	0.027

The most striking feature of these results is that now also the filter error in the trend component (slightly) decreases as more periodic components are estimated. One possible explanation might be that including a constant term comes down to imposing the restriction that one of the frequencies to be estimated is  $\omega=0$ . After all, a constant term is the same as a cosine function of frequency zero. Imposing this restriction might prove inefficient because some other low frequency might give a better fit and thereby a lower filter error. It is also possible that without the restriction, the frequencies of the estimated periodic components are better matched with the assumed (trend) frequency range because less compensating effects occur between the components. Indeed a deeper look at the filtering results shows that in the first iteration without including a constant term, a better fit is achieved compared to when a constant term is included. To see this compare the results to those in Table 5.3. The error in the cycle gets slightly worse for most values of  $m$ . However for the case  $m=20$ , the lowest filter error of all parameter settings thus far is obtained. So also for the cycle component it seems best to exclude the constant term, provided sufficient periodic components are estimated.

### 5.5.6 Logarithmic transformation

Excluding a constant term, using sufficiently small stopping criteria and estimating as much periodic components at the same time as possible, it is possible to almost perfectly filter out the cycle component. The filter error for the exponential trend however could not be reduced far enough in the previous experiments. As a possible solution, in the sixth experiment we first calculated the (natural) logarithm ( $\ln(x)$ ) of the time series, then filtered this transformed time series and then calculated the inverse logarithmic transformation ( $\exp(x)$ ) of the filtered series. The reason that this might work is that the logarithm transforms an exponential trend into a *linear* trend while a linear trend, as we have seen, can be filtered well from a time series because it is a special case of a cosine function having a very low frequency. Note that because of this transformation the normally additive decomposition of the time series

becomes a multiplicative decomposition in terms of the original time series. If  $\ln(x_t)$  instead of  $x_t$  is filtered into two components  $a_t$  and  $b_t$  then it holds

$$\ln(x_t) = a_t + b_t \Leftrightarrow x_t = \exp(a_t + b_t) = \exp(a_t) \cdot \exp(b_t) \quad (5.5.2)$$

Taking the exponents of the components  $a_t$  and  $b_t$  and multiplying them rather than adding them therefore leads to the original time series. The filter errors in the trend with and without including a constant term for different values of  $m$  are shown in Table 5.5. These results should be compared to the second column of Tables 5.3 and 5.4.

Table 5.5 RSS/TSS in filtered trend and cycle components using the zero phase frequency filter for different numbers  $m$  of periodic components (a) with and without including a constant term when first applying a (natural) logarithmic transformation. (stopping criterion cyclical descent  $10^{-3}$  and overall algorithm  $10^{-5}$ )

$m$	<i>Constant</i>	<i>No Constant</i>
1	0.002	0.000
5	0.000	0.000
10	0.002	0.000
15	0.004	0.000
20	0.004	0.000

These results confirm that by means of a logarithmic transformation much better filtering results for the trend can be obtained. When no constant term is included, the filter errors are even zero for all values of  $m$ . The key problem when filtering a time series with an exponential trend is that the frequencies of the periodic components that can adequately describe the exponential trend are inconsistent with the trend definition in the frequency domain applied here (i.e. all periodic components with a period length longer than the sample size). By linearizing it, the logarithmic transformation causes that all the information about the trend is adequately contained within the trend frequency pass-band  $[0, 1/T]$  and can be filtered correctly without being torn apart.

Because the logarithmic transformation is non-linear, its effects cannot be analyzed directly based on its phase and gain function. We do know that the transformation maintains the ordering within a time series (i.e. it is a monotonic transformation). It is therefore not to be expected that the logarithmic transformation induces substantial phase shifts. Frequency shifts however can occur. One extreme example is a linear trend with an infinitely small frequency which is transformed into a “periodic” component with still a small frequency, though a little higher, which may disturb the filtering results<sup>36</sup>. For higher frequencies the effects will be less “extreme”. To see this consider the logarithmic transform of a some (positive valued) perfect cosine function. The shape of the periodic behavior changes a little while the peaks and troughs stay in place. Because of its no longer perfect periodic behavior, in a filtering context the information of the cosine function will be spread on a (small) interval of frequencies instead of on a single frequency. If the pass-band of the filter is wide enough the filtering consequences will be small.

To summarize, using a logarithmic trend will in some cases significantly improve the filtering results, while in other cases it might worsen the results. The transformation

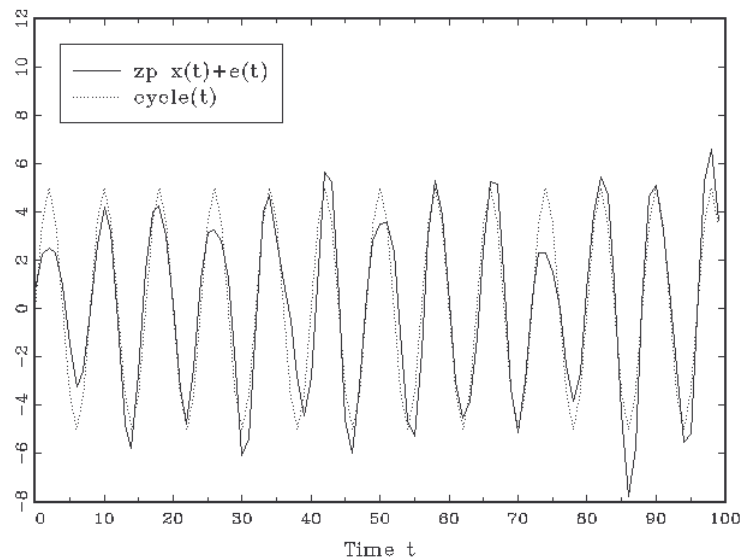
<sup>36</sup> To see this note the bended shape of the (natural) logarithmic function  $\ln(x)$ .

should therefore always be applied with care and only if the properties of the time series to be filtered support the use of the transformation (i.e. in case of exponential trends).

### 5.5.7 Stochastic time series

Real life economic time series will of course never show such deterministic behavior as the time series (5.5.1) used in the previous experiments. As the seventh and final experiment we therefore tried to filter the cycle component from the original time series (5.3.16) with the linear trend, to which a Normally distributed error term  $\varepsilon_t$  with zero mean and a variance of four was added. The filtered and the “ideal” cycle component are shown in Figure 5.24.

Figure 5.24 Filtered time series (5.3.16) with an added error term  $\varepsilon_t \sim N(0,4)$  using the zero phase frequency filter for  $\omega_1=2\pi/20$  and  $\omega_2=2\pi/4$ . ( $m=20$ , no constant term included, stopping criterion cyclical descent  $10^{-4}$  and overall algorithm  $10^{-5}$ )



The figure shows that the filtered cycle component significantly deviates from the “optimal” filter output. The same will hold for all other components we would try to isolate from the time series. These disturbing effects of the error terms on the “optimal” filter output however are a logical consequence of the construction of the time series and are therefore not to be considered an error of the filtering algorithm. After all, we know that the spectrum of a white noise process  $\varepsilon_t$  shows an evenly distributed variance mass over all frequencies in the frequency interval  $[0,\pi]$ . For this also see section 4.5.1. This means that on average *all* frequencies will be present in a realization of such a random process. When filtering a time series which includes a white noise error term, the filter output will include the fluctuations from the realization of the random process which fall within the pass-band of the filter. Because of this, it is no longer fair to compare the filter output to the deterministic cycle component. Furthermore, note that when analyzing economic time series, the stochastic character of the time series can explicitly be taken into account, for example by estimating a spectral density on the filter output.

### 5.5.8 Main findings and discussion

The objective of the experiments with the zero phase frequency filter as performed in this section was to learn more about the behavior of the filter and to find out which parameter settings best to use in actual applications of the filter. The main findings of these experiments are the following.

1. The filter produces reasonably stable output in the sense that the previous values of a filtered series do not change very much as more data of the series comes available.
2. The stopping criteria of both the filter algorithm itself and the cyclical descent algorithm that is applied in each of its iterations should be set as small as possible (given the numerical precision of the computer used for the calculations) to increase the filter precision and reduce the filter error as far as possible.
3. Estimating and filtering as much periodic components at the same time as possible reduces the filter error. Especially when more periodic components are estimated, a sufficiently low stopping criterion is important because of the in general slow convergence of the cyclical descent algorithm. Of course with this and the previous point one always has to find a balance with the computational effort required.
4. It is better not to include a constant term in the estimation process because this is only a special type of a periodic component with frequency zero. Including a constant term is therefore the same as applying a restriction on one of the periodic components to be estimated which may be inefficient and lead to poorer filtering results.
5. In case of exponential trending behavior, as is the case for many economic (index) time series, it is better to use the (natural) logarithmic transformation and its (exponential) inverse. Hereby an exponential trend is transformed into a linear trend which can be filtered correctly. Besides of course the scale, the filtering results at other frequencies will only be disturbed very little by applying the logarithmic transformation.

Given these main findings, the following parameter settings are preferred to be used for the zero phase frequency filter as described in section 5.4.4.

- (a)  $m=20$  periodic components.
- (b) Do not include a constant term  $\mu$ .
- (c) The minimal value of  $M_j/M_0$  is  $10^{-5}$ .
- (d) The stopping criterion for the cyclical descent algorithm  $10^{-4}$ .
- (e) The number of points in the grid is 500.
- (f) The resolution of the DDFT is  $2^{16}$  frequencies.

Both the experiments from this section and the (many) applications of the filter in Part II and IV lead to the conclusion that filtering a time series should *never* be seen as some automated process, no matter how good the filtering technique applied and how thoroughly the optimal parameter settings were chosen. Instead one should



always check the “stability” of the filtering results when other parameter settings are used, for example a lower number of periodic components. It is also recommended to check whether the filtering results are consistent with what is known about the filtered time series from a simple visual or numerical analysis performed before the actual filtering. Note that the spirit of both these recommendations is very similar to those given with the process of selection the appropriate model order for the purpose of parametric spectral estimation as given in section 4.7.7. One could say that both filtering real life time series and selecting model orders remain more of an art than a science, no matter how much effort one puts in designing the optimal approach. More on art versus science is said in sections 6.1.1 and 18.1.3.

## 5.6 Spurious cycles

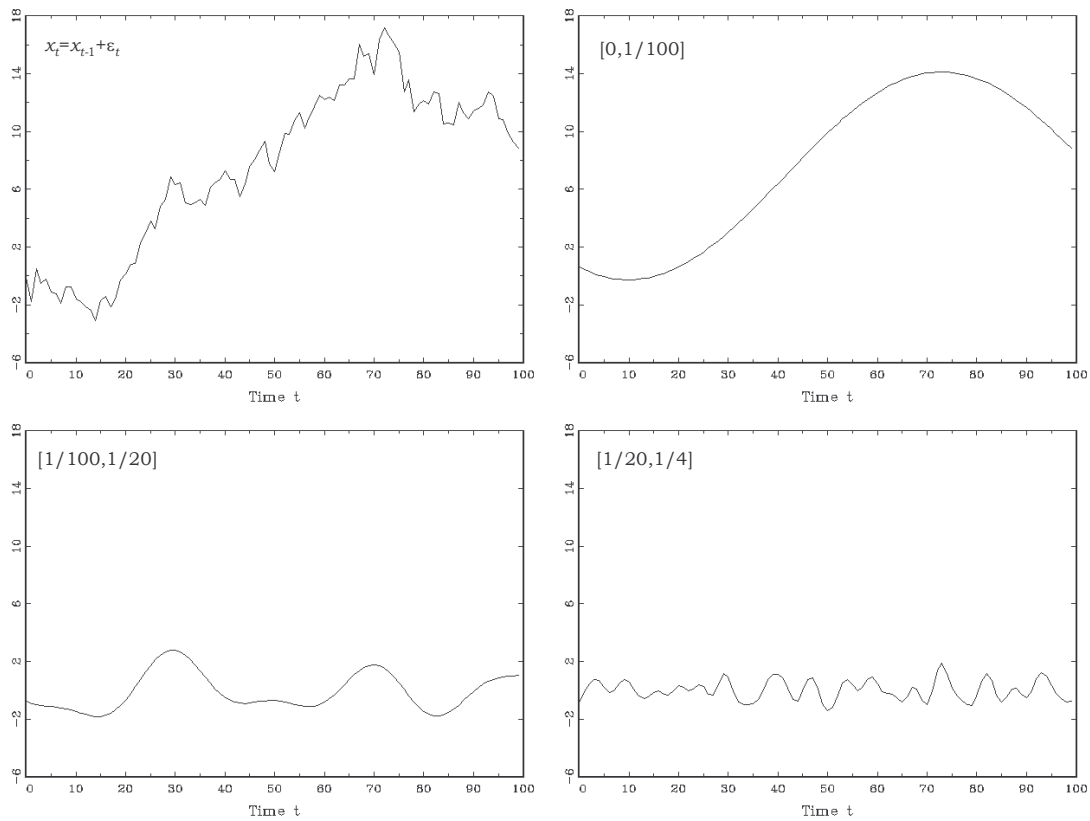
In the introduction of this chapter we referred to the extensive literature on how all kinds of filtering techniques may lead to “spurious cycles” in the filtered time series. That is, using these techniques, practitioners may conclude to have found (pseudo) cyclical behavior in time series that is not really there. In this section the very relevant question is explored whether or not the zero phase frequency filter as described and tested in the previous sections also can produce such spurious cycles. Almost all of the literature on this subject concerns the case of filtering time series that contain a stochastic trend. The simplest of such a process is the *random walk* process

$$x_t = x_{t-1} + \varepsilon_t \quad (5.6.1)$$

where  $\varepsilon_t$  is a normally distributed white noise process with zero mean and variance  $\sigma^2$ . In Figure 5.25 a sample of this process of size  $T=100$  is plotted. The figure also plots the resulting time series after filtering this sample by the zero phase frequency filter for the pass-bands  $[0, 1/100]$ ,  $[1/100, 1/20]$  and  $[1/20, 1/4]$ . Let’s for a moment forget that we know the format of the true Data Generating Process (DGP) and look at the sample of the process as an empirical time series of some economic variable. In that case the first filter aims to isolate the trend using the definition of all non-periodic components as before. The objective of the second filter would be to isolate long term fluctuations in the time series with a period length of between a hundred years and twenty years. The third filter is targeted at isolating business cycle fluctuations with a period length between twenty years and four years. Note that the pass-bands of these filters are adjacent and that only the range of high frequency fluctuations with a period length of between four and two years are left out of consideration here.



Figure 5.25 Sample of a random walk process and the filtered time series using the zero phase frequency filter for different pass-bands.



Two important things should be noted about these filtered time series. The first is that while the original sampled time series is clearly non-stationary because of its stochastic trend, the bottom two filtered time series could very well be seen as stationary time series without any trending behavior. This observation is consistent with what we know from section 5.3.2 about filters with a FRF which is (exactly) zero valued at the zero frequency (i.e.  $G(\omega)=0$  for  $\omega=0$ ). Filters satisfying this condition were shown to always delete both first and second order stochastic trends from a time series. The second observation is that although the filter removes the stochastic trend from the time series, the resulting filtered time series will in general *not* be the sample of the white noise process  $\varepsilon_t$  that was used for the construction of the process. This can easily be seen from (5.3.22) which states that for a filter with  $G(0)=0$ , the filter output can be written as

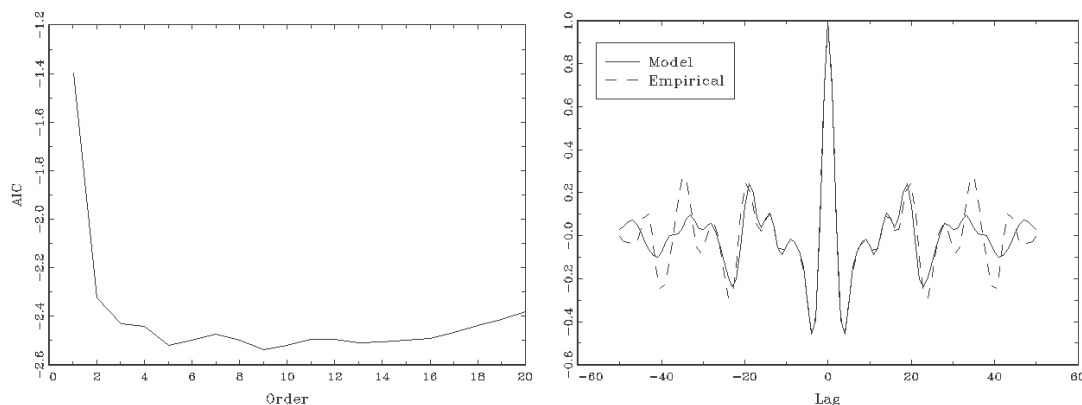
$$y_t = G(L)x_t = H(L)(x_t - x_{t-1}) = H(L)\varepsilon_t \quad (5.6.2)$$

where  $H(L)$  is some remainder polynomial in terms of the lag operator  $L$ . The filtered time series will therefore only be equal to the white noise process  $\varepsilon_t$  if  $H(L)=1$  which will of course only be the case if the filter has exactly the FRF of the first differencing filter shown in Figure 5.1. In all other cases the filtered time series will be a stationary weighted sum of past and future values of  $\varepsilon_t$ . It is therefore a misconception that applying a filter with  $G(0)=0$  always produces exactly the supposed underlying stationary process.

The question is now what such a filtering exercise means with respect to the problem of finding spurious cycles. The best way to analyze this is again in the frequency domain. In practical applications of the filter, the dynamic properties of a filtered series can be analyzed by means of estimating spectral densities. The most appropriate candidate from the two stationary filtered series is the series with the alleged business cycle type fluctuations. That is the time series at the bottom right corner of Figure 5.25. The series with the long periodic components is not that well suited for spectral analysis. Although it certainly is stationary, it shows too “little” fluctuations to be considered as a stochastic process having a spectral density. Also note that for this series the types of fluctuations can almost directly be seen from the time series itself. Estimating a spectral density to uncover the dynamic properties is therefore superfluous and will only lead to confirmation of the results already obtained from a simple visual inspection. Note that similar to the discussion in the introduction of section 5.5 about how the concept of a trend depends on the sample size, also the concept of considering (and modeling) a time series as a stochastic process is somehow related to the sample size. If, for example, we would have a sample of  $T=1000$  instead of  $T=100$  of the  $[1/100, 1/20]$  filtered series, it would be much more plausible and also useful to see it as a (stationary) stochastic process.

For the  $[1/20, 1/4]$  filtered series a spectral density was estimated by means of the (Yule-Walker) Maximum Entropy spectral estimation technique as described in section 4.6. As a primary source for the order of the AR model, the order selection criterion AIC was used. The results of the order selection process are shown in Figure 5.26.

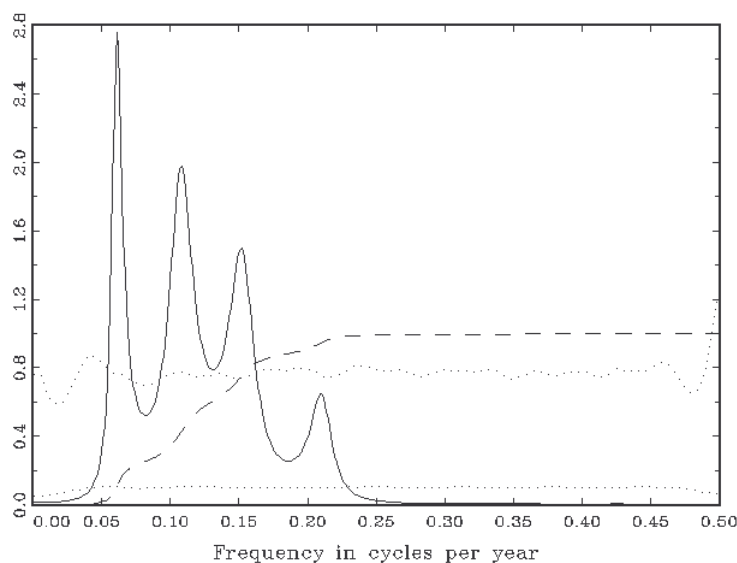
Figure 5.26 Value of the AIC for orders one through twenty for the  $[1/20, 1/4]$  filtered random walk (left) and the empirical autocorrelations together with the autocorrelations from an AR(15) model estimated by the YW technique (right).



The left picture in Figure 5.26 shows the value of AIC for orders one through twenty. The absolute minimal value of the AIC occurs for order  $p=9$  indicating that an AR(9) model is the best one to estimate. Local minimums are at the orders  $p=5$  and around  $p=14$ . As it turns out however the autocorrelations from an AR(9) model are (for lags higher than nine) not very consistent with the empirical autocorrelations estimated from the sample. Also see the remarks with respect to the process of selecting an appropriate model order given in section 4.7.7. In a practical application one should therefore have looked for a higher order model which is more consistent with the sample autocorrelations. Only for order  $p=15$  this seems to be the case. This is illustrated by the right picture in Figure 5.26 which compares the sample

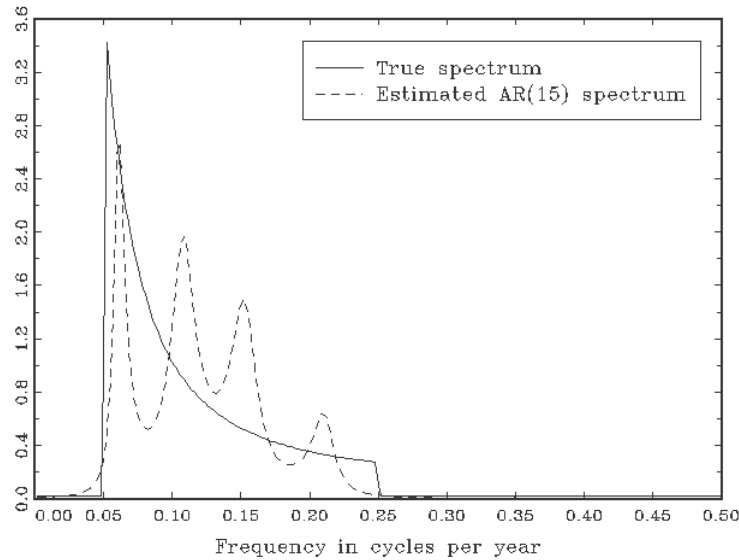
autocorrelations to the autocorrelations of an estimated AR(15) model. The two autocorrelation patterns are now pretty much in line. The model of order fifteen therefore seems best at picking up the sample autocorrelations and extending these outside the sample. Figure 5.27 shows the (normalized) spectral density of the estimated AR(15) model on the  $[1/20, 1/4]$  filtered time series, including the corresponding 95% confidence interval of a white noise process. Also see section 4.8.1 for how the confidence interval is obtained.

Figure 5.27 Spectral density of the  $[1/20, 1/4]$  filtered random walk estimated by means of an AR(15) model using the YW estimation technique together with the corresponding 95% confidence interval of a white noise process.



The first thing to note about this estimated spectral density is that there is virtually no spectral mass outside the pass-band  $[1/20, 1/4] = [0.05, 0.25]$  showing that the zero phase frequency filter has correctly deleted all periodic components outside this pass-band. The second observation concerns of course the dominant peaks in the spectrum with the height of the peaks decreasing as the frequency increases. Also, most of the peaks reach outside the confidence interval of a white noise process indicating them to be statistically significant peaks. Generating other samples of the random walk, again filtering the time series and again estimating a spectrum leads to similar shapes of the spectral density. The only difference is the number of peaks. Based on these findings one could state that the zero phase frequency filter can indeed produce spurious cycles if the true process has a stochastic trend. After all, the peaks in a the spectrum indicate pseudo periodic behavior of the underlying stochastic process while we know that the random walk does not contain any specific cyclical behavior. However care should be taken when posing this conclusion. To arrive at the estimated spectrum two steps were taken. The first was the filtering of the time series using the zero phase frequency filter. The second step was estimating a spectrum on this filtered time series using the (Yule-Walker) Maximum Entropy spectral estimation technique. So the observed spurious cycles may either come from the filtering or from the spectral estimation technique. To be able to make this important distinction, Appendix B.16 shows that the *theoretical* (instead of the *empirical*) spectrum of the  $[1/20, 1/4]$  filtered random walk is as shown in Figure 5.28.

Figure 5.28 Theoretical spectrum of the  $[1/20, 1/4]$  filtered random walk together with its Maximum Entropy AR(15) estimate.



Contrary to the estimated spectrum, the theoretical spectrum of the filtered series shows no sign of any pseudo periodic behavior in the sense of having peaks around some frequencies. With its exponential decay from the start towards the end of the pass-band this spectrum is just a piece of the spectral density of a random walk process. The spectrum of a random walk process can be approximated by the spectrum of an AR(1) process with an autoregressive parameter  $\beta_1$  extremely close to one. See section 4.5.3 for the practical calculations. Such a spectrum starts at infinity for frequency  $\omega=0$  and exponentially decays towards zero as the frequency increases towards  $\omega=0.5$ . Note that the shape of the theoretical spectrum of the filtered series is a direct consequence of the  $H(L)$  polynomial from (5.6.2). Because the zero phase frequency filter has the FRF of an ideal band-pass filter, its theoretical filter output is just as shown in Figure 5.28. In theory therefore the filter should *not* lead to spurious cycles as the empirical results indicate. Note that this is contrary to the spurious cycles produced by for example the Hodrick-Prescott filter in case of stochastic trends as shown by for example Harvey and Jeager (1993). Because of the smooth approximating shape of its FRF (see Figure 5.5 in section 5.2.4) applying the Hodrick-Prescott filter to the spectrum of a random walk in theory produces a similar type of peak as the zero phase frequency filter only with a much more rounded shape at the edges of the pass-band<sup>37</sup> looking more like a single peaked spectrum indicating pseudo periodic behavior. The fact that the estimated AR(15) spectrum in as sense cycles around the smooth theoretical spectrum of the filtered time series may very well be because the special shape of the theoretical spectrum is difficult to approximate using an autoregressive process. For example Priestley (1981) shows that such spurious peaks are typical for an autoregressive approximation of what really is a moving average process. Of course this assumes that the true DGP is known, which will never be the case in empirical applications.

<sup>37</sup> A pass-band filter can be implemented with the basically one sided Hodrick-Prescott filter by first applying a low-pass filter followed by a high-pass filter or vice versa.

The previous results warrant the conclusion that *theoretically* speaking, because of its ideal FRF, the zero phase frequency filter does *not* suffer from the danger of producing spurious cycles unlike most other filtering approaches. *Practically* however when applying the filter followed by the Maximum Entropy spectral estimation technique, spurious cycles certainly can occur. What does this mean for the practical applications of these filter and spectral estimation techniques? The simple answer would be to say that filtering time series using band-pass filters should be avoided in the case of random walk processes or processes with other types of stochastic trends. In such cases only the first order differencing operator should be applied to obtain the “original” driving stationary stochastic process. However there are two strong objections to this simple answer.

First, in practice it is never known for certain whether, and if so, what type of stochastic trend drives a process. Starting with the work of Nelson and Plosser (1982) a vast amount of literature exists showing that economic time series contain stochastic trends. However some years ago Ben-David and Papell (1996), Cheung and Chinn (1997) and Diebold and Senhadji (1996) all conducted tests with new longer time series data from Maddison (1995). All find that the longer time series strongly reject the stochastic trend hypothesis in favor of a deterministic (linear) trend. Murray and Nelson (2000) on the other hand present a number of arguments that cast doubt on these conclusions of linear trends. What matters here is that the discussion about economic variables containing stochastic or deterministic trends has not been settled yet and probably never will.

The second, related, objection against the simple answer of just using the “appropriate” differencing operator instead of some other filtering technique, is that this requires the postulation of a model of the stochastic process *before* a thorough empirical analysis of the data has taken place. From the discussion in section 2.1 it is clear that I am in favor of first finding empirical regularities about the data before postulating any kind of model at all to describe these regularities, as the standard approach to scientific research prescribes. In this sense, reconsider the fact that the entire discussion about economic time series containing stochastic or deterministic trends is built around the class of theoretical time series models from the well known Box and Jenkins (1970) methodology. The whole concept of a “unit root” only exists in the context of the class of ARIMA models from this methodology.

Because of the two objections mentioned, we will not pursue elaborate statistical testing procedures to decide on the type of trends in economic time series. However, when filtering time series and estimating spectral densities, we should constantly be aware of the potential dangers of finding spurious cycles. By analyzing several time series over different historical time periods while at the same time checking the stability of the estimated spectral densities for consecutive orders of the AR model, the possibility of finding spurious results is reduced. After all, the more often some empirical regularities are encountered, the more confidence one gets in this being true regularities rather than some spurious phenomenon.

## *Part III*

# Stylized Facts

*“Financial archaeology involves digging through reams of financial data in search for answers.”*

Philippe Jorion and William N. Goetzmann (2000)

Part III presents the results on the first research objective as formulated in section 1.3. The eleven chapters of Part III can be divided into three parts. The first part consists of Chapter 6 which describes how the three techniques from Part II work together in a specific methodology developed for the research presented here. The second part consists of Chapter 7 until Chapter 16. These chapters contain the results from investigating the empirical properties of ten important macroeconomic variables, using the methodology from Chapter 6. These results are presented in a standard format. The third and last part consists of Chapter 17 which summarizes the results in terms of almost ninety stylized facts and thereby provides the “answer” to the first research question.





## 6 Methodology Part III

Part III contains the results on the first research objective as formulated in section 1.3. In short, the objective is to gather broad empirical information about the behavior of macroeconomic variables based on one and the same methodology. In case such information is found to be robust with respect to for example the historical time period or the country of origin, we speak of stylized facts. Although this research question is originally derived from and placed in the specific context of ALM scenario models, the results have relevance in many other lines of economic research and applications. The stylized facts found here are for example very relevant for macroeconomic model building and testing in general. Part II contains the relevant theory and experiments with respect to the three central techniques: Vector AutoRegressive (VAR) models, spectral analysis and filtering techniques. In this first chapter of Part III these techniques are used to construct a specific methodology suited for the objective of collecting stylized facts about macroeconomic behavior by analyzing a representative range of time series. It should be stressed that the three techniques are not used separately in a competitive sense. Instead, the three techniques work together to form the specific methodology developed for the research presented here.

### 6.1 General approach

For studying macroeconomics it is never enough to investigate one single variable. Macroeconomics is a multivariate process turning up in many different variables which are all interrelated in some way. The next chapters therefore present the analysis of a large number of time series of different macroeconomic variables, each of which constitute a distinct part of macroeconomic behavior. The variables are given in Table 6.1. Some variables, such as national product and industrial production, relate to the real sector of an economy while others such as interest rates and equity indices relate to the financial sector. Where possible, the variables are corrected for differences in price levels. The production indices are volume indices while also the wage and equity indices have been corrected for the development of the price index and therefore hold in real terms. Only the interest rates hold in nominal terms because (long) time series of real interest rates are not available. In section 13.2.2 some information about the long term behavior of real interest rates is subtracted from the nominal interest rates and consumer price inflation series. Also some other derived series are studied such as the term spread (difference between long and short term interest rate) and the equity risk premium (only the long term behavior, just as for the real interest rates).

As long as possible annual time series for these variables were collected for the Netherlands, the United Kingdom and the United States. Table 6.1 shows the starting years of the available samples. All time series end in 1999. The time series themselves and a description of their sources can be found in Appendix F. To obtain the longest possible samples, annual time series were used. Some may object that data on variables that long ago (sometimes dating two centuries back) are no longer representative for what macroeconomic behavior looks like today and will look like in

the near future. This may very well be so but this can only be concluded after a thorough analysis of the data itself. Time series dating back as far as possible are needed for three reasons. First, it is an explicit objective of this research to obtain information about the *long term* behavior of macroeconomics. Second, an objective is also to investigate possible *changes* in the macroeconomic process over time. The third and final reason for using very long time series is to be able to better assess whether the results are true empirical *regularities* or that they are merely a coincidence for some specific historical time period. Only in this way it is possible to speak of stylized facts. For the same reason variables from more than one country are studied. Each of the three countries studied belong to what are called the developed countries in the world today. They have always had a capitalist type of economic system and have taken part in the Industrial Revolution from the very beginning. The choice for the United Kingdom and the United States is rather obvious because of their size, data availability and the fact that they have a prominent position in the empirical macroeconomic literature. The choice for the Netherlands seems somewhat strange from this point of view. However, there are several arguments to support this choice. First, a major application of the results from this research will be scenario analysis for ALM purposes in the Netherlands. Second, the Dutch economy is strongly linked to the German economy which is the largest economy in Europe. The results for the Netherlands can therefore be seen as rather representative for the whole of Europe. Analyzing long term German data itself is rather cumbersome because of the many major historical events that took place in that country<sup>38</sup>. The third and final reason for including the Netherlands is that the Dutch economy has always been a very open trading economy that heavily depends on the state of the world economy. The results for the Netherlands can therefore probably also have a more general relevance.

Table 6.1 Start of the various annual macroeconomic time series, all ending in 1999.

	<i>Netherlands</i>	<i>United Kingdom</i>	<i>United States</i>
National Product Index	1870	1855	1870
Industrial Production Index	1921	1855	1860
Employment	1911	1855	1890
Consumer Price Index	1813	1600	1820
Real Industry Wage Index	1926	1829	1820
Short Term Interest Rates	1828	1820	1831
Long Term Interest Rates	1814	1700	1798
Default Spread	-	1929	1857
Real Equity TRR Index	1824	1800	1870

The results for each of the variables are presented in a separate subsequent chapter. In general such an analysis consists of three steps. These are

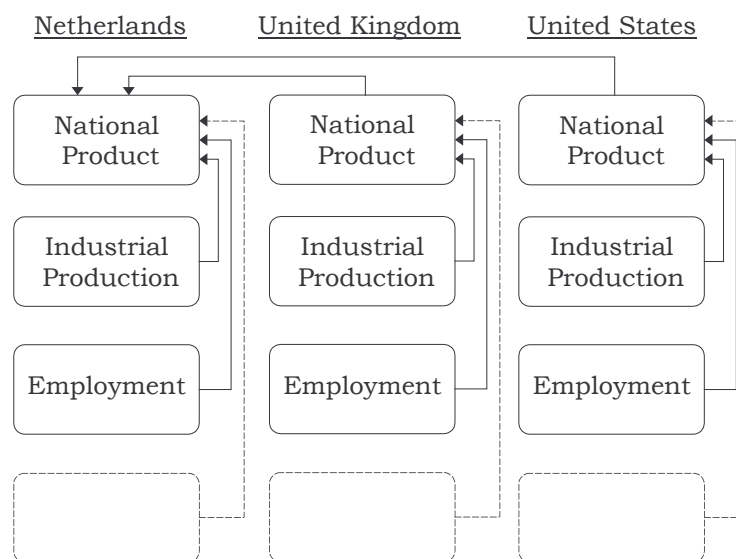
1. Data analysis
2. Filtering results
3. Spectral analysis

<sup>38</sup> For example the extreme effects the two World Wars have had on the German economy and, more recently, the union of East and West Germany.

In the next sections of this chapter, each of these steps is discussed in more detail with respect to both their objective and how they are related. The results are presented in as compact a format as possible. In principle the results for each of the three countries are equally important. This is why the chapters in the main text contain the *numerical results* for the three countries. This also enables a direct comparison of the results and an assessment of having found true regularities or not. For the spectral analysis, also a number of important *graphical results* are presented. In the main text this is only done for the Netherlands. The graphical results for the other two countries can be found in respectively Appendix D and E.

As we know, the relations between the different variables are a very important aspect of macroeconomics. However, if one would have to investigate all bivariate relations within the in total, say, thirty variables (three countries times ten variables) this would result in having to study 465 relations. Because this is obviously impossible within a reasonable amount of time, a kind of chain linking approach is used which is illustrated in Figure 6.1. The international relations are only studied by means of the relations between the behavior of the national product for the three countries. The national product is often seen as most representative for the actual state of an economy<sup>39</sup>. For both the United Kingdom and the United States it is investigated how their national product relates to that of the Netherlands (i.e. the Netherlands are used as a reference point). For each of the other variables “only” the relations with the national product of the same country are studied. Especially in the case of spectral analysis, with such a chain linking approach the results for the investigated relations can be used to derive relations that are not analyzed explicitly. If we for example find that at some frequency the long term interest rate lags the national product by four years while the short term interest rate lags the national product by only two years, than one can easily derive that the short term interest rate leads the long term interest rate by two years at this frequency.

Figure 6.1 The investigated relations between the variables in the various countries.



<sup>39</sup> Note that for describing the business cycle state of an economy, often the industrial production instead of the national product is used. However, here the national product is chosen as a reference variable because of the much longer historical time series available in case of the Netherlands. For this see Table 6.1.

The general objective is to gather broad information about macroeconomic behavior on an as uniform and methodological sound basis as possible. The broadness both relates to the type of variables, the many historical time periods and the different frequencies (ranging from the very long run to shorter term business cycle fluctuations). The applied techniques enabled to achieve this objective within a reasonable amount of time. This however does not alter the fact that basically the investigation of a single time series is already a substantive research project in itself. It is therefore not possible to provide very elaborate (theoretical) discussions of the results. Instead, with each variable a discussion of the results is presented in a bulleted format. Besides the main findings, these bulleted remarks also relate the results found here to those found by others in previous research, without having the illusion of being complete. Furthermore, only a few theoretical explanations for the results are given because the objective is the collection of empirical regularities which should precede the postulation and testing of theories as indicated in Chapter 2. However, room was left for some obvious and intuitive discussion of the results.

### 6.1.1 Art versus science - I

Before continuing with the discussion of the methodology used for the data analysis, frequency domain filtering and spectral analysis, it is important to note that, despite the many modeling techniques used, it was tried as much as possible *not* to let the analysis of the time series turn into some automated process. At various stages in the analysis of the time series, *judgement* has been used. A first example of this is the exclusion of some parts of the available samples because of aberrant behavior of the time series. This holds for example for the two World War periods and for some periods with clearly poor quality data. A second example concerns the filtering procedure described in section 6.3. In some cases the pass-bands for the various components were slightly altered to obtain more intuitively appealing filtering results. A final example concerns the order selection for the Maximum Entropy spectral analysis described in section 6.4. Judgement was used to combine the various sources of information into the final model order. In general one can say that the methodology described here should be used as a set of tools by the investigator and should not be put to work in total isolation.

That the results in the following chapters are not obtained from some fully automated process should also be clear from the fact that, where needed, extra analyses of special topics are included. Examples are the level effect in the business cycle volatility of interest rates in sections 12.2.1 and 13.2.1, the long term behavior of real interest rates in section 13.2.2, the analysis of the equity risk premium in section 16.2.1 and the analysis of the long and short term inflation hedge of equities in section 16.4.3.

The next sections of this chapter discuss the steps of data analysis, frequency domain filtering and spectral analysis in more detail. Following the separate chapters for each of the variables, Part III closes with a summary of the results in which the results from the individual chapters are combined to enable a direct overview.

## 6.2 Data analysis

The first step in the analysis of each of the variables is a simple graphical and numerical inspection of the time series for each of the countries. The purpose of such a data analysis is to quickly obtain a feel for the behavior of the variable at hand and to obtain conventional information to which the less trivial information from the subsequent filtering and spectral analyses can be compared.

The *graphical analysis* consists of a joint plot of the original time series of the variable at hand for the various countries. In the case of indices, such as for example the national product, the series always pass through a common point in order to make a fair comparison of for example the long term growth rate in the various countries. In general this is the starting year of the shortest time series. In the case of exponential trending behavior, also the (natural) logarithms of the variables are plotted to bring out the properties more clearly because of the smaller scale and the transformation of the exponential trend into a linear trend. The actual filtering and spectral analysis described in the next two sections is also done on these transformed time series.

The *numerical analysis* consists of a number of conventional statistics of (a transformation of) the time series for the various countries, calculated on different time periods. Here and also in the numerical results from the other parts of the analysis (filtering and spectral analysis), numbers are reported in percentages only up to one decimal. Because of the in general small sample sizes, reporting more decimals would suggest precision that is not actually there. Reporting only one decimal also brings out the results more clearly. The actual calculations however are always done with the highest precision. The presented *statistics* are the average (growth in case of indices), the standard deviation and a correlation. For variables other than the national product this correlation is always with the growth of the national product of the same country. In case of the national product, the correlation is with the national product of the Netherlands. In all cases it is a contemporaneous correlation. An alternative would be to (also) report the maximum correlation for some lead or lag. This is not done here because more detailed information on the correlation and phase structure between the variables is given by the spectral analysis as described in section 6.4. Furthermore, the ratio between the standard deviation (of the transformation) of the variable and the growth of the relevant national product is given to enable a direct comparison of the volatilities of the various variables. The *time periods* for which these statistics are calculated are the longest period possible for the relevant time series, the longest period for which data for both the relevant variable and the national product are available, the pre World War I period (*prewar* period), the period between the two World Wars (*interwar* period) and the period starting after World War II and ending in 1999 (*postwar* period). A similar division into time periods is used in the other parts of the analysis in terms of the filtering results and spectral analysis. In the numerical results, often a further split is made within the postwar period. Besides the period as a whole, separate statistics are reported for the first half (until 1975) and the second half (1976-1999).

## 6.3 Filtering

After the data analysis as described in the previous section, a filtering technique is used to dissect each of the time series for the relevant variable into several components which enable a separate analysis of the various aspects of the behavior of the variable. In the following sub-sections first the desired decomposition is described. Next, the actual filtering of the time series to perform such a decomposition is described. This section closes with how some extra calculations with respect to the volatility of the filtered series are performed.

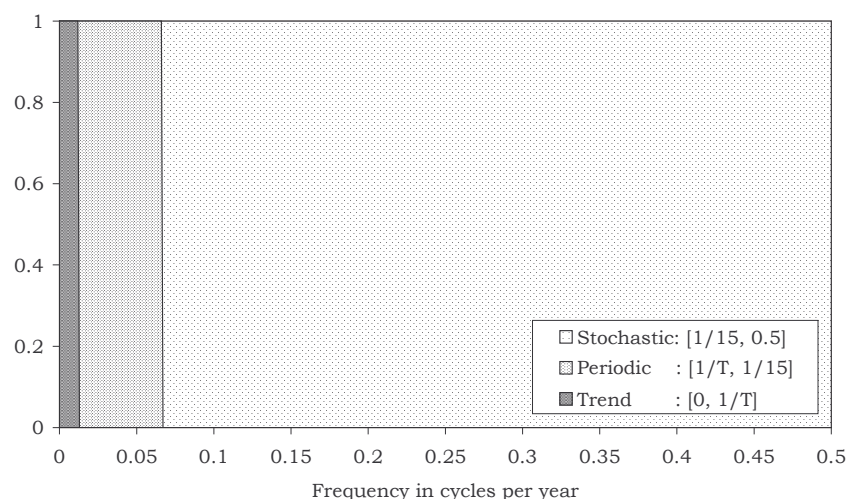
### 6.3.1 Decomposition

At a conceptual level, the decomposition applied is similar to the original decomposition of Tinbergen into a trend, a cycle, a seasonal and a random component as described in section 2.2.2. Because we use annual data, seasonal fluctuations do not play a role here. Also, we already know that truly random type fluctuations from a white noise process have a spectral density that evenly distributes the variance over the entire frequency range. It is therefore not possible to identify random fluctuations as a separate component of a time series. Instead randomness can have its effect on all components of a time series. This brings us to the following decomposition.

$$\text{Time Series} = \text{Trend} + \text{Periodic} + \text{Stochastic} \quad (6.3.1)$$

It is easiest to formalize, motivate and explain this decomposition in the frequency domain based on Figure 6.2.

Figure 6.2 Decomposition of a time series in the frequency domain.



#### *Trend*

The trend of a time series is defined as that part of the time series that is not periodic. In the frequency domain this means that the trend consists of all fluctuations within the time series which have a period length longer than the sample size  $T$ . Note that the sample size measured in time rather than in the number of



observations is relevant here. So, the trend type “fluctuations” are not able to complete a full cycle within the sample of size  $T$ . This also includes the average value of the time series as a fluctuation of infinite length. In the frequency domain this formalizes as all fluctuations of the time series within the frequency range  $[0, 1/T]$  as shown in Figure 6.2. Note that such a trend definition implies that the relevant frequency range depends on the sample size. This is not that strange because the same component that may in a short sample be seen as the trend may in a much longer sample no longer be seen as a trend. As soon as a component is able to complete at least one full cycle within the sample it becomes what shall be called a periodic component.

#### *Periodic component*

The periodic component of a time series is defined as all fluctuations within the time series that are able to complete at least one full cycle within the sample of size  $T$  but which have a period length longer than fifteen years. In the frequency domain this formalizes as all fluctuations of the time series within the frequency range  $[1/T, 1/15]$  as shown in Figure 6.2. Given that long term annual time series are analyzed, one could say that the periodic component consists of all long term fluctuations in the time series, such as possibly the Kondratieff type fluctuations with an alleged length of some forty-five to sixty years. To specifically investigate the existence of such a long wave within the frequency range of the periodic components we also zoom in on the frequency range  $[1/70, 1/30]$ . That is, all fluctuations in a time series with a period length between thirty and seventy years. This is a broad range around the length of long waves as found by others indicated in section 2.2.3. Because the shortest time series analyzed has a length of about seventy years, this way there can be no overlap between the trend and this *long wave component* either. Fluctuations with a period length shorter than fifteen years certainly also can be seen as periodic components. However, it is worthwhile to define such fluctuations as a separate so called stochastic component.

#### *Stochastic component*

The stochastic component (which is not the same as a white noise component) is defined as all fluctuations within the time series in the frequency range  $[1/15, 0.5]$  as shown in Figure 6.2. These are all fluctuations with a period length ranging from the shortest of two years (highest frequency) until fifteen years. These special type of periodic fluctuations are termed as “stochastic” because for these fluctuations it is worthwhile to see them as stochastic variables of which the properties can be analyzed further by for example spectral analysis. Note that whether or not to view a variable as stochastic is purely a modeling choice. Stochastic models are needed for variables that show “enough” fluctuations within the sample size that are furthermore random enough to prevent a direct assessment of the variable’s behavior. In such cases stochastic models can be useful to reveal the behavior of the variable under investigation. It would for example be of little use to analyze the long term fluctuations from the previously described periodic component by means of a stochastic model. After all, given the available sample size, we are only able to observe a few completed cycles for these components. Just looking at the filtered periodic components is enough to extract the relevant information. The motivation



for putting the split between the periodic and stochastic component at fifteen years consists of the following arguments.

1. Fifteen years is a comfortable upper limit on all types of alleged (endogenous) *business cycle fluctuations* mentioned in section 2.2.3. Only the Kuznets cycle falls outside this range with a period length of some fifteen to twenty-five years but as explained there, this type of cycle probably had exogenous causes. So the stochastic component intends to focus on possible business cycle fluctuations classified as the Juglar cycle (seven to eleven years) and the Kitchin cycle (three to five years).
2. If the period length would increase above fifteen years, the stochastic modeling would become more difficult because of the few completed cycles within the samples. Within the postwar period, fluctuations with a period length of fifteen years can only complete a little more than three cycles.
3. Because the Frequency Response Function of the popular first order differencing filter suppresses most of the fluctuations with frequencies below  $1/15=0.067$  (see Figure 5.1 in section 5.2.1), most of the volatility of returns or growth rates calculated from the first order differences (of the logarithm) of the variables, as used in conventional modeling approaches, is preserved in the stochastic component. More on this will follow in section 6.3.3.

### 6.3.2 The actual filtering

In the previous sub-section, the desired decomposition into trend, periodic and stochastic components in the frequency domain is defined. An additional requirement of the decomposition is of course that it does not induce any phase shifts which can disturb the inference about the lead / lag relations between the variables. These requirements (the frequency intervals in Figure 6.2 and the requirement of no phase shifts) directly translate into the Frequency Response Functions (FRF) of the corresponding ideal band-pass filters to be applied. For the implementation of such filters on finite samples of data a very adequate filtering technique available is the *zero phase frequency filter* as described in section 5.4. For the parameter settings of the filter, those given at the end of section 5.5 are used. In doing so, also the recommendations given there is followed not to let filtering become some automated process, to test the stability of the filter output with respect to other parameters settings and to always confront the filter output with the properties of the original time series (as obtained from the preliminary data analysis). In some cases the pass-bands for the various components can be slightly altered to obtain more intuitively appealing filtering results.

The filtering of both the trend, periodic and long wave component is always performed on the entire available sample. Sometimes the two World Wars show very exceptional fluctuations which can disturb the filtering process. In that case the disturbing period(s) are replaced by a simple linear interpolation to be able to correctly perform the filtering exercise. Because the two World Wars can also have very disturbing effects on the filtered stochastic components, just as with the data analysis, the available sample is always split into three periods: the prewar, the interwar and postwar period which are filtered separately. With respect to the stochastic components also note that in some exceptional cases some other lower

upper limit of the stochastic fluctuations than fifteen years is used. This is sometimes necessary for the interwar period when the specific historical events that occurred during that period introduce disturbing high amplitude fluctuations with a long period length into the stochastic component. A similar remark holds with respect to the filtering of the long wave component. Sometimes a slightly different pass-band was applied for the filter than the standard  $[1/70, 1/30]$  interval to bring out the long term behavior of the variable under investigation more clearly. The exact frequency range applied is always shown in the legend of the graphical filtering results.

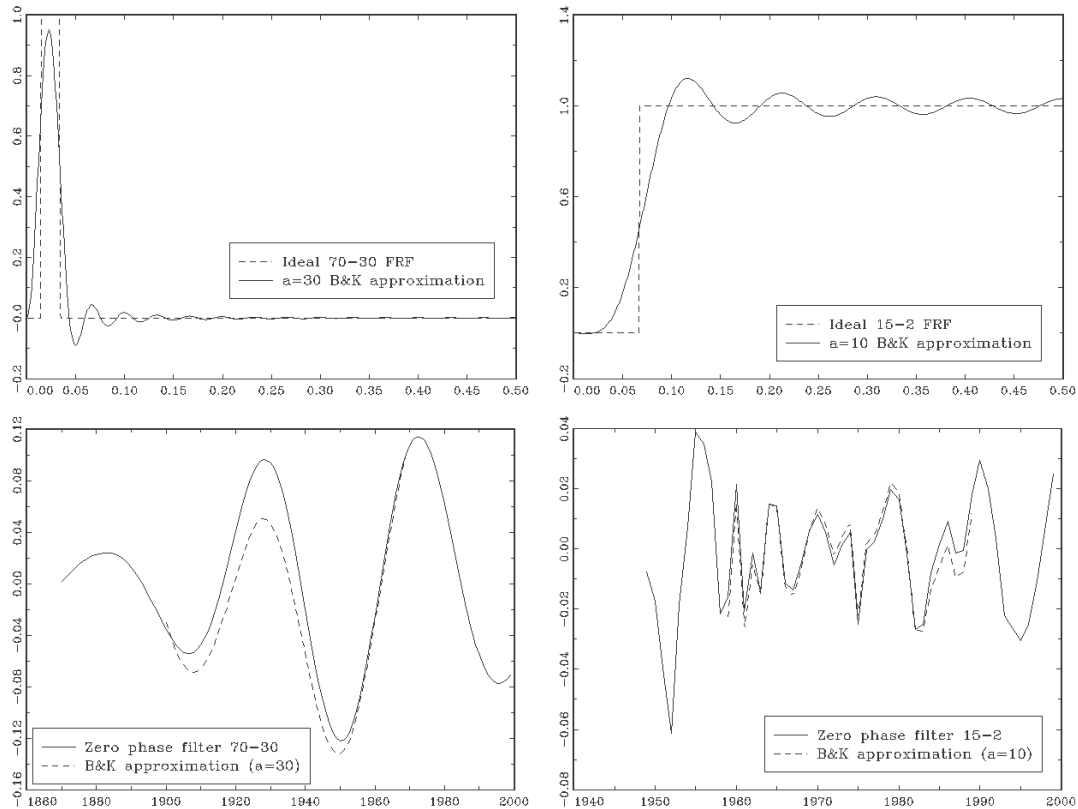
Section 5.5 describes a number of simulation experiments to test the zero phase frequency filter. All these experiments were performed on artificially constructed time series of which the true components were known. Here, a final test of the filter is performed by applying it to a real life time series and comparing its filter output with the output of both the Baxter-King and the Christiano-Fitzgerald filter as described in respectively section 5.3.2 and 5.3.3. After the zero phase frequency filter, these filters come closest to implementing the FRF of an ideal band-pass filter. Also, these filters have been successfully applied by others. This makes it interesting to see whether these two filters on the one hand and the zero phase frequency filter on the other hand, having the same objective but operating rather differently, come up with similar filter output. The (natural) logarithm of the National Product index for the Netherlands has been filtered for both the long wave component in the frequency range  $[1/70, 1/30]$  for the full 1870-1999 period and the stochastic component in the frequency range  $[1/15, 0.5]$  for the postwar period 1949-1999. Note that because of the use of the logarithmic transformation, such filtered components (and their statistics such as reported standard deviations) have the easy interpretation of relative deviations from the underlying trend<sup>40</sup>.

Let's first compare the results from the zero phase frequency filter to those of the Baxter-King filter. In the first case (of the long wave component) a Baxter-King filter with parameter  $\alpha=30$  is used which means losing thirty observations on each side of the sample. In the second case (of the stochastic component)  $\alpha=10$  is used, losing ten observations on each side of the sample. Figure 6.3 shows both the Baxter-King approximation of the FRF of the relevant ideal band-pass filters and the filter output from applying the two filters. The results clearly show that the zero phase frequency filter and the Baxter-King filter give very similar filter output on the joint part of the sample while the zero phase frequency filter very naturally extends the filtered series all the way to the beginning and end of the sample. The minor differences in the amplitude of the fluctuations (and not in the dating of peaks and troughs because both filters do not cause phase shifts) are very likely caused by the differences in the actual FRF applied as shown in the top two pictures.

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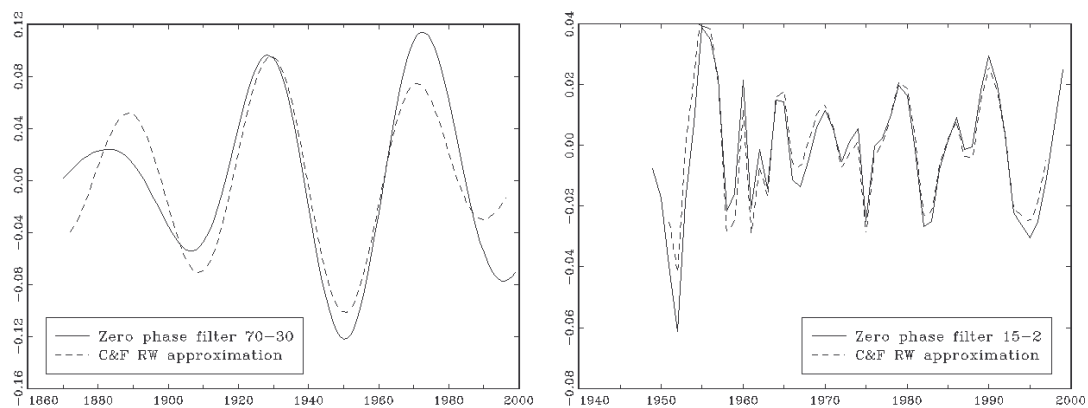
<sup>40</sup> To see this, suppose that the logarithm of some time series  $x_t$  has been decomposed by a linear filter into a trend component  $trend_t$  and a cycle component  $cycle_t$  as  $\ln(x_t) = \ln(trend_t) + \ln(cycle_t)$ . In that case  $\ln(cycle_t) = \ln(x_t) - \ln(trend_t) = \ln(x_t / trend_t) = \ln(1 + (x_t - trend_t) / trend_t) \approx (x_t - trend_t) / trend_t$  which illustrates this point.

Figure 6.3 Comparison of output of zero phase frequency filter and Baxter-King filter applied on the (natural) logarithm of a National Product index for the Netherlands. The left hand side shows the case of filtering for the long wave component in the frequency range  $[1/70, 1/30]$  for the period 1870-1999. The right hand side shows the case of filtering for the stochastic component in the frequency range  $[1/15, 0.5]$  for the postwar period 1949-1999.



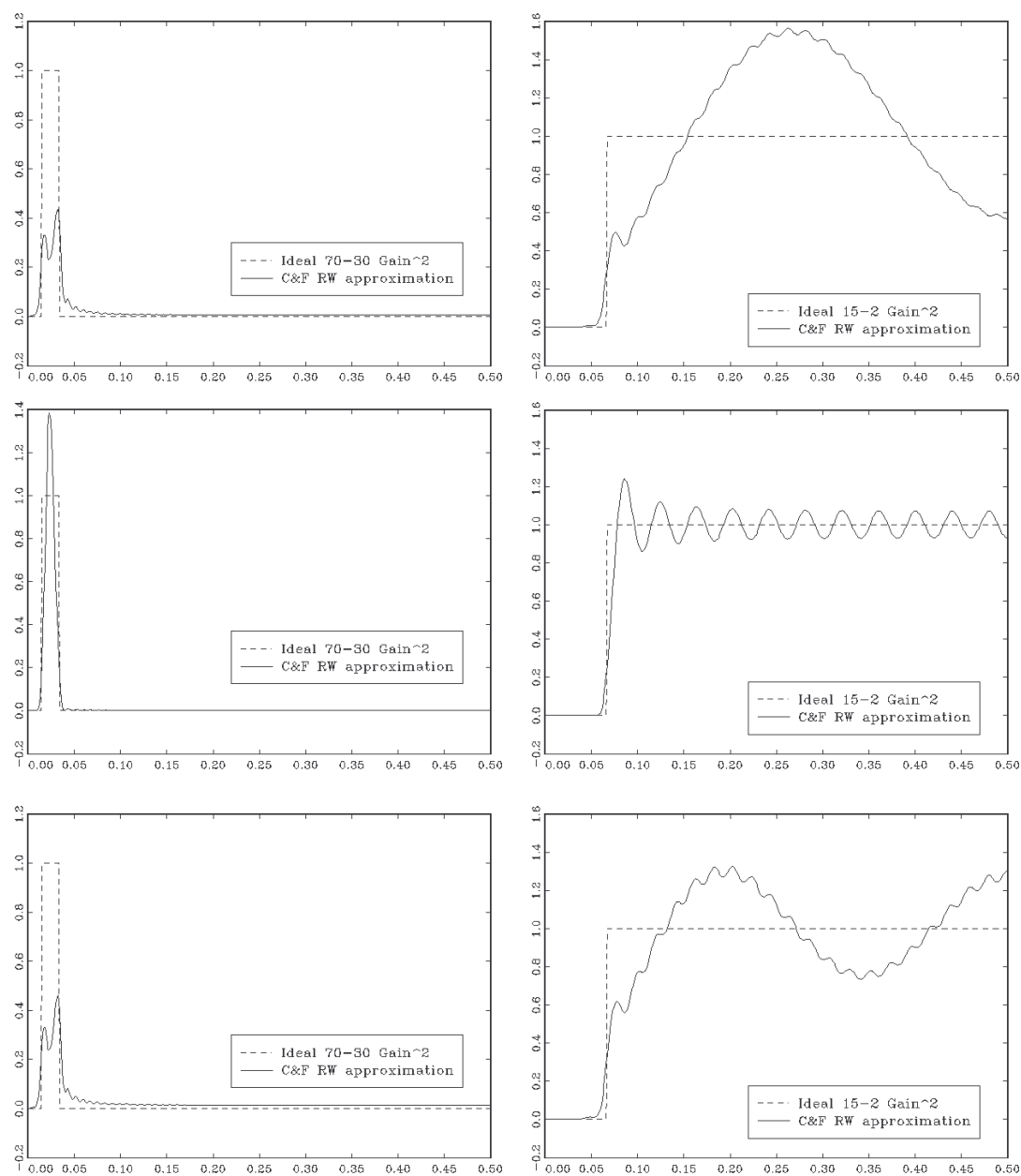
Let's now compare the results from the zero phase frequency filter to those of the Christiano-Fitzgerald filter. Figure 6.4 shows the filter output from applying the two filters. Because the weights of the Christiano-Fitzgerald filter vary over time (i.e. are non-stationary) it is not sufficient to look at only one FRF or Power Transfer Function (PTF). Also see section 5.3.3. Therefore, Figure 6.5 shows the Christiano-Fitzgerald approximation of the PTF of the relevant ideal band-pass filters for three situations: in the beginning of the sample, in the middle of the sample and at the end of the sample. The results clearly show that also the zero phase frequency filter and the Christiano-Fitzgerald filter give very similar filter output. Compared to the Baxter-King filter, the Christiano-Fitzgerald loses less observations, but still two on both sides of the sample. The biggest differences occur away from the middle of the sample. The bigger differences in those areas of the sample were to be expected because from section 5.3.3 we know that in those cases the quality of the Christiano-Fitzgerald approximation of the FRF (PTF and phase) of the ideal band-pass filter deteriorates. As far as the PTF is concerned, this is confirmed by the results from Figure 6.5. Especially for the filter with the  $[1/70, 1/30]$  pass-band frequency range, at the end of the sample, the Christiano-Fitzgerald filter is not capable of adequately approximating the PTF of the ideal filter<sup>41</sup>. It is therefore no coincidence that in Figure 6.4, the biggest differences in filter output occur at both ends of the sample for the long wave component. Because we know that both the Baxter-King and the Christiano-Fitzgerald filter *do not* and the zero phase frequency filter *does* exactly apply the FRF of the ideal band-pass filter, we should have more confidence in the output from the latter filter. Furthermore, the zero phase frequency filter does not suffer from the loss of any observations at the ends of the sample and the drawbacks of the Christiano-Fitzgerald filter mentioned at the end of section 5.3.3. That the results from the three filters are to a large extent comparable, merely confirms our confidence in the zero phase frequency filter while at the same time it shows that the other two filters can also provide acceptable, though not optimal, results.

Figure 6.4 Comparison of output of zero phase frequency filter and Christiano-Fitzgerald filter applied on the (natural) logarithm of a National Product index for the Netherlands. The left hand side shows the case of filtering for the long wave component in the frequency range  $[1/70, 1/30]$  for the period 1870-1999. The right hand side shows the case of filtering for the stochastic component in the frequency range  $[1/15, 0.5]$  for the postwar period 1949-1999.



<sup>41</sup> That the approximation seems worse for low frequency ranges could be caused by the fact that the Christiano-Fitzgerald filter assumes (low frequency) trends to have been removed prior to filtering the time series. For reasons explained at the end of section 5.3.3, here we did *not* remove the trend.

Figure 6.5 PTF (squared gain) of Christiano-Fitzgerald filter for the filters applied to obtain the results in Figure 6.5 compared with the PTF of the ideal band-pass filters. The top row shows the results in the beginning of the sample, the middle row in the middle of the sample and the bottom row at the end of the sample.



In each of the separate chapters for the various macroeconomic variables the results of filtering the time series are presented both graphically and numerically. The *graphical analysis* is always for the Netherlands<sup>42</sup> and shows

1. the (transformation of) the original time series together with the trend component,
2. the periodic component of the variable together with its long wave component and the long wave component of the corresponding national product,
3. the stochastic component of the variable together with the stochastic component of the national product and
4. the long wave component together with the long wave component in the same variable for the United Kingdom and the United States.

The comparisons with the same components of the national product are included to visualize the relations within a single country while comparing the long wave in the various countries comes from the fact that the long wave is believed to be an international phenomenon. The complete graphical results for the United Kingdom and the United States are contained in respectively Appendix D and E.

The *numerical analysis* part of the filtering results is similar to that of the data analysis as discussed in section 6.2. Instead of on (a transformation of) the original time series it is now performed on the filtered periodic, long wave and stochastic components of the time series for each of the three countries. Because these filtered components all have an average of zero (the time series average is contained in the trend component) an average value is not reported. The results therefore consist of the standard deviation, the ratio of the standard deviation with that of the same component in the national product and the correlation with this series, calculated for the various time periods. For each variable there is also a table which summarizes the peaks and troughs of the long wave component (these are also indicated in the graphical results). This enables the estimation of the peak-to-peak and trough-to-trough period length and also the possible lead or lag of the long wave in the relevant variable over the long wave in the corresponding national product index. These numbers are reported both for the separate countries as for the average peaks and troughs calculated on the three countries. After all, the long wave is believed to be a world wide phenomenon.

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<sup>42</sup> The only exception is in Chapter 15 for the default spread for which no sufficiently long historical time series is available. In that specific case the US results are included in the main text.

### 6.3.3 Volatility of filtered series

In conventional econometric (Box and Jenkins) time series analysis it is common to estimate stochastic models for the (stationary) *first order (or sometimes second order) differences* of time series. This holds both for the modeling of (natural) logarithms of indices<sup>43</sup> such as national product or an equity index and for the modeling of interest rates. In the research presented here we apply the high-pass [1/15,0.5] filter to obtain the so called stochastic component. Now suppose that a stochastic model would indeed be estimated for such a filtered component. Then an interesting question becomes how these two approaches differ with respect to the volatility? Because of the linearity of the filters (both the high-pass and the first order differencing filter are linear filters) it does not matter in what order they are applied on a time series, as long as they are both applied. For this also see end of section 5.1. It should therefore make no difference whether we first calculate the first order differences and then apply the high-pass filter to the resulting series or, the other way around, if we apply first order differencing on the filtered stochastic component of the time series. Because the stochastic component has already been calculated, for convenience the latter approach is chosen. In order to investigate the differences in volatility, the standard deviation of the historical first order differences of the original (log transformed) time series is calculated and compared to the standard deviation of the first order differences of the filtered stochastic component. For similar historical periods as for the data analysis and the filtering results, both these standard deviations and their ratio are reported.

Note that the volatility of the first order differences of the filtered stochastic component will always be smaller than the full volatility of the first order differences of the original time series because the FRF of the first order differencing filter leaves a part of the volatility below the frequency  $1/15=0.067$  intact (see Figure 5.1 in section 5.2.1) while the [1/15,0.5] high-pass filter completely suppresses these fluctuations. In itself this smaller volatility is not a bad thing. It is a logical consequence of the chosen filtering approach. However, very large differences are not desirable. Suppose for example that one is used to modeling total returns on an equity index, calculated as the first order differences of the logarithm of this index, with a standard deviation of say 20%. Although this is not very likely, if modeling the high-pass [1/15,0.5] filtered stochastic component would lead to a volatility of the returns of only 10% then it would be difficult to feel very comfortable with such an alternative modeling.

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<sup>43</sup> Because  $(x_t - x_{t-1}) / x_{t-1} \approx \ln(x_t) - \ln(x_{t-1})$  for small changes in  $x_t$ , the growth of an index is often calculated as the first order differences of the natural logarithm of the index time series.



## 6.4 Spectral analysis

The variance of the *stochastic component* is reported with the filtering results for the various samples. The most ideal instrument for further investigation of the *dynamics* that constitute this total variance is spectral analysis. Because of the small (sub) sample sizes it is recommended to use a parametric approach for the estimation of the spectral densities. Therefore the *Maximum Entropy spectral estimation* technique as described in section 4.6.4 is applied. Based on the outcomes of the extensive Monte Carlo experiment described in section 4.7, the *Yule-Walker* technique is used for the estimation of the autoregressive models involved. In general the other estimation techniques such as OLS and the Burg method lead to very similar estimates with respect to the location of the peaks in the spectral densities. The major difference is that these methods, especially OLS, tend to overestimate the amount of periodicity present in the data. That is, the peaks in the estimated spectral densities are too sharp to be consistent with the sample autocorrelations. One could say that the OLS method yields too “non-stationary” spectral estimates which is solved by the Yule-Walker method by imposing a stationarity restriction. See section 4.7.7 for more information on this point.

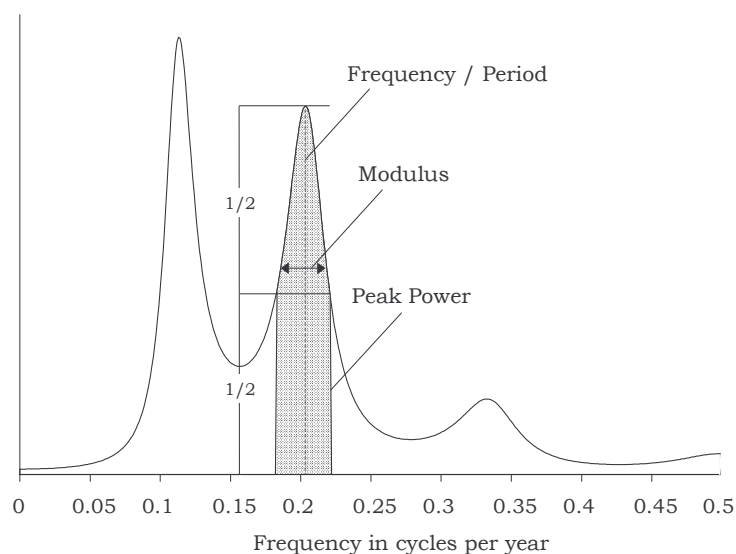
Following the recommendations from section 4.7.7, for the *order selection* procedure a combination is used of the information coming from the (local) minimums of the various order selection criteria, the order above which the information in the corresponding spectral densities does not significantly change anymore and the order that gives the “best” extrapolation of the sample autocorrelations. The most weight for selecting the model order is put on how well the model fits the sample autocorrelations in order to keep the analogy with conventional non-parametric spectral estimators as high as possible. In general, judgement was used to combine the various sources of information into the final model order.

When estimating the spectral densities, a split is made between *static* and *dynamic* estimation. The following sub-sections describe in more detail what this distinction means and what kind of results are reported. In general, just as with the data and filtering analysis, the results consist of both a graphical and a numerical part.

### 6.4.1 Static

For the static spectral analysis separate spectral densities are estimated for the filtered stochastic components for the prewar, interwar and postwar period, to the extent of course that the original time series cover these time periods. For the estimation of the *auto-spectra* univariate autoregressive models are used to avoid the feed-across effect as described at the end of section 4.7.7. With respect to the *graphical results*, the resulting (normalized) spectral density and the integrated spectrum are shown, together with the corresponding 95% confidence interval of a white noise process to be able to assess the statistical significance of possible peaks in the spectrum. The confidence intervals are obtained by the procedure described in section 4.8.1. The *numerical results* consist of the following measures to characterize the estimated auto-spectra as best as possible. The link between the measures and the possible peaks in a spectral density is illustrated in Figure 6.6.

Figure 6.6 Various measures of a peak in an auto-spectrum.



### *Peak Period*

Without exception, the estimated autoregressive models contain complex roots which cause peaks in the corresponding spectral densities. The reported periods are the period lengths (in years) of these complex roots calculated as the reciprocal of the frequency of the roots. These frequencies are the center of the peaks in the spectral densities and are therefore representative for the kinds of pseudo period behavior present in the variable under investigation. Calculating these peak periods from the roots of the autoregressive model is easier and more precise than calculating them numerically from the maximums of the spectral densities (as would have to be done in the case of non-parametric spectral estimation).

### *Peak Power*

The peak power aims to give an indication of how important the various peaks in a spectral density are in explaining the total variance of the stochastic component. Contrary to the period and also the modulus to which we turn next, it has to be calculated numerically for each of the complex roots. The peak power is defined as the (relative) amount of variance that is between the first frequency to the left of the peak period and the first frequency to the right of the peak period at which the spectral density has dropped to half the height of the peak. This frequency area is always bounded by the limits of the frequency range  $[0, 0.5]$  and by the frequency exactly in the middle between two adjacent peak frequencies. The last is to prevent that the areas for the peak power of two adjacent peaks overlap and that thereby a part of the total variance is attributed to more than one peak in the spectral density. The peak power can be interpreted as a kind of  $R^2$ . If it is for example 0.25 this says that the relevant peak in the spectrum describes about 25% of the total variance.

### *Modulus*

The modulus aims to measure the amount of periodicity as indicated by the sharpness of a peak in the spectrum. The example in section 4.5.3 illustrates that it is the modulus of a complex eigenvalue (the inverse of a complex root) that determines the sharpness of the corresponding peak in the spectral density. The closer it is to one, the more distinct pseudo periodic behavior of the relevant frequency will be present in the stochastic process at hand. The closer it is to zero the broader the corresponding peak will be. An alternative measure for the amount of periodicity, which is not used here, is the more conventional *spectral bandwidth* as described by Priestley (1981, p. 513). This is the width of the interval that is used to calculate the peak power as defined in Figure 6.6. Note that calculating the modulus of a spectral peak is only possible in the case of autoregressive spectral estimation and not for the non-parametric approaches.

Besides a variable's auto-spectrum, also its *cross-spectrum* with the national product of the relevant country is estimated. So, as illustrated in Figure 6.1, a country's national product is the reference point for the multivariate relations. In case of the national product for the United Kingdom or the United States the cross-spectrum with the national product of the Netherlands is estimated. The cross spectra for each of the samples are estimated by means of a bivariate VAR model on the stochastic component of the variable at hand and the relevant national product. The order of the model is always the same as the order of the univariate AR model used for estimating the auto-spectrum (of the variable under investigation, so not of the national product). Because the cross-spectra are determined by the cross correlations between the two variables, it is also checked whether the resulting model reasonably fits the sample cross correlations. In general the auto-spectra from the VAR model look very much like the auto-spectra from the AR models. However due to the feed-across effect, roots from one variable may turn up in the spectrum of the other variable, although in general with a low peak power. This complicates the automated identification of the relevant eigenvalues from the VAR model. The cross-spectra are therefore evaluated at the frequencies of the complex roots of the AR model used for the estimation of the auto-spectra of the variable under investigation. Because of their ease of interpretation, the *coherence* and *phase* representation of the cross-spectra are used. These two types of spectra are not graphically reported. Because they in general have a smooth shape it is sufficient to include the following two measures in the numerical results for each of the peaks in the auto-spectra.

### *Coherence*

The coherence measures the phase corrected (absolute) correlation between the fluctuations of the indicated period lengths in the variable under investigation and the relevant national product. It is calculated as the value of the coherence spectrum evaluated at the frequencies of the complex roots of the AR model used for the estimation of the auto-spectra for which also the peak period, peak power and modulus, as described before, are reported.

### Phase

The phase measures the lead / lag relations at the indicated period lengths between the variable under investigation and the relevant national product. It is calculated as the value of the phase spectrum evaluated at the frequencies of the complex roots of the AR model used for the estimation of the auto-spectra. The phase is expressed in years and is always between a lead (positive phase) of half a period length and a lag (negative phase) of half a period length.

An exception to this procedure has to be made if in a sub-sample there are more data available for the variable under investigation than there are for the national product. For example the prewar sample of the stochastic component of the Dutch consumer price index is 1813-1913 while the Dutch national product series only starts in 1870. In that case, peak period, peak power and modulus are all based on the univariate spectral estimates for the full 1813-1913 period. The coherence and phase are calculated from the cross-spectrum estimated by the VAR model of the same order on the largest possible joint sample, 1870-1913 in the example, but at the frequencies indicated by the auto-spectrum estimated on the full 1813-1913 sample. The underlying assumption is that the estimated auto-spectrum does not differ very much whether it is estimated on the full 1813-1913 sample or on the smaller 1870-1913 sample. In that case it is the most efficient to estimate as much measures as possible on the largest available sample.

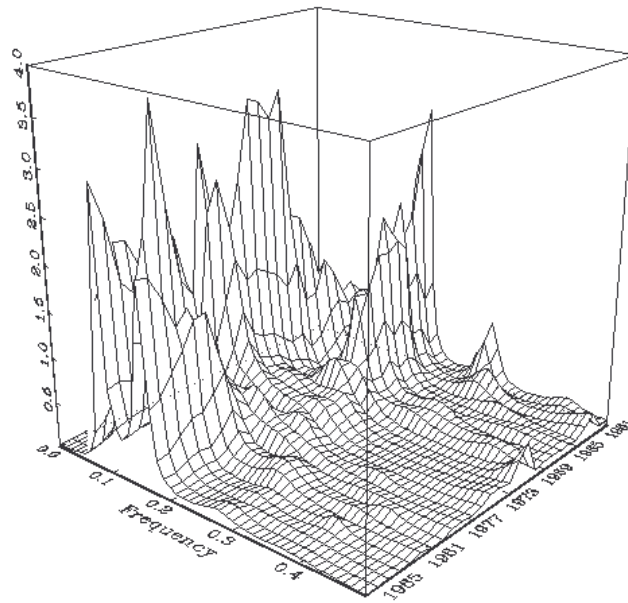
### 6.4.2 Dynamic

One of the explicit objectives as stated in the first research question in section 1.3 is to investigate possible *changes* in the macroeconomic process in the course of time as well. Information on such possible changes is of course already obtained by estimating separate (static) spectral densities for the prewar period, interwar and the postwar period. To zoom in on this aspect, also a *dynamic spectral analysis* is performed by means of analyzing *rolling window spectral estimates*<sup>44</sup>. Such an analysis is performed for the postwar period only. The interwar sample is too short to allow for such an analysis while the data for the prewar period is in general of too poor a quality to obtain such delicate information. Furthermore, the postwar period is of course the most relevant for inference about the direction in which the macroeconomic process can change in the coming decades. Finally, only the postwar period is studied in this dynamic manner to limit the already extensive amount of work. What the rolling window spectral estimates behold is best illustrated by Figure 6.7.

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<sup>44</sup> Priestley (1981) refers to such spectral densities as *evolutionary spectra*.

Figure 6.7 Rolling window spectral estimates.



What is depicted here are twenty-seven consecutive estimated (normalized) spectral densities for the postwar period for some economic variable. Each of the estimates is based on a different sub-sample consisting of twenty-five observations. The first uses the 1949-1974 sample, the second the 1950-1975 sample, the third the 1951-1976 sample and so on until the final sub-sample of 1975-1999. The year reported on the right axis is always the middle of the relevant sub-sample. For the estimation of the spectral densities, again the Yule-Walker estimation technique is used. Note that, because of the even shorter samples than with the static spectral estimation, the superior performance of the Yule-Walker technique in small samples, as pointed out by the Monte Carlo experiment in section 4.7, is of extreme importance here to obtain reasonable spectral estimates. All the twenty-seven spectral densities use the same AR *model order* which is selected by a rolling window version of the same approach as with the static estimation. Because of the small sample size, criteria such as the AIC have the tendency to indicate a very low model order while comparing the corresponding model autocorrelations to the sample autocorrelations show very marked differences. Just as with the static analysis, often the fit with the sample autocorrelations was decisive for the choice of the model order. A model order high enough was chosen to ensure that for each of the sub-samples a good enough fit was obtained. Note that in principle the resulting model order can be different than the one used for the static estimation of the spectral density for the full postwar period.

Although a careful investigation of a picture such as in Figure 6.7 may certainly reveal interesting information, it is not to be the most informative *graphical analysis*. From this picture we can for example see the continuous presence of two peaks in the spectrum, one around a period length of ten years (frequency 0.10) and one around a period length of five years (frequency 0.20). We also see that in the middle of the sample the importance of the second peak in describing the total variance has decreased. However for example we cannot see exactly how the period lengths of the peaks have evolved over time. It is also hard to say something about

changes in the amount of pseudo periodicity present in the process (i.e. the sharpness of the peaks). Furthermore such normalized spectra tell us nothing about changes in the total variance. Therefore such pictures are not included in the graphical results. Instead, for each of the consecutive spectral densities the same measures are calculated as for the static analysis, i.e. peak period, peak power, modulus, coherence and phase for each of the complex roots present in the corresponding AR polynomials are calculated. The evolution of these measures is reported in separate pictures.

Two further remarks are in order here. First, contrary to the static results, here the non-normalized peak power is used because also the rolling window variance is included. Non-normalized peak powers can be directly compared to this total variance. Second, it is somewhat arbitrary to determine how the peaks in consecutive spectral densities should be linked to one another in the course of time. However, if the spectral densities are relatively constant over time in terms of the number of complex roots present in the corresponding AR models, as for example in Figure 6.7, it suffices to simply order the roots in terms of their frequency / peak period and connect all the first roots from each sub-sample, all second roots, and so on. In general the first three roots are selected, as far as these are present of course. In case this procedure results in an awkward linkage of the peaks in the spectral densities, this is in general easily observed from the graphical results. Especially large jumps in the rolling window peak periods are very informative on this point and also indicated how to alter the linkage of the spectral peaks.

The approach for the multivariate spectral measures is in principle the same as for the static estimates. For each of the sub-samples a bivariate VAR model is estimated with the stochastic component of the relevant national product from which the coherence and phase at the peak frequencies of the corresponding (univariate) auto-spectrum of the variable under investigation are calculated. The only difference is that sometimes a specific value of the phase is increased or decreased by a full period length. This especially occurs when the rolling phase shows very large jumps between a phase somewhere near a lead of half a period length and a lag of half a period length which are essentially the same. By adding or subtracting a full period length a more fluent evolution of the rolling window phase can be obtained.

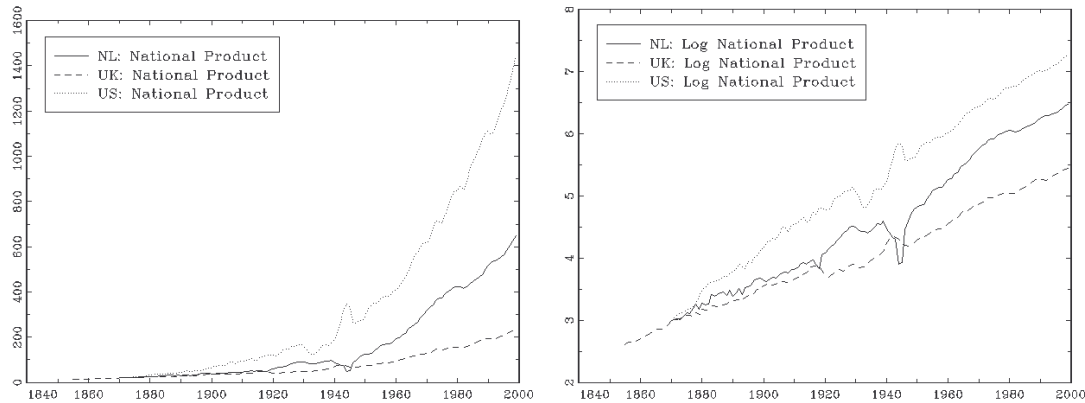
Finally, in the *numerical part* of the dynamic spectral analysis the average of the various rolling window spectral measures (peak period, peak power, etc.) is reported. Here the normalized peak power is reported again. The main objective of reporting such averages is to see whether these are very different from the static estimates. If this turns out to be the case it may be so that the static estimates do not give very useful information because for example very large changes occurred in the underlying stochastic process during the sample period. In that case, at least some further analysis is required to see which type of information is the most informative (the static or the average dynamic).



## 7 National Product

### 7.1 Data analysis

Figure 7.1 Real (volume) National Product indices and their (natural) logarithm. The series pass through a common point in 1870 (start of NL and US series).



- The national product indices give an indication of the total value of all goods and services produced in the various countries within one year. They hold in real (volume) terms which means they are corrected for price changes. In this sense they can be seen as volume indices. They are not corrected for the growth in a country's population (i.e. they are not per capita indices). Long term population growth is rather steady at about 1% per year and hardly changes the results presented in this chapter.
- In general one would like to analyze the various constituents of a national product index like consumption, investments, government expenditure (consumption and investments) and the balance of exports and imports. Except for the industrial production indices analyzed in the next chapter, unfortunately long term data for such variables are not available.
- One common feature of the time series that is hard to miss from Figure 7.1 is their *exponential trend*. For example for the NL the size of the national product in 1999 is more than thirty times as big as at the start of the sample in 1870. This comes down to an average<sup>45</sup> (real) growth of 2.7% per annum which together with other statistics can be found in Table 7.1. With 2.0% the long term average growth in the UK is lower than in the NL while the US has experienced a higher growth of 3.2%. These differences in average growth are reflected in the slope of the logarithmic transformations of the series and thereby also in their value at the end of the sample.
- The averages in Table 7.1 also show that there have been long periods (several decades) during which the *average growth* was markedly lower or higher than the long term averages. For example the average growth was higher during the first half of the postwar period (until 1975) whereas it was lower during the second half of the postwar period, especially for the NL. These periods are known

<sup>45</sup> The averages reported in the tables are conventional arithmetic averages. In this case, because of the low volatility, the geometrical averages are virtually identical.



respectively as “the golden age of fast growth” and the period of the “productivity slowdown”. Note that it is generally believed that the countries involved in the second World War needed about five years to recover from it. Therefore the high growth rates during the three decades following the war cannot solely be attributed to this recovery effect. Maddison (1989) reports and discusses statistics for OECD countries during these periods while Boeschoten (1992) presents a separate analysis for the NL. Hobsawm (1994) discusses the history of the 20<sup>th</sup> century in terms of three episodes: (I) 1914-1945, “the age of catastrophe”, (II) 1945-1973, “the golden age” and (III) 1973-1991, “the landslide”.

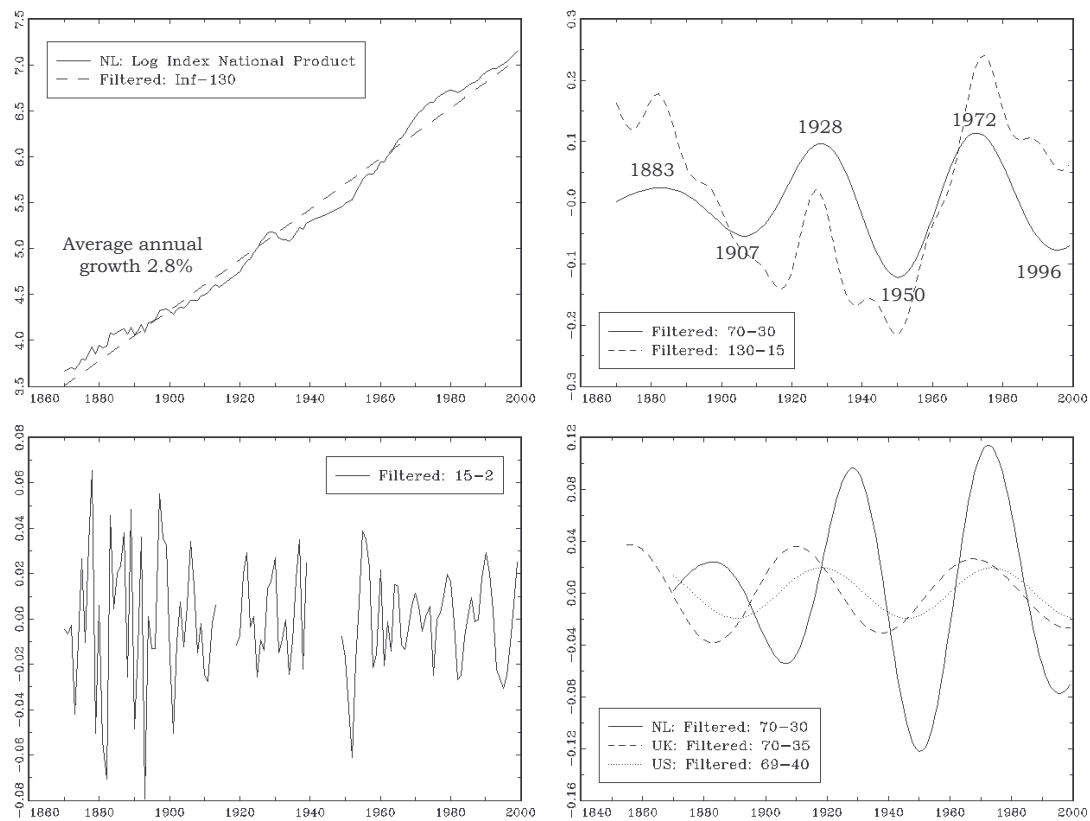
- Specific *historical events* have had strong effects on the volume of the national product. In the NL, World War I and especially World War II led to a sharp decrease in production while for the UK and the US the period of World War II shows a strong increase in national production, probably because of the weapon and other war industry. Negative effects of the Great Depression during the 1930's are also clearly visible in the data, especially for the US.
- Besides the trends and the long term deviations in average growth around these trends, also *shorter term fluctuations* of the national product indices are visible. Before World War II there were many years in which the volume of national product actually decreased. During the postwar period such events hardly occurred anymore and these were replaced by periodic setbacks in the growth rate rather than in the level of the national product. The smoother development of the indices during the latter period already shows that the volatility of the growth of the national products has decreased over time. This is confirmed by the standard deviations for the various sub-periods as reported in Table 7.1 and is consistent with the alleged stabilization of the economy as reported by others. Also see section 2.2.4.
- Especially during the postwar period, the *volatilities* of the growth rate for the various countries are very similar. That is, the ratio of the standard deviations is about one which indicates that the national product is not more or less volatile in one country than it is in the other.
- The *correlation* between the growth of the national product for the various countries is positive for almost every sub-period which indicates some extent of international connection between the fluctuations of the indices. In general the correlations are not extremely high which may be consistent with some sort of lead / lag relation between the country's economies. The further back in time, the lower the correlations which might be because of poor quality data or because of less international relations at that time.

Table 7.1 Statistics of the growth (delta logarithm) of National Product indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the NL.

<i>Netherlands</i>				<i>United Kingdom</i>				<i>United States</i>			
<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>	<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>	<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>
1871-1999	2.7%	7.5%	-	1856-1999	2.0%	3.5%	-	1871-1999	3.3%	5.5%	-
1871-1913	2.2%	5.0%	-	1871-1999	1.9%	3.5%	0.5	1871-1913	3.9%	4.5%	0.7
1920-1939	2.8%	3.7%	-	1871-1913	1.7%	3.6%	0.7	1871-1913	3.9%	4.5%	0.9
1950-1999	3.4%	2.3%	-	1920-1939	1.6%	3.8%	1.0	1920-1939	1.8%	6.9%	1.9
1950-1975	4.4%	2.4%	-	1950-1999	2.4%	2.0%	0.9	1950-1999	3.3%	2.4%	1.1
1976-1999	2.3%	1.5%	-	1950-1975	2.8%	2.1%	0.9	1950-1975	3.7%	2.7%	1.1
				1976-1999	2.0%	1.8%	1.2	1976-1999	3.0%	1.9%	1.3
							0.3				0.5

## 7.2 Filtering results

Figure 7.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a National Product index for the NL and its long wave component compared to UK and US data (bottom right).



- For filtering the trend, periodic and long wave components from the Dutch time series, linear interpolation was applied for the periods 1916-1919 and 1939-1949 to remove the disturbing effects of the two World Wars. Note that this interpolation is visible in the top left picture of Figure 7.2. Also for the UK and US, linear interpolation was applied, possibly for (slightly) different periods.
- The *trend component* for each of the three countries (see Appendix D and E for the figures with the UK and the US results) consists of a linear trend. The average annual growth rates indicated by these trends are almost identical to the actual average growth rates as reported in the data analysis. For the NL for instance, the long term average growth was 2.7% while the slope of the trend component is 2.8%.
- Because the linear trends were not imposed in some way but are the result of a filtering exercise in the frequency domain, on the one hand these findings can be seen as “support” for the linear trend hypothesis for national product indices as also found by for example Diebold and Senhadji (1996). On the other hand, a filtering exercise is not a formal statistical test. See the end of section 5.6 for more references on the discussion about linear versus stochastic trends.

- A general dominant property of the *periodic components* for each of the three countries is roughly speaking a high level during the 19<sup>th</sup> century, a low level during the first half of the 20<sup>th</sup> century and again a high level during the second half of the 20<sup>th</sup> century. This pattern is probably caused by the sequence of special historical events during the first half of the 20<sup>th</sup> century which was a period of great turbulence. For example World War I, the Great Depression, the Russian Revolution and World War II took place During this period. Maddison (1989) classifies this period as one of great conflict and disharmony in the world.
- Besides this very long historical swing and some higher frequency fluctuations, the periodic components also contain *long wave components*. For the NL the amplitude of this long wave component (around 10%) is much larger than for the UK and the US (around 3%). Such long wave fluctuations were already observed in the data analysis in terms of long periods of higher and lower average growth rates. Table 7.5 indicates an average peak-to-peak and trough-to-trough *period length* of respectively 51 and 52 years. At around 45 years it is approximately 10 years shorter than for the UK and US. A comparison of the dating of the peaks and troughs in this table with the approximate dating from the literature as given in Table 2.1 shows that these are very consistent with the peaks and troughs in the long wave components for the UK and US. Furthermore with respect to the long wave component, the NL *lag* the other two countries, especially at the end of the 19<sup>th</sup> century and at the beginning of the 20<sup>th</sup> century. This could be because of late participation of the NL in the Industrial Revolution at the end of the 19<sup>th</sup> century and the fact that the NL were one of the countries that held on the gold standard longer than other countries during the 1930's. See for example Griffiths (1996) and Van Zanden (1996) for these two causes. Furthermore, it could be that the UK and the US have been the leading countries in the world in adopting new fundamental innovations as indicated in Table 2.1. Especially during the 19<sup>th</sup> century and the beginning of the 20<sup>th</sup> century, the peaks and troughs in the long wave component of the UK clearly *lead* those in the US and the NL. This could be because of the UK's leading role in the Industrial Revolution. Finally note the low level of each of the long wave components at the end of the sample (1999) which may indicate that at the start of the 21<sup>st</sup> century the world may find itself in the upgoing phase of the fifth Kondratieff cycle initiated by the Information and Communication Technology (ICT) as already suggested in section 2.2.3.
- The *stochastic components* for the postwar period show peaks and troughs of business cycle fluctuations that are consistent with historical knowledge. Well known recessions around 1975, at the beginning of the 1980's and at the beginning of the 1990's are clearly visible. These recessions appear to be approximately ten years apart. Some higher frequency fluctuations are visible as well. For the NL the stochastic component also shows a clear M-shaped pattern during the 1980's as described in section 2.2.3.
- In each of the three countries the high value of the stochastic component at the end of the sample (1999) indicates a relatively high level of national product<sup>46</sup>.
- The results in Appendix D and E show that during the postwar period there has been a high degree of consistency between the stochastic fluctuations in the various countries. Careful investigation of the peaks and troughs in the relevant

<sup>46</sup> At the moment of writing, the summer of 2002, we know that this period of relative prosperity has been / will be followed by a recession in most of the Western countries.

pictures shows that both the UK and the US tend to lead the NL by one or two years. This is consistent with the literature from which we know that business cycles in the UK and the US behave rather synchronized while both tend to lead the business cycles in European countries such as the NL. See for example Maddison (1982), Bjørnland (2000) and Schoenmaker (1999) for these issues. More detailed information on the exact nature of these *international lead / lag relations* should come from the spectral analysis in the next section.

- In general the prewar and interwar periods show a higher *volatility* than the postwar period. During the postwar period the maximum deviations from the underlying trend are somewhere in the range of 3 to 4%. With a growth of the underlying trend of around the same size, it is clear that actual decreases in national product hardly occurred during that period, resulting in the *growth cycles* as described in section 2.2.3. During the prewar and interwar period the maximum deviations can be up to 6 or 7% thereby at some times causing substantial decreases in the national product. The lower volatilities, as indicated by the results in Table 7.2, confirm the alleged stabilization of economies during the postwar period when compared to the prewar and interwar period.
- During the entire postwar period the volatility of the stochastic components is about 10% higher in the UK and the US than it is in the NL. During the second half of the postwar period this difference is even 40%. This could indicate that business cycle fluctuations in these countries are more pronounced than they are in the NL. This should however be confirmed by the results for other variables such as the industrial production. Note that the average growth rates from Table 7.1 do not consistently confirm that this higher volatility has also led to a higher average growth rate (risk premium). On average one could say that during the postwar period the standard deviation of the stochastic (annual) fluctuations in the national product are somewhere around 2.0%. This means that about 95% of the business cycle deviations in national product indices are between  $-4\%$  and  $+4\%$  of the underlying trend.
- The results in Table 7.3 show that the stochastic components consistently describe around 90% of the total volatility of the conventional growth of the national product. Only during the interwar period for both the NL and the US this is lower at around 70%.
- In general the standard deviations / amplitudes of the stochastic component are much lower than those of the periodic and long wave components. From this one could conclude that the stochastic fluctuations are therefore not that important. However, the fact that they describe almost the entire volatility of the growth rates of the national product is a first argument against such a conclusion. Furthermore, note that the importance of a component depends on the horizon applied in an analysis. This means that the volatility of a component with period lengths around forty years can only manifests itself on a horizon of several decades while the volatility of a component with a period length of say five years is already very visible within a couple of years.
- Just as with the data analysis, the *correlations* between the various countries are positive but not extremely high, probably because of international lead / lag relations.

Table 7.2 Statistics of the filtered periodic, long wave and stochastic components of National Product indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the NL.

	Netherlands				United Kingdom				United States			
	Filt	Hist	Ratio	Corr	Filt	Hist	Ratio	Corr	Filt	Hist	Ratio	Corr
Periodic	1870-1999	12.4%	-	-	1870-1999	8.2%	0.7	0.8	1870-1999	9.5%	0.8	0.3
Long Wave	1870-1999	6.3%	-	-	1870-1999	2.3%	0.4	0.1	1870-1999	1.4%	0.2	0.6
Stochastic	1870-1913	3.3%	-	-	1870-1913	2.6%	0.8	0.2	1870-1913	3.4%	1.0	-0.1
	1919-1939	1.8%	-	-	1919-1939	3.9%	2.1	0.0	1919-1939	4.5%	2.5	0.5
	1949-1999	2.0%	-	-	1949-1999	2.1%	1.1	0.5	1949-1999	2.2%	1.1	0.2
	1949-1975	2.2%	-	-	1949-1975	1.7%	0.8	0.5	1949-1975	2.1%	0.9	0.2
	1976-1999	1.8%	-	-	1976-1999	2.5%	1.4	0.6	1976-1999	2.4%	1.4	0.7

Table 7.3 Standard deviation of growth (delta logarithm) of National Product indices for various sub-periods (Hist) versus standard deviation of first order differences of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States		
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio
1870-1913	4.8%	5.0%	1.0	1855-1913	3.4%	1.0	1870-1913	4.4%	4.5%
1919-1939	2.5%	3.7%	0.7	1919-1939	3.5%	3.6%	1919-1939	5.2%	6.9%
1949-1999	1.9%	2.3%	0.8	1949-1999	1.9%	2.0%	1949-1999	2.1%	2.3%
1949-1975	2.2%	2.4%	0.9	1949-1975	2.0%	2.1%	1949-1975	2.3%	2.6%
1976-1999	1.4%	1.5%	0.9	1976-1999	1.8%	1.8%	1976-1999	1.9%	1.9%

Table 7.4 Peaks and troughs of the long wave component in National Product indices. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the National Product index for the NL (to the extent that the “same” peaks and troughs are present).

	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>Peak-Peak Trough-Trough Lead</i>	
Netherlands	Peak Trough Lead	1883  -	45 1907 -	1928 43 -	44 1950 -	1972 46 -	45 45 -
United Kingdom	Peak Trough Lead	1856  27	54 1883 24	1910 55 18	58 1938 12	1968 61 4	56 58 14
United States	Peak Trough Lead	   	 1890 17	1918 56 10	56 1946 4	1974 53 -2	56 55 5
Average	Peak Trough	1870  	49 1893	1919 51	53 1945	1971 53	51 52



## 7.3 Spectral analysis

### 7.3.1 Static

- All estimated static spectral densities for the various countries and sub-periods from Figure 7.3 show very little spectral mass below the frequency  $1/15=0.067$ . This is consistent with the fact that the underlying stochastic components were obtained by a filter with pass-band  $[1/15, 0.5]$ .
- In general three types of peaks can be observed in the spectral densities. The *first* is a peak around a *period length* of between seven and eleven years. This peak can be observed for almost every period and country with the exception of the interwar period in the US. Another “exception” is the prewar period in the UK, for which the period length of more than thirteen is relatively long. The peak typically has a *modulus* of around 0.9 indicating a high degree of pseudo periodicity. With a *peak power* of typically 0.3 the corresponding fluctuations describe around 30% of the total variance of the stochastic components. Without exception the spectral peak reaches outside the plotted 95% confidence interval indicating that these fluctuations can be considered *significantly different* from the fluctuations at the same frequencies in a completely random white noise process. With the exception of the prewar period, these fluctuations show a high degree of correlation in terms of a *coherence* of approximately 0.9. Note that this is much higher than the conventional correlation numbers reported in the previous sections because a coherence is a correlation that is corrected for possible phase differences. The *phase* consistently indicates a lead for both the UK and the US of around one and a half year over the NL at these frequencies. So these results confirm the *international lead / lag relations* between the various countries as mentioned in the previous sections. Judged by their frequency, these type of fluctuations may in general be classified as *Juglar cycles* as describe in section 2.2.3.
- The *second* type of peak that is present in the spectral densities has a *period length* of some four to five years. Although frequently visible, this peak is not as consistently present for the various periods and countries as the first type of peaks. Also it never reaches outside the white noise confidence interval and can therefore formally not be considered statistically significant. In general the *modulus* is around 0.8, thereby indicating less pseudo periodic behavior than the previous peaks. Also, the *peak power* is a little lower, indicating these fluctuations describe around 20% of the total variance. Furthermore the international correlation between these type of fluctuations is somewhat lower as indicated by a *coherence* of around 0.6. Especially for the more recent sub-period the associated *phase* indicates a lead of the UK and the US of less than one year. The lag of about two years for the prewar period in the UK is not very informative because of the low coherence of only 0.4 while the other lag of two years for the interwar period for the UK may be a consequence of the turbulence during this period. Judged by their frequency these type of fluctuations may in general be classified as *Kitchin cycles* as describe in section 2.2.3.
- The *third* type of peak that is sometimes present in the spectral densities are rather high frequency fluctuations with a *period length* of between two and three years. Although they can have a reasonably high *modulus*, their *peak power* is in

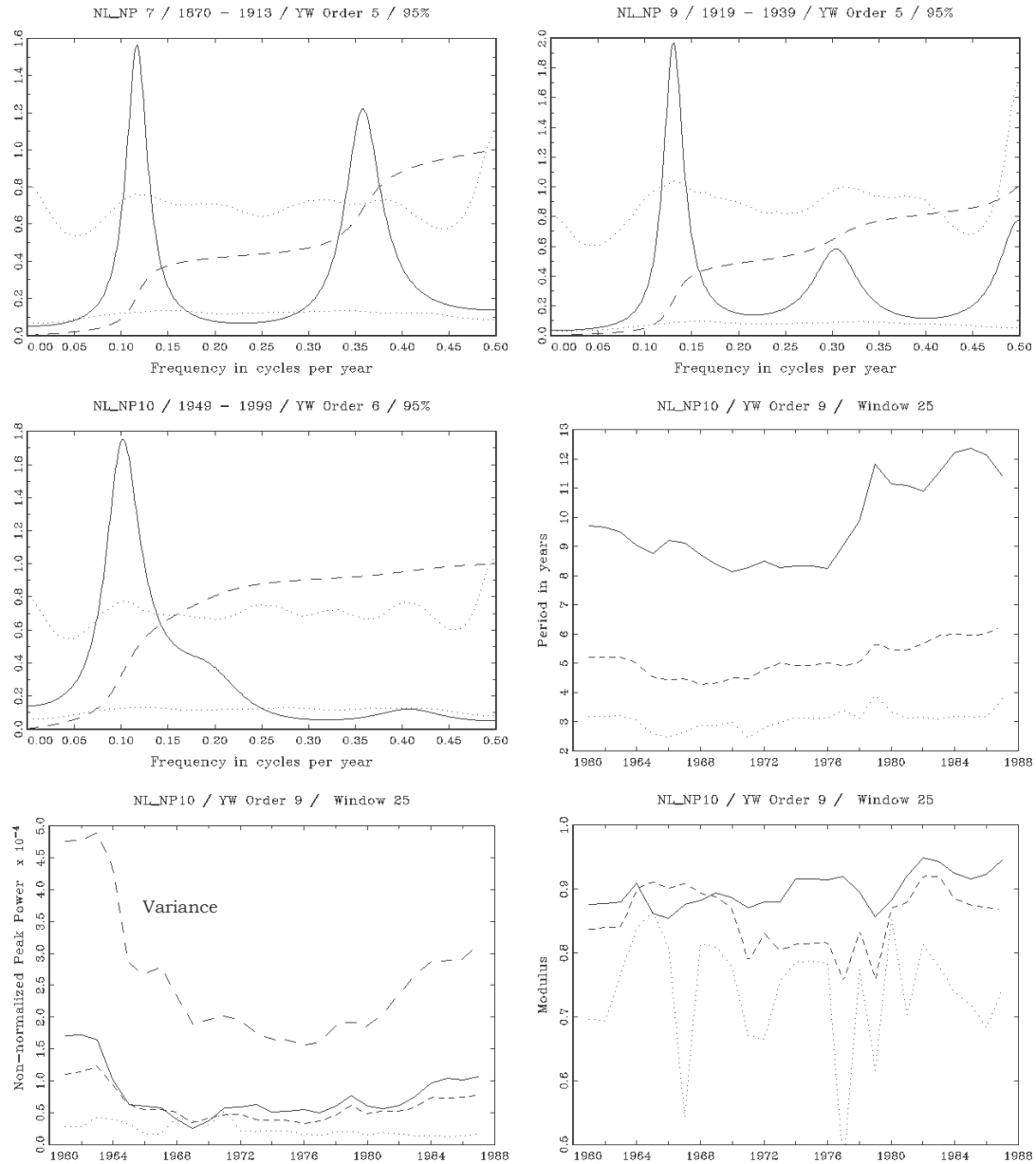
general rather low which indicates that they do not contribute very much to the total variance of the stochastic components. One exception is the prewar period for the NL where it reaches a peak power of 0.3 which is even higher than for the Juglar type fluctuations during that period. This peak also appears to be statistically significant at the 95% confidence level. A further look into the data sources suggests that these exceptional fluctuations may be caused by a poor quality of the data<sup>47</sup>. In general one could therefore say that these high frequency fluctuations consist of rather infrequent and incidental shocks or even measurement errors in the national product indices.

- With respect to possible *changes* in the estimated spectral densities between the various sub-periods one can say that the postwar period reveals more consistent spectral densities for the three countries with higher coherences and lower peak powers for the very high frequency fluctuations. This can both indicate an improved quality of the data, a higher degree of international synchronization and less incidental shocks.

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<sup>47</sup> The description of the sources in Maddison (1995) indicate that for the Dutch 19<sup>th</sup> century data rather preliminary results from Smits and Van Zanden were used that were still being recalculated at the time of the data collection.

Figure 7.3 Static and dynamic estimates of the spectral density of the stochastic component of a National Product index for the NL for various sub-periods. The static estimates (first three pictures) show the normalized auto-spectra (solid lines), integrated normalized auto-spectra (dotted lines) and the 95% white noise confidence intervals for the auto-spectra (fine dotted lines) for respectively the prewar, interwar and postwar period. The dynamic estimates (last three pictures) show the properties of the spectral peaks (eigenvalues) of the rolling window estimates of the auto-spectra during the postwar period.



### 7.3.2 Dynamic

- Comparing the *average measures* of the postwar rolling window spectral estimates as given in Table 7.6 to the measures of the corresponding static estimates from Table 7.5, shows these are almost identical. This indicates that the static estimates in a sense capture the average shape of the spectral densities during the entire period. If one would have to point out one difference it would be the higher coherence for the Kitchin peaks than in the static estimates. However the fact that the various rolling window measures from Figure 7.3 show marked differences during the postwar period indicates that the properties of the stochastic component have certainly not constantly been the same.
- With respect to the length of business cycle fluctuations in each of the countries, at least to some extent, a lengthening of the *period length* for both the Juglar and the Kitchin peaks has taken place. This finding is consistent with previous studies that reported longer growth cycles as indicated in section 2.2.4.
- In the UK and the US the *variance* of the stochastic components has almost doubled during the course of the postwar period. At the end of the sample in the NL also an increase in variance has occurred but this is preceded by a strong initial decrease in the variance which is mainly caused by some very high amplitude fluctuations immediately following World War II. The increase in variance is mainly caused by the increased (non-normalized) *peak power* of the Juglar type fluctuations and, to a lesser extent, also by the Kitchin type fluctuations. Note that such a finding of an increasing variance during the postwar period runs contrary to the alleged stabilization of western economies.
- The *modulus* of the Juglar peak seems to have increased from a little below to a little above 0.9 indicating an increased regularity in the cyclical fluctuations. With respect to the Kitchin peak, the modulus for the UK has steadily decreased from 0.9 to around 0.8 while for the NL and the US there seems to have been a similar decrease until the middle of the period which has been followed by an increase back to around 0.9.
- The *coherence* of the Juglar peak has clearly increased until levels close to 1.0 for both the UK and the US indicating an increased (phase corrected) correlation between the three economies, probably because of so called globalization effects. For the US this also holds for the Kitchin peak but for the UK this coherence has decreased to almost 0.6 which may be consistent with its decreased modulus.
- With the exception of some higher values, probably caused by incidental observations, there is no clear pattern to be observed from the *phase* at both the Juglar and Kitchin frequencies. The lead of the UK and the US is rather stable somewhere around one year. The only exception is a small lead or even a lag of the Kitchin fluctuations at the start of the sample.

Table 7.5 Statistics of static spectral estimates of the filtered stochastic components of National Product indices for various sub-periods. The coherence and phase hold relatively to the NL. The phase is to be interpreted as the expected lead over the NL.

Netherlands					United Kingdom					United States						
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase		
1870-1913	8.5#	0.2	0.9	-	1855-1913*	13.3#	0.2	0.9	0.3	-0.5	1870-1913	9.6#	0.2	0.9	0.9	4.7
	2.8#	0.3	0.9	-		5.0	0.2	0.8	0.4	-2.2		4.5	0.2	0.8	0.4	0.9
						3.1	0.2	0.8	0.8	0.1		2.9	0.2	0.8	0.5	-0.4
						2.2	0.2	0.8	0.8	0.1						
1919-1939	7.6#	0.3	0.9	-	1919-1939	10.3#	0.3	0.9	0.9	1.4	1919-1939					
	3.3	0.2	0.8	-		4.7	0.2	0.8	0.8	-1.9		5.9#	0.5	0.8	0.8	0.1
						2.5	0.1	0.9	0.4	0.1						
1949-1999	9.9#	0.4	0.9	-	1949-1999	11.1#	0.4	1.0	1.0	1.4	1950-1999	10.9#	0.3	0.9	0.8	1.4
	4.9	0.2	0.7	-		4.8	0.1	0.9	0.7	0.1		4.7	0.2	0.8	0.6	0.4
	2.5	0.1	0.8	-		2.9	0.1	0.8	0.7	0.2		2.9	0.1	0.6	0.3	0.2

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1855-1913 frequencies.

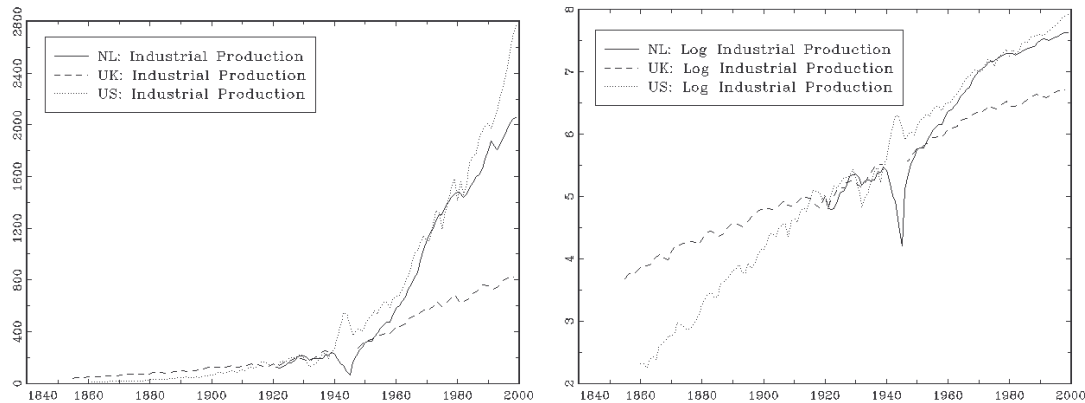
Table 7.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of National Product indices for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 7.5.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
9.8	0.3	0.9	-	-	11.5	0.3	0.9	0.9	1.7	10.4	0.4	0.9	0.8	1.2
5.1	0.2	0.9	-	-	4.9	0.2	0.9	0.8	0.6	4.7	0.2	0.8	0.7	0.7
3.1	0.1	0.7	-	-	2.8	0.1	0.8	0.8	0.2	2.7	0.1	0.7	0.8	0.3

## 8 Industrial Production

### 8.1 Data analysis

Figure 8.1 Real (volume) Industrial Production indices and their (natural) logarithm. The series pass through a common point in 1921 (start of NL series).



- Just as the national product, the industrial production indices are real (volume) indices which are not corrected for population growth. They measure the value of all goods produced in a country's industries during a year.
- Just as the national product indices, the series clearly show an *exponential trend*. In general the corresponding average growth numbers from Table 8.1 show that the *average growth* of industrial production has been higher than the growth of the national product. For the longest available sample in the NL and the US the difference is respectively 0.6% (the average growth of the national product for the 1922-1999 period in the NL is 3.0%) and 0.7%. For the postwar period the difference for both countries is 0.6%. For the UK with 2.1% the long term growth of industrial production has been approximately the same as the growth of the national product. The higher average growth of industrial production implies that other components of the national product must have known a lower average growth. One reason for the higher growth in industrial production might be the transformation into manufacturing economies that has taken place during the past centuries. The ordering of the three countries according to the average growth of industrial production is similar to the ordering according to the growth of the national product. Again the US has shown the highest growth, (closely) followed by the NL while the UK has known (by far) the lowest growth rates.
- The industrial production indices show alternating long periods of high and low growth, for example during the first and second half of the postwar period. In general the differences between the low and high average growth rates seem larger than those observed for the national product. That is, the amplitude of the long term fluctuations in the industrial production indices seems larger than in the national product.
- The *shorter term fluctuations* are more volatile in the industrial production indices than they are in the national product indices. As a direct consequence actual decreases in the level of industrial production frequently occurred, also during

the postwar period. Note that this can be seen as an argument against making a distinction between business cycles and growth cycles as being two different phenomena as already argued in section 2.2.3.

- The standard deviation ratios in Table 8.1 show that for the NL and the UK the *volatility* of the industrial production growth is about one and a half times as high as for the national product. For the US the ratio is somewhere around a factor two. This higher volatility of industrial production growth is probably because it is more directly linked to the capital investments (Juglar) and supply investments (Kitchin) that take place in business enterprises. If these indeed are the sources of business cycle fluctuations then it seems logical that business cycle fluctuations turn up more pronounced in industrial production than in for example the national product. In the latter variable, dampening effects of for example (private) consumption and (anti-cyclical) government expenditures occur, causing a lower volatility. A comparison of the absolute volatility between the countries shows that the volatilities in the NL and the UK seem rather comparable, whereas the volatility in the US has been about twice as high as in the other countries. Finally, the decrease in the standard deviations between the prewar and postwar periods is (slightly) supportive for an alleged stabilization of the economies.
- The consistently high *correlations* with the growth of national product show that industrial production moves very much in line with the national product. Therefore no major leads or lags are to be expected in the subsequent steps of the analysis. Because of this and the fact that business cycle fluctuations turn up more pronounced in the industrial production than they do in the national product, industrial production is sometimes used as a coincidental indicator of the state of an economy instead of the national product. The correlations are lowest during the prewar period, probably because of a poor quality of the data. Note that a less globalized world economy during that period is not an issue here because these correlations measure relations at a national level.

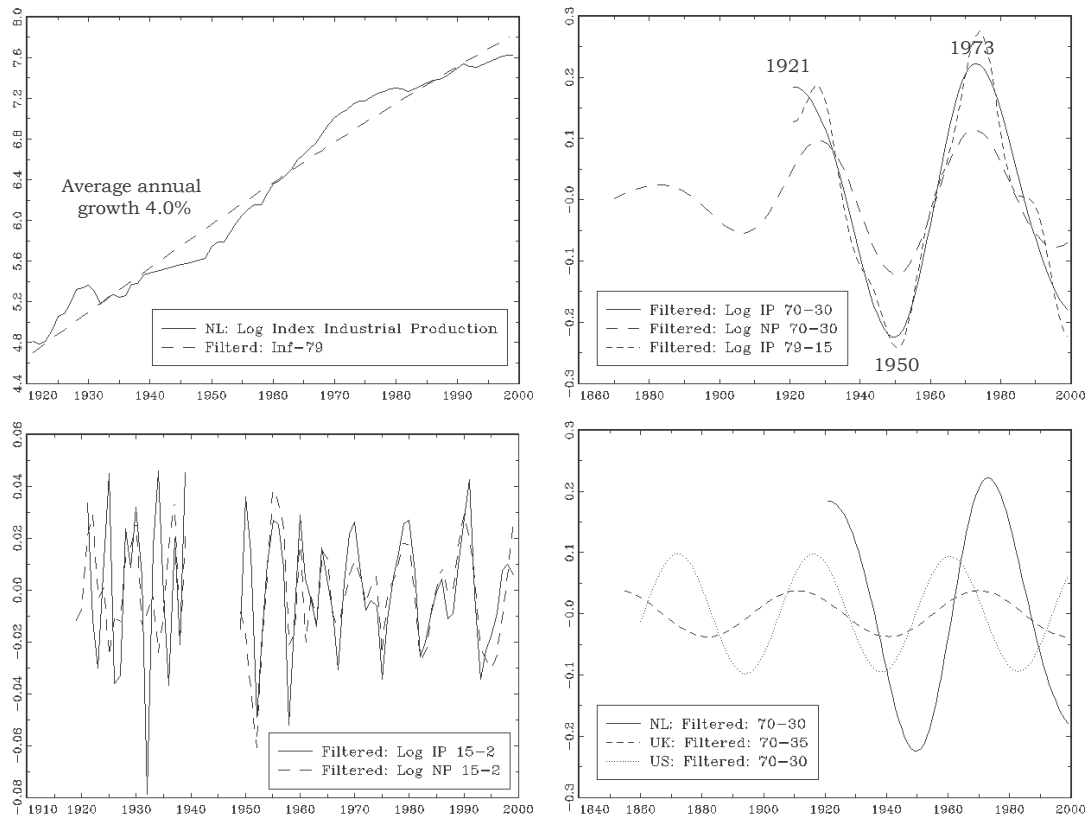


Table 8.1 Statistics of the growth (delta logarithm) of Industrial Production indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

Netherlands					United Kingdom					United States				
	Avg	Stdev	Ratio	Corr		Avg	Stdev	Ratio	Corr		Avg	Stdev	Ratio	Corr
1922-1999	3.6%	13.5%	1.6	0.9	1856-1999	2.1%	4.7%	1.3	0.4	1871-1999	4.0%	10.1%	1.8	0.8
					1856-1913	2.3%	4.0%	1.2	0.5	1871-1913	4.8%	9.0%	2.0	0.7
1922-1939	3.7%	6.8%	1.8	0.6	1920-1938	3.1%	8.3%	2.1	0.7	1920-1939	2.6%	15.7%	2.3	0.8
1950-1999	4.0%	3.3%	1.5	0.8	1950-1999	2.0%	3.1%	1.6	0.9	1950-1999	3.9%	5.6%	2.3	0.9
1950-1975	5.9%	3.2%	1.3	0.8	1950-1975	2.6%	3.2%	1.5	0.9	1950-1975	4.2%	6.1%	2.2	0.9
1976-1999	1.9%	1.9%	1.3	0.8	1976-1999	1.4%	2.8%	1.6	0.8	1976-1999	3.5%	5.0%	2.6	0.8

## 8.2 Filtering results

Figure 8.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Industrial Production index for the NL and its long wave component compared to UK and US data (bottom right).



- For each of the three countries linear interpolation was applied for some specific periods to avoid disturbing effects on the filtered trend, periodic and long wave components caused by specific historical events as for example the two World Wars.
- The *trend component* for each of the three countries shows a linear trend of which the average growth is very consistent with the long term average growth rates from the data analysis. Only for the NL the trend shows a 0.4% higher growth rate than the actual historical statistics. This is very likely caused by the interpolation applied over the World War II period for the filtering purposes. This is supported by the fact that the 4.0% slope of the trend is exactly the same as the average growth for the combined 1922-1939 and 1949-1999 periods (excluding World War II). Note that even for the relatively short 1921-1999 sample for the NL, the filtering process results in a linear trend without in some way imposing additional information<sup>48</sup>. This is noteworthy because most of the relatively recent literature indicating linear instead of stochastic trends in national product indices (see the end of section 5.6) use much longer samples of data starting somewhere in the second half of the 19<sup>th</sup> century.

<sup>48</sup> For example by explicitly imposing a linear trend. Another example is Reijnders (1990) who introduces information from other, longer, time series to avoid what he calls *perspective distortion* as described in section 5.5.1.

- For the NL the *periodic component* is virtually identical to the long wave component because the relevant pass-bands used in the filtering process are virtually identical due to the only 79 observations in the sample which is close to the upper limit of 70 years for the long wave component. In case of the longer samples for the UK and the US, just as with the national product, the periodic component shows a pattern of relatively high values during the prewar and postwar period with lower values during the interwar period.
- The *dating* in Table 8.4 of the peaks and troughs in the *long wave component* for the NL and the UK is practically the same as for the national product in those countries and is therefore also consistent with the approximate dating of the Kondratieff type cycles that emerge from the literature. This is also the case for the US with the exception of the dating of the last peak and trough which are approximately 15 years earlier than in the corresponding national product index. However, the comparison with the periodic component from Appendix E, shows that the postwar part of the long wave component is not very consistent with the postwar part of the periodic component. The latter indicates a later dating of the most recent trough in the long term behavior of the industrial production index. Therefore, in a sense the aberrant dating of the long wave for the US at the end of the sample may be due to a filter error. In general therefore no clear *lead* or *lag* of the long wave in industrial production over the national product can be detected. The average peak-to-peak and trough-to-trough *period lengths* are respectively 48 and 53 years. With respect to the *amplitude* of the long wave component, the UK shows a virtual identical amplitude as for the national product while for both the NL and the US the amplitude is respectively two and five times as large. For this, see both the graphical filtering results and the volatility ratios of the long wave components in Table 8.2. These findings confirm what was already found in the data analysis.
- Almost without exception the *stochastic component* in the industrial production indices moves exactly in line with the same component in the national product indices. This is confirmed by the very high correlations of around 0.8 as reported in Table 8.2. Only the interwar period for the NL shows a much lower correlation. The volatility ratios indicate that for the UK and the US, business cycle fluctuations in industrial production are between one and a half and twice as volatile as in the national product. The same was found in the data analysis. However, for the NL during the postwar period the volatility ratio indicates only a 10% higher volatility while from the data analysis this was still about 50%. Overall, these ratios are somewhat lower than those reported in the literature. For example, based on research for ten countries, Backus and Kehoe (1992) report (Hodrick-Prescott filtered) investments to be two to four times as volatile as the national product. Also Englund et al. (1992) summarize results for (Hodrick-Prescott filtered) quarterly investment data from the postwar period for various countries as obtained by Kydland and Prescott (1990), Danthine and Girardin (1989) and Brandner and Neusser (1990) to give a volatility ratio of around three. One reason for these differences might be that here industrial production is analyzed which is of course not exactly the same as investments. An international comparison of the volatilities, just as for the national product, shows that the volatility for the UK and the US is higher than for the NL. Such differences might depend on the exact nature of the goods produced in an economy. For example

the NL are known to be a more trading type of economy in which the service sector plays a very important role which is known to be less sensitive to business cycle fluctuations.

- The results in Table 8.3 show that stochastic components describe almost the entire volatility of the conventional growth rates.

Table 8.2 Statistics of the filtered periodic, long wave and stochastic components of Industrial Production indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands				United Kingdom				United States			
		Stdev	Ratio	Corr		Stdev	Ratio	Corr		Stdev	Ratio	Corr
Periodic	1921-1999	15.0%	1.1	0.7	1855-1999	6.4%	0.8	0.6	1870-1999	12.5%	1.3	0.9
Long Wave	1921-1999	14.7%	1.9	1.0	1855-1999	2.6%	1.1	1.0	1870-1999	6.8%	5.0	0.4
Stochastic	1921-1939	3.3%	1.7	0.2	1855-1913	4.3%	1.7	0.6	1870-1913	7.5%	2.2	0.8
	1949-1999	2.1%	1.1	0.7	1919-1938	7.3%	1.9	0.8	1919-1939	10.3%	2.3	0.8
	1949-1975	2.3%	1.0	0.7	1949-1999	2.8%	1.3	0.8	1950-1999	4.2%	1.9	0.9
	1976-1999	1.8%	1.1	0.8	1949-1975	2.3%	1.4	0.8	1950-1975	4.3%	2.1	0.9
					1976-1999	3.2%	1.3	0.8	1976-1999	4.0%	1.7	0.9

Table 8.3 Standard deviation of growth (delta logarithm) of Industrial Production indices for various sub-periods (Hist) versus standard deviation of first order differences of filtered stochastic components (Filt).

Netherlands				United Kingdom				United States			
	Filt	Hist	Ratio		Filt	Hist	Ratio		Filt	Hist	Ratio
				1855-1913	3.9%	4.0%	1.0	1870-1913	8.9%	9.0%	1.0
1921-1939	5.1%	6.8%	0.8	1919-1938	8.1%	8.2%	1.0	1919-1939	14.4%	15.7%	0.9
1949-1999	2.4%	3.3%	0.7	1949-1999	2.9%	3.0%	1.0	1949-1999	5.4%	5.6%	1.0
1949-1975	2.9%	3.2%	0.9	1949-1975	2.9%	3.1%	0.9	1949-1975	5.7%	6.1%	0.9
1976-1999	1.8%	1.9%	0.9	1976-1999	2.8%	2.8%	1.0	1976-1999	5.1%	5.0%	1.0

Table 8.4 Peaks and troughs of the long wave component in Industrial Production indices. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>Peak-Peak Trough-Trough Lead</i>
Netherlands			1921	52	1973	52
Peak				1950	49	49
Trough			7	0	-1	1
Lead						
United Kingdom			1911	59	1970	59
Peak				1941	58	59
Trough		1882	59			
Lead			1	-1	-2	-1
United States		1872	44	1916	45	45
Peak					1961	
Trough		1894	44	1938	45	45
Lead			-4	2	8	7
					13	
					16	
Average		1872	44	1916	52	48
Peak					1968	
Trough			55	1943	51	53
Lead		-3	5	3	3	2
					4	

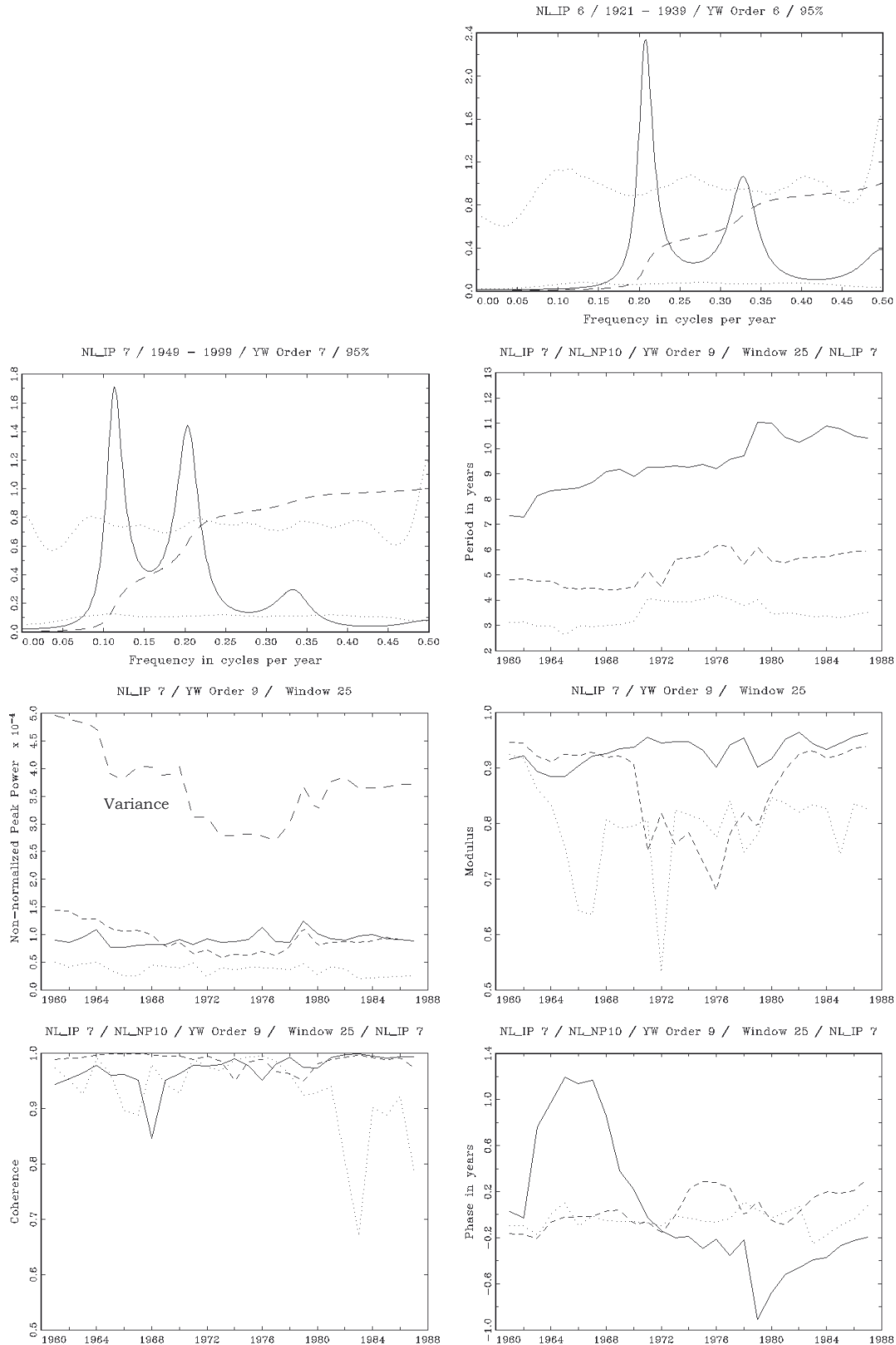
## 8.3 Spectral analysis

### 8.3.1 Static

- In general the same three types of peaks can be observed in the various estimated spectral densities as for the national product indices. Besides the high frequency fluctuations of between two to three years, two further spectral peaks are present for most of the combinations of time period and country, indicating pseudo periodic behavior at both the Juglar and Kitchin frequencies. The *period lengths* of these peaks are consistently around respectively ten and five years. A striking difference with the spectral densities for the national product is the more dominant presence of the Kitchin type fluctuations. Both the *modulus* (0.9) and *peak power* (0.3) are higher, indicating more distinct periodic behavior and describing a higher percentage of the total variance. Also, almost without exception, the Kitchin peak is significant at the 95% confidence level while this was never the case for the national product. The *coherences* with the relevant national product indices are close to one for both peaks, with the exception of the interwar period in the NL. All *phases* are close to zero, indicating no significant lead or lag for the industrial production as already suspected in the data analysis. Alternatively, the small values for the phase might also be interpreted as being leads or lags of a few months. Although in reality this may certainly be the case, it is hard to draw such precise conclusions based on the annual data used here.
- Comparing the results for the three sub-periods reveals no clear *changes* in the stochastic behavior of the industrial production. At least one of the two spectral peaks of the Juglar and Kitchin type fluctuations is also present in the prewar and interwar data with very similar properties as for the postwar period. Especially for the UK, Table 8.5 shows extremely consistent spectral densities across the prewar, interwar and postwar period (the two high frequency roots for the prewar period have zero peak power).



Figure 8.3 Static and dynamic estimates of the spectral density of the stochastic component of a Industrial Production index for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



### 8.3.2 Dynamic

- The *averages* of the rolling window measures for the postwar period, as given in Table 8.6, show that the static estimates adequately reflect the average shape of the spectral density during that period.
- The *period length* of the Kitchin peak for each of the countries has steadily increased by approximately one year during the postwar period. For the NL the Juglar cycle shows a strong increase from around seven to more than ten years while this period length remained steady around ten years for the other two countries.
- The changes in the overall *variance* show a mixed picture across the three countries. The NL show a strong initial decrease until the middle of the sample followed again by a material increase. Both the UK and the US show an increasing variance with a stabilization at the end, which for the UK this even turns into a declining variance again. However, for each of the separate countries the pattern for the industrial production is pretty much the same as for the national product index. For the UK the peak powers of the Juglar and Kitchin peaks both follow the pattern in the total variance. For the NL and the US however, the peak power of the Juglar peak show a steady increase. For the NL the pattern in the total variance seems to be caused by the pattern in the peak power of the Kitchin peak while for the US the Kitchin peak power shows a subsequent increase and decrease, just as for the UK.
- The *modulus* of the Juglar peak has increased over time in each of the three countries. For the UK and especially for the US the modulus of the Kitchin peak has remained rather stable over time while for the NL it has been very low in the middle section of the sample.
- The *coherence* with the national product index has in general been high for both spectral peaks during the entire sample. At the end of the sample it has often increased up to the maximum level of one. The main exception is the coherence of the Kitchin peak for the US which has slightly decreased over time.
- With respect to the *phase* it seems that for most of the countries and spectral peaks a change has taken place from a (sometimes extensive) lag towards a slight lead of less than a year over the national product index. Only for the Juglar peak in the NL the reverse seems to have happened. Despite the obvious shortcomings of analyzing annual data with respect to such small lead / lag relations, the conclusion of a recent (slight) lead of the industrial production over the national product seems warranted.

Table 8.5 Statistics of static spectral estimates of the filtered stochastic components of Industrial Production indices for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase			
1921-1939	4.8#	0.3	0.9	0.4	2.1	1855-1913*	8.5#	0.4	1.0	1.0	-0.2	1860-1913	9.5#	0.3	0.9	1.0	0.6
	3.0#	0.2	0.9	0.8	-0.4		5.4#	0.2	0.9	0.8	-0.7						
							3.4	0.0	0.8	0.9	-0.1		3.5	0.2	0.8	0.9	0.2
							2.2	0.0	0.7	0.4	-0.2		2.6	0.1	0.7	0.8	-0.3
1949-1999	8.9#	0.2	0.9	0.9	0.2	1919-1938	10.0#	0.3	0.9	1.0	0.6	1919-1939					
	4.9#	0.3	0.9	1.0	-0.2		4.5	0.2	0.8	0.9	0.0		6.4#	0.3	1.0	1.0	0.3
	3.0	0.1	0.9	0.8	-0.1		2.4	0.1	0.9	0.9	0.1		3.2	0.2	0.8	1.0	0.1
							10.1#	0.3	0.9	1.0	0.3	1949-1999	10.3#	0.2	0.9	1.0	-0.2
							4.9#	0.2	0.9	1.0	0.1		4.6#	0.3	0.8	0.9	0.0
							3.0	0.1	0.8	0.9	-0.1		2.7	0.3	0.7	0.9	-0.2

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1855-1913 frequencies / period lengths.

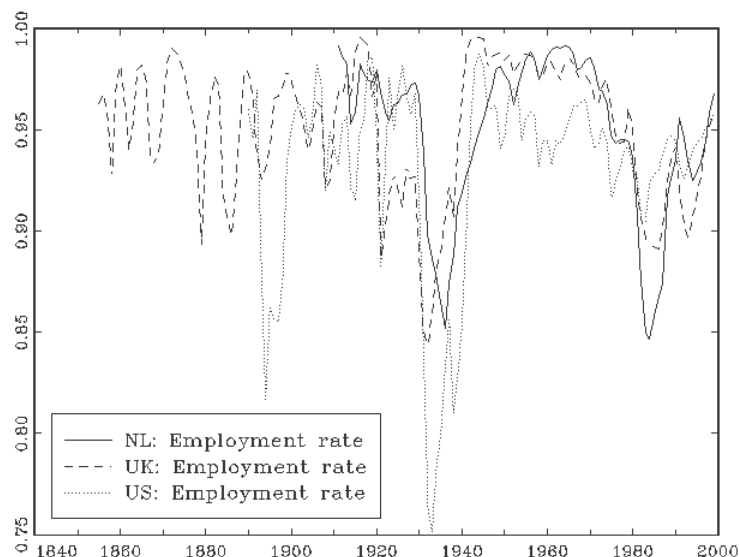
Table 8.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of Industrial Production indices for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 8.5.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
9.4	0.3	0.9	1.0	0.0	10.1	0.2	0.9	0.9	-0.4	9.8	0.3	0.9	1.0	-0.1
5.3	0.3	0.9	1.0	0.0	4.9	0.3	0.9	1.0	0.1	4.7	0.3	0.8	1.0	0.0
3.5	0.1	0.8	0.9	0.0	2.9	0.1	0.8	0.9	-0.1	2.7	0.2	0.8	1.0	-0.1

## 9 Employment

### 9.1 Data analysis

Figure 9.1 Employment rates.



- It is more convenient to work with the employment rate rather than the unemployment rate because the first more naturally compares to for example the national product (if the national product is high, the employment rate is also high and vice versa). It is simply calculated as one minus the unemployment rate. Note that the employment rate is not the same as the participation rate.
- Contrary to the two index time series analyzed in the previous sections, the employment rates do not show trending behavior. Instead they show both long and short term fluctuations around an *average value* of about 95% (i.e. an unemployment rate of 5%).
- The employment rate has been substantially higher and lower than this average value for longer periods of time. As an example consider the first half of the postwar period during which the average employment rates for the NL and the US were approximately 98% while during the second half of the postwar period they were as low as 92% on average. So in general one could say that due to long term movements, employment rates can be about 3% higher or lower than their long term average. Note that the observed long term fluctuations in the employment rates seem to match well with the approximate dating of the long waves as given in Table 2.1.
- The original time series in Figure 9.1 also show periods with rather exceptional employment rates, to some extent caused by specific *historical events*. For example during the Great Depression in the US employment rates reached the extremely low level of 75%. Striking are also the low employment rates in the NL at the start of the 1980's and in the US at the end of the 19<sup>th</sup> century.
- With respect to the *shorter term fluctuations*, employment rates at first sight seem to move pretty much in line with the general dating of recent business cycles. For

example the recessions in the middle of the 1970's and at the start of the 1980's and 1990's are clearly visible. This is confirmed by the *correlations* with the growth of the national product indices as reported in Table 9.1. These correlations are somewhere around 0.4 indicating a positive, though not perfect, relation. The subsequent sections might therefore reveal some lead / lag relation with respect to the employment rates. In general the positive correlations confirm the conventional knowledge that the employment rate is a procyclical variable.

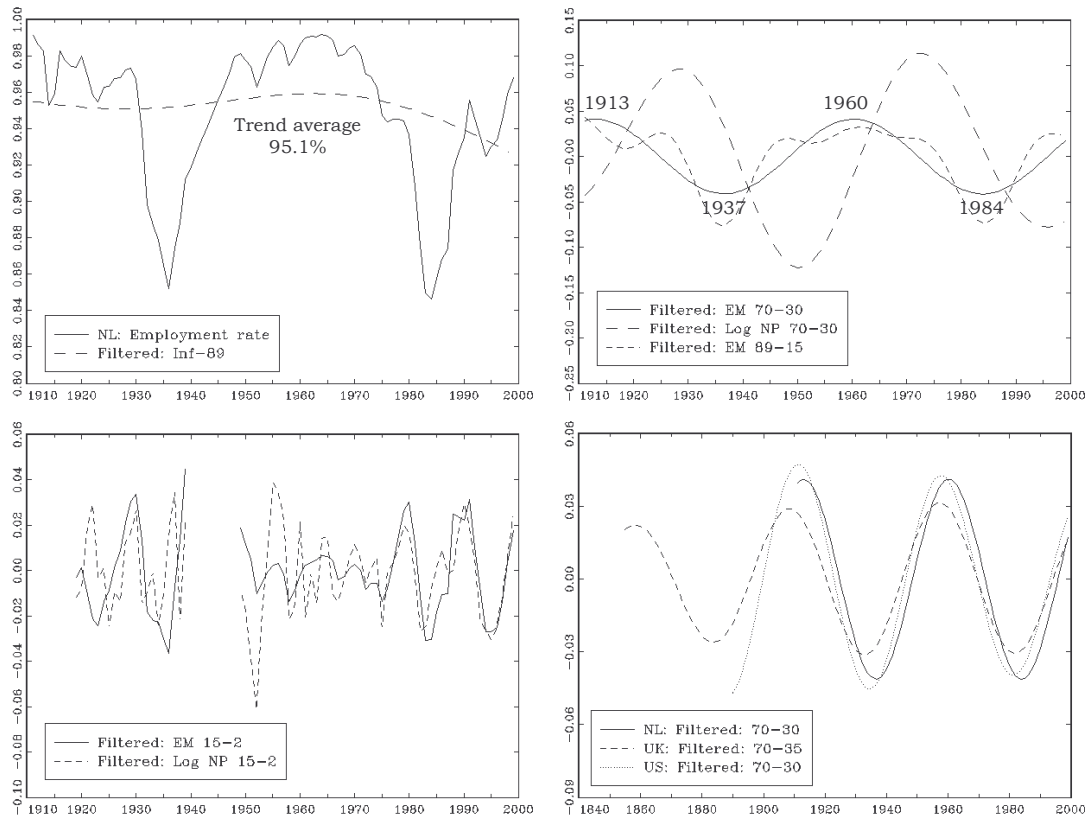
- For the *volatility ratios* a mixed picture emerges from the results in Table 9.1. During some periods the employment rate volatility is much larger than that of the growth of the national product while at other times it is only about half as big. A closer look shows that the ratio for most countries and time periods is somewhere around 0.8. Major exceptions are the second half of the postwar period in the NL and the UK and the interwar period in the US.

Table 9.1 Statistics of the Employment rate for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

Netherlands					United Kingdom					United States				
	Avg	Stdev	Ratio	Corr		Avg	Stdev	Ratio	Corr		Avg	Stdev	Ratio	Corr
1911-1999	94.9%	3.9%	0.5	0.2	1855-1999	95.1%	3.4%	1.0	0.1	1890-1999	93.1%	4.8%	0.8	0.2
					1855-1913	95.6%	2.3%	0.7	0.2	1890-1913	92.9%	4.4%	0.8	0.3
1919-1939	93.4%	4.3%	0.8	0.3	1919-1939	90.9%	3.4%	0.7	-0.1	1919-1939	89.0%	7.9%	1.2	0.2
1950-1999	95.1%	4.0%	1.8	0.5	1950-1999	95.2%	3.2%	1.6	0.2	1950-1999	94.3%	1.5%	0.6	0.4
1950-1975	98.0%	1.1%	0.4	0.4	1950-1975	97.9%	0.7%	0.3	0.3	1950-1975	95.1%	1.3%	0.5	0.4
1976-1999	92.0%	3.6%	2.4	0.3	1976-1999	92.4%	2.3%	1.3	-0.2	1976-1999	93.5%	1.3%	0.7	0.3

## 9.2 Filtering results

Figure 9.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of the Employment rate for the NL and its long wave component compared to UK and US data (bottom right).



- Although some of the fluctuations in the employment series seem rather exceptional no interpolation was applied to facilitate the filtering process. Only for the NL the 1940-1946 data are obtained by means of linear interpolation because of missing values.
- As expected, the *trend component* for each of the three countries is rather flat and closely resembles the long term average employment rates of about 95%.
- The *periodic component* can show some wild fluctuations, in some cases caused by specific historical events. In general however the underlying “trend” in the periodic component seems to be captured well by the *long wave component*. Figure 9.2 and Table 9.4 show that the dating of the peaks and troughs in the long wave components of the employment rate is very similar across the three countries. Also the amplitudes are comparable and somewhere around 3%, which is consistent with what was found in the data analysis. With the exception of especially the early cycles in the UK, the long wave in the employment rates seems to *lead* the long wave in the national product by 10 to 15 years. This lead is approximately a quarter of a cycle which means that the dating of the peaks and troughs of the long wave in the *growth* of the national product and the employment rate are approximately the same. To see this, note from Figure 5.1 in section 5.2.1. that the value of the phase of the first order differencing operator for low frequencies is about 0.25. One hypothesis which explains this long wave



lead in employment rates is that “starting up” new basic innovations such as in Table 2.1 requires a lot of manpower before, in later phases, the full (capital) exploitation of the innovation can take place. As an example note that during the past decade many new jobs were created in the ICT sector while the true benefits of these innovations in terms of (the growth of) the national product may be yet to come. The average peak-to-peak and trough-to-trough *period lengths* are respectively 50 and 48 years.

- With respect to the *stochastic component* the following observations can be made. First, the volatility ratios for most of the countries and sub-periods indicate that business cycle fluctuations in employment are less volatile than they are in the national product. This finding is consistent with research by others. For example Blackburn and Ravn (1992) report a postwar volatility ratio of 0.7 while also the summary in Englund et al. (1992) indicates a somewhat lower volatility of employment for various countries. See section 8.2 for the exact literature used for this summary. Second, especially for the US the business cycle fluctuations in the employment rate are well in line with those in the national product. During the postwar period for the NL and the UK there are indications of a slightly lagging behavior. Third, compared to the conventional correlations reported in the previous section, the correlations between the stochastic component in the employment rates and the national product are (much) higher. The reason is probably that the long term fluctuations are dominantly present in the employment rates and thereby in a sense lower the correlations present at the business cycle frequencies because of the lead in the long wave component. The filtering process therefore reveals the true (short term) relations between the variables.
- Because the employment rate is not an index type variable such as the national product and the industrial production, it is not that informative to evaluate the volatility of the growth (first differences) of the employment rate. Instead however it is interesting to see which part of the conventional volatility of (the level of) the employment rate itself is described by the stochastic component. The results in Table 9.3 show that at about 70% the stochastic component describes a smaller portion of the original variance of the employment rate than we have seen for (the growth of) the national product and industrial production indices. Again, this is probably because of the strong presence of the long term fluctuations which are outside the frequency range of the stochastic component. Note that this is confirmed by the fact that the volatility ratio for the separate first and second half of the postwar period is higher than it is for the entire postwar period which covers all phases of the long term fluctuations during that time period.

Table 9.2 Statistics of the filtered periodic, long wave and stochastic components of Employment rates for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands				United Kingdom				United States			
		Stdev	Ratio	Corr		Stdev	Ratio	Corr		Stdev	Ratio	Corr
Periodic	1911-1999	3.5%	0.3	-0.1	1855-1999	2.8%	0.4	0.1	1890-1999	4.2%	0.4	0.7
Long Wave	1911-1999	2.9%	0.4	-0.2	1855-1999	2.0%	0.9	0.7	1890-1999	3.0%	2.2	0.1
Stochastic					1855-1913	2.1%	0.8	0.5	1890-1913	2.3%	0.7	-0.7
	1919-1939	2.2%	1.2	0.2	1919-1939	2.7%	0.7	0.9	1919-1939	5.0%	1.1	0.4
	1949-1999	1.5%	0.8	0.5	1949-1999	1.2%	0.6	0.7	1950-1999	0.9%	0.4	0.9
	1949-1975	0.7%	0.3	0.3	1949-1975	0.5%	0.3	0.6	1950-1975	0.8%	0.4	0.8
	1976-1999	2.1%	1.2	0.8	1976-1999	1.7%	0.7	0.7	1976-1999	1.0%	0.4	1.0

Table 9.3 Standard deviation of Employment rates for various sub-periods (Hist) versus standard deviation of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States				
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio		
1919-1939	2.2%	4.3%	0.5	1855-1913	2.1%	2.3%	0.9	1890-1913	2.3%	4.4%	0.5
1949-1999	1.5%	4.0%	0.4	1919-1939	2.7%	3.4%	0.8	1919-1939	5.0%	7.9%	0.6
1949-1975	0.7%	1.0%	0.7	1949-1999	1.2%	3.2%	0.4	1949-1999	0.9%	1.5%	0.6
1976-1999	2.1%	3.6%	0.6	1949-1975	0.5%	0.7%	0.7	1949-1975	0.8%	1.3%	0.7
				1976-1999	1.7%	2.3%	0.7	1976-1999	1.0%	1.3%	0.8

Table 9.4 Peaks and troughs of the long wave component in Employment rates. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

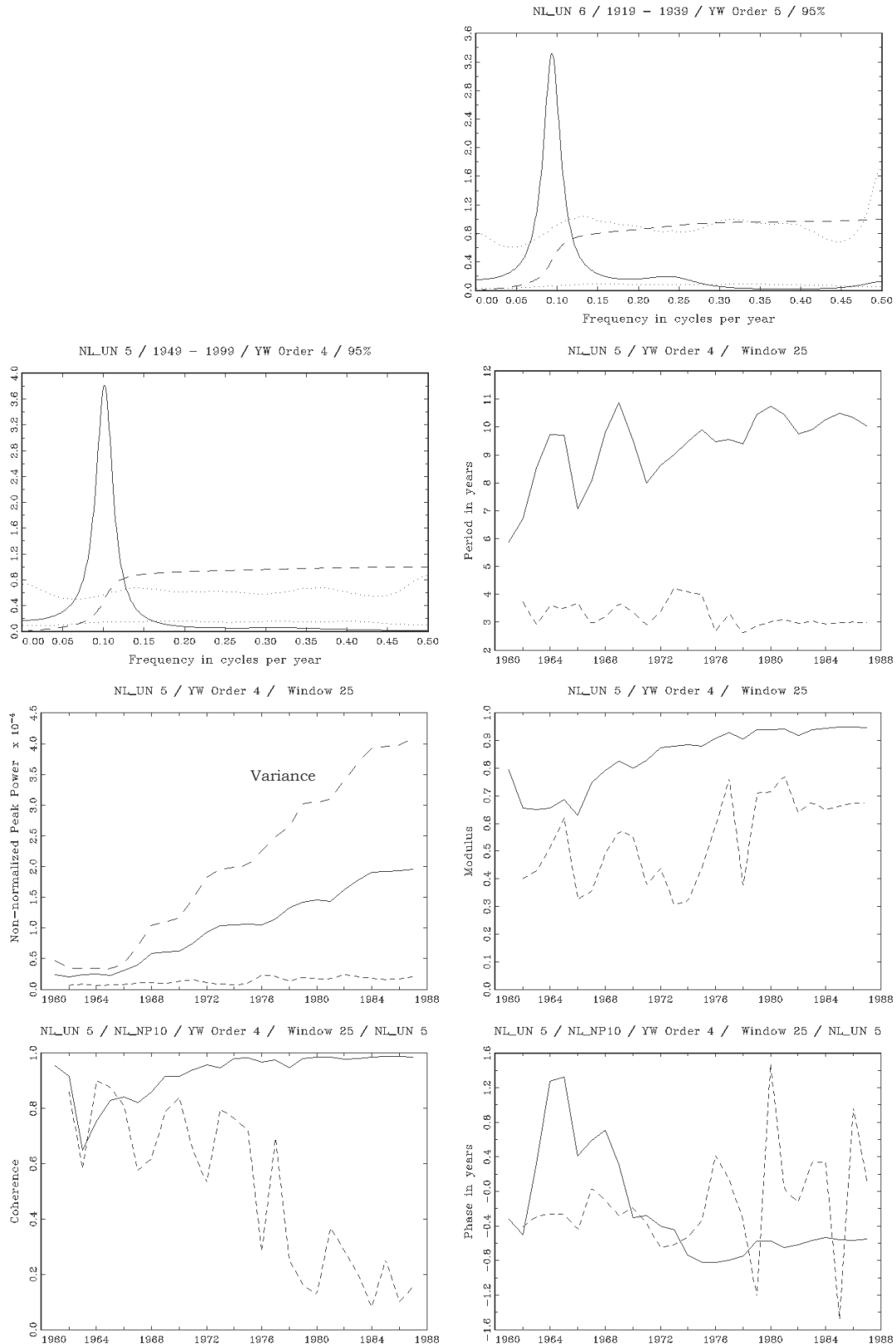
	1	2	3	4	5	Peak-Peak Trough-Trough Lead
Netherlands			1913	47	1960	47
Peak						47
Trough						47
Lead			15	13	12	13
Peak		1858	50	49	1957	50
Trough			1884	49	1933	49
Lead		-2	2	5	11	5
Peak			1911	47	1958	47
Trough			1890	44	1934	46
Lead			0	7	12	11
		1858	53	1911	48	50
Peak						50
Trough			1887	48	1935	48
Lead		12	6	8	10	11
				13	16	11
						11

## 9.3 Spectral analysis

### 9.3.1 Static

- The estimated spectral densities for all countries and sub-periods are dominated by a peak at the Juglar frequencies. In all cases it is around a *period length* of ten years and significant at the 95% confidence level. With a *peak power* of between 30% and 50% it described a large portion of the total variance of the stochastic component. The consistently high *modulus* of 0.9 or more indicates clear pseudo period behavior. Also the *coherence* is consistently at a very high level, thereby confirming the strong correlation already reported in Table 9.2. Note that at around 0.9 the coherence numbers are higher than these conventional correlations because here a correction has been made for possible *phase* shifts. For the postwar period in the NL and the UK, lags of six months up to a year are reported while for the US no significant lead or lag is detected. This is both consistent with the visual inspection of the filtered time series in the previous section and with what is known from the literature. Blackburn and Ravn (1992) report a (slight) lag for the UK during the postwar period. For example Stock and Watson (1989) use employment, together with industrial production, in an index of coincidental indicators for the US, which indicates that apparently they found no significant lead or lag for the employment rate. If present, a lag of the employment rate may in general be explained by the time needed to take on new personnel during an expansion phase, by the time needed to close existing employers contracts during recession phases and possibly also by companies waiting to see whether a dawning recession will indeed set in.
- There are also some slight indications of the presence of a Kitchin peak in the static spectral density estimates. Contrary to the Juglar peak, it is not present in all periods for all countries. It is never significant and has a lower *peak power* of between 10% and 20% of the total variance. The *coherence* is in general high and is most of the times accompanied by a phase indicating a lag of a few months. Apparently the inventory type fluctuations associated with the Kitchin peak have little effect on the employment rates. One reason might be that business enterprises are more familiar with these type of fluctuations and look through them without actually laying off personnel.
- With the exception of the (absolute) volatilities, no significant changes can be detected in the spectral densities estimated for the prewar, interwar and postwar period. Just as with the industrial production, especially the UK shows extremely consistent spectral densities across these periods.

Figure 9.3 Static and dynamic estimates of the spectral density of the stochastic component of the Employment rate for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



### 9.3.2 Dynamic

- The *average measures* calculated from the rolling window spectral estimates, as reported in Table 9.6, are very similar to the static estimates. The only exceptions are perhaps the average phases of the Juglar peak in the NL and the US. For the NL it shows a smaller lag while for the US it shows a larger lag than was the case in the static estimates.
- In the NL a lengthening of the Juglar type fluctuations has clearly taken place, just as was the case for the national product and the industrial production index. For the UK and the US the *period length* of the Juglar peak shows no clear changing pattern and has stayed somewhere around ten years. The period length of the Kitchin peak in the UK and the US has increased about one year while for the NL no such statement is possible because it has hardly been observed during the entire postwar period.
- In each of the three countries the *variance* of the stochastic component has steadily increased. This increase seems mostly caused by the peak power of the Juglar type fluctuations.
- With a value of about 0.70 the *modulus* of the Juglar peak started off rather low in each of the three countries and steadily increased to a value of around 0.95 indicating very regular fluctuations towards the end of the sample. For the Kitchin peak in the UK and the US no clear pattern is visible.
- The *coherence* at both the Juglar and Kitchin frequencies has been high throughout the entire sample but has increased to values close to one at the end of the sample.
- The *phase* of the Juglar peak in the UK has been around a lag of about one and half year during the entire sample. For the US this was also the case during the first half of the postwar period while the very small lag appears to have been only something of the recent decades. During the second half of the postwar period for the NL there has consistently been a lag of about half a year while during the first half strangely enough a lead of about a year can be observed. The latter is however accompanied by a lower value of the coherence of almost 0.6 making inferences about the phase somewhat suspect. During the recent decades a reduction of the lag of the employment rate has taken place in all three countries. One possible explanation could be an increased flexibility in employment contracts. However, the lag at the Kitchin frequencies in the UK and the US has stayed approximately the same throughout the postwar period at around half a year. The fact that in general the phases of the static estimates are comparable with the phases from the rolling window estimates at the end of the sample is probably because of the higher variance at the end of the sample which makes the properties of this period dominate the overall autocorrelations and thereby also the static estimates.

Table 9.5 Statistics of static spectral estimates of the filtered stochastic components of Employment rates for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase			
1919-1939	10.7 <sup>#</sup>	0.4	0.9	0.7	-0.2	1855-1913*	8.4 <sup>#</sup>	0.3	0.9	0.9	-0.1	1890-1913	9.1 <sup>#</sup>	0.3	0.9	0.9	-3.6
	4.1	0.1	0.8	0.9	-2.0		4.7	0.2	0.8	0.5	-0.4		3.7	0.3	0.8	0.8	1.8
							2.5	0.1	0.7	0.8	0.0		2.6	0.1	0.7	0.9	1.2
1919-1939						1919-1939	10.3 <sup>#</sup>	0.3	0.9	1.0	0.3	1919-1939	10.4 <sup>#</sup>	0.5	0.9	0.8	1.6
							4.3	0.2	0.8	0.9	0.0		3.4	0.1	0.6	0.9	0.3
							2.4	0.1	0.8	0.9	0.1						
1949-1999	9.9 <sup>#</sup>	0.5	0.9	0.8	-0.7	1949-1999	9.9 <sup>#</sup>	0.4	1.0	1.0	-1.1	1950-1999	9.8 <sup>#</sup>	0.3	0.9	1.0	-0.1
	3.0	0.1	0.6	0.2	0.0		4.9	0.2	0.9	1.0	-0.6		4.5	0.2	0.8	1.0	-0.2
							2.4	0.0	0.7	0.2	-0.6		2.9	0.2	0.7	0.9	-0.1

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1855-1913 frequencies / period lengths.

Table 9.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of Employment rates for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 9.5.

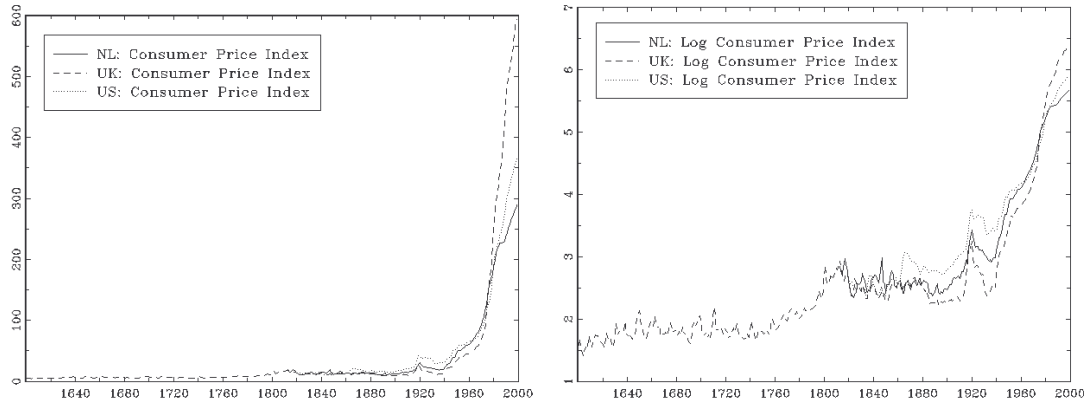
Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
9.3	0.5	0.8	0.9	-0.2	9.9	0.4	0.8	0.9	-1.3	9.8	0.3	0.9	1.0	-0.5
3.3	0.1	0.5	0.5	-0.2	4.9	0.3	0.9	1.0	-0.6	4.5	0.3	0.8	1.0	-0.3
					2.5	0.1	0.7	0.6	-0.6	2.7	0.1	0.7	1.0	-0.1



## 10 Consumer Prices

### 10.1 Data analysis

Figure 10.1 Consumer Price indices and their (natural) logarithm. The series pass through a common point in 1820 (start of US series).



- The consumer price index series analyzed here aim to represent the evolution of the price level of some “standard” package of consumer goods. In general these prices are influenced by both monetary factors (the amount of money circulating in an economy) and real factors (the laws of supply and demand).
- At first sight, the *trend* of the price indices seems to be of the exponential type. However, the logarithmic transformations of the time series do not show the corresponding linear trends. Note that a priori, exponential trending behavior of prices is not to be expected. After all, prices do not represent technological progress or demographic growth such as for example the national product. They do not represent some cumulation of wealth either as for example an equity total return index does. In each of the three countries there seems to have been a structural break in the long term trend underlying the prices somewhere around the start of the 20<sup>th</sup> century. The 19<sup>th</sup> century was a period of almost no long term price changes. That is, the *inflation* rate was around 0%. The 20<sup>th</sup> century however has shown average inflation rates of approximately 3% for the NL and the US while the average increase in UK consumer prices was about 4%. It is therefore best to speak of *piecewise log linear trends* for the price index series with a break around the year 1900. For what it is worth, the overall long term average inflation rate is approximately 1.5%.
- Around these very long term trends there clearly have been alternating periods, comprising several decades, of relatively high and low price changes. For example for the NL, the first and second half of the postwar period have shown average inflation rates of respectively 4.6% and 3.0%. With respect to these long term fluctuations in prices, structural changes seem to have occurred as well. From the second World War onwards, long periods of *deflation* that existed before that time have not occurred anymore. Note that just as with the “distinction” between business cycles and growth cycles (see section 2.2.3) these long term fluctuations

may very well be the same phenomenon, only with different levels of long term price changes (trends) underlying them.

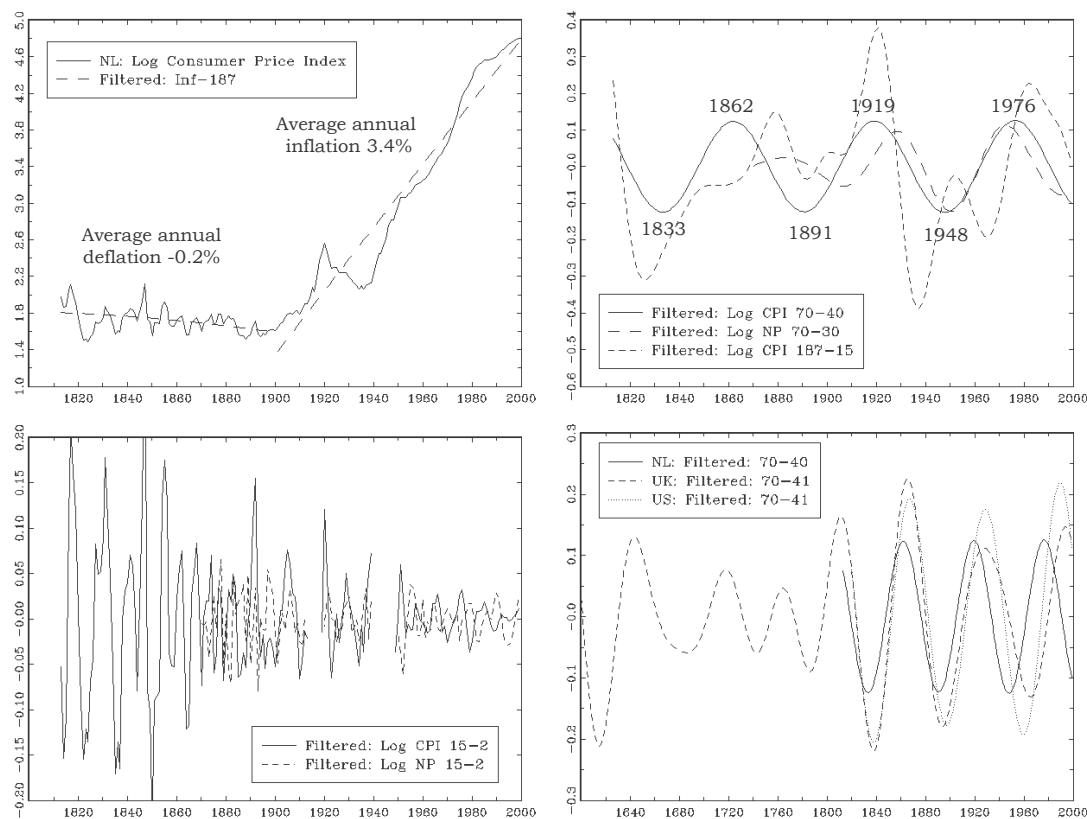
- The most prominent *historical event* visible in the time series is the Great Depression in the 1930's during which consumer prices strongly decreased for several years in a row.
- Marked changes have also occurred with respect to the *shorter term fluctuations* in the prices. First, starting around 1900 the *volatility* is much smaller than before that time. Note that this (visual) decrease in short term volatility does not show up very clearly in the standard deviations reported in Table 10.1. This is probably because the inflation rates also contain some of the long term fluctuations present in the indices. We will return to this in the next section. Second, during the postwar period prices have shown a more smooth development than before that time. Besides the lower volatility this is also caused by the increased *auto correlations* of the inflation rates as reported in Table 10.1. This "stickiness" of prices during the postwar period is part of the conventional wisdom about the behavior of prices. For example Backus and Kehoe (1992) find that in most countries prices have been substantially more persistent since World War II than in earlier periods. Third, as a result of both the decreased volatility, the increase in the underlying long term inflation rate and the increased persistence of prices, *deflation* has occurred less frequent during the postwar period than before that time. Simply counting the number of years in which prices decreased reveals that during the prewar period prices decreased in about 47% of the years while during the postwar period this is the case in only 3% of the years.
- Strongin and Petsch (1997) argue that in 1979 the Federal Reserve Bank (FED) in the US has changed its monetary policy from rather passively reacting to already occurred inflation into a more proactive prevention of inflation. The result was a substantial increase in the stability of inflation. The standard deviations in Table 10.1 for the US do not directly confirm this. A closer look at the data however reveals a standard deviation of inflation of 3.2% for the 1950-1979 period and of 2.4% for the 1980-1999 period. For the NL and the UK the standard deviation also decreased, respectively from 2.9% to 1.7% and from 5.9% to 3.2%.
- The *volatility ratios* from Table 10.1 show that especially for the UK the standard deviation of price changes has been much higher than that of the growth of the national product.
- The *correlations* with the growth of the national product indicate yet another important change in the behavior of prices. During the postwar period this correlation is consistently negative while during earlier periods it has been positive most of the times and sometimes zero. These changing correlations could indicate a change in the lead / lag relations of prices compared to the national product. More on this will be said in the next sections.
- Finally, for example Longue and Willet (1976), Okun (1971), Chowdhury (1991) and Barro (1995) indicate a positive (cross section) relation between the *average inflation rate* and its *volatility*. The results in Table 10.1 are supportive of this finding in the sense that during the postwar period the UK has known the highest average rate of inflation while at the same time it has also had the highest standard deviation. Note that section 20.4.2 describes this so called *level effect* for the volatility of the stochastic component in the Consumer Price Index for the NL.

Table 10.1 Statistics of the growth (delta logarithm) of Consumer Price indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index. The final column for each country shows the first order auto correlations.

Netherlands					United Kingdom					United States							
	Avg	Stdev	Ratio	Corr	Auto		Avg	Stdev	Ratio	Corr	Auto		Avg	Stdev	Ratio	Corr	Auto
1814-1999	1.5%	7.4%	-	-	0.2	1601-1999	1.2%	8.3%	-	-	0.2	1821-1999	1.8%	5.6%	-	-	0.4
1814-1899	-0.4%	9.2%	-	-	0.1	1601-1899	0.2%	8.1%	-	-	0.1	1821-1899	0.2%	6.2%	-	-	0.2
1871-1999	2.5%	5.3%	0.7	0.1	0.3	1856-1999	2.7%	7.6%	2.2	0.2	0.4	1871-1999	2.3%	4.7%	0.9	0.3	0.7
1871-1899	0.1%	5.9%	1.0	0.4	-0.1	1856-1899	-0.7%	4.9%	1.4	0.1	-0.1	1871-1899	-0.6%	3.2%	0.8	0.2	0.3
1900-1999	3.2%	4.9%	0.6	0.0	0.5	1900-1999	4.1%	8.1%	2.3	0.2	0.4	1900-1999	3.1%	4.8%	0.8	0.2	0.6
1871-1913	0.7%	5.2%	1.1	0.3	0.0	1856-1913	-0.4%	4.4%	1.3	0.0	-0.1	1871-1913	0.3%	3.1%	0.7	0.3	0.4
1920-1939	-1.6%	4.9%	1.3	0.0	0.2	1920-1939	-3.1%	9.1%	2.4	0.3	0.0	1920-1939	-1.5%	4.5%	0.7	0.7	0.3
1950-1999	3.8%	2.7%	1.2	-0.1	0.7	1950-1999	5.9%	5.0%	2.6	-0.3	0.6	1950-1999	3.9%	2.9%	1.2	-0.1	0.7
1950-1975	4.6%	3.0%	1.3	-0.3	0.5	1950-1975	5.5%	5.7%	2.8	-0.3	0.4	1950-1975	3.3%	2.8%	1.0	-0.1	0.6
1976-1999	3.0%	2.1%	1.4	-0.2	0.9	1976-1999	6.3%	4.1%	2.4	-0.3	0.8	1976-1999	4.6%	2.9%	1.5	-0.1	0.8

## 10.2 Filtering results

Figure 10.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of Consumer Price index for the NL and long wave component compared to UK and US data (bottom right).



- From the simulation results in section 5.5.6 we know that the applied zero phase frequency filter is not very good at correctly filtering exponential trends from a time series. A general solution suggested and tested there and applied on for example the national product and industrial production indices, is to first apply a (natural) logarithmic transformation which transforms an exponential trend into a linear trend. Linear trends have a clear representation in the frequency domain (i.e. extremely low frequency fluctuations) and can therefore be filtered correctly. As noted in the previous section, prices do not show either a simple linear or exponential trend. The logarithmic transformation of the series results in a pattern that seems best characterized by a combination of two very different linear trends with a (structural) break around the year 1900. This is why for each of the three countries a separate filtering of the trend was applied for the pre-1900 and post-1900 period of the logarithmic transformation of the series. Filtering of the trend for the entire sample at once, as expected, resulted in trends that do not capture the true behavior of the series very well. Note that, because just looking at Figure 10.2 already clearly shows the break, we did not perform a formal statistical test on the structural break around 1900. Because from the simulation experiments we know that direct filtering of exponential type trends mainly disturbs the filtering of the trends themselves and not so much the filtering of other components, for the periodic and long wave components the

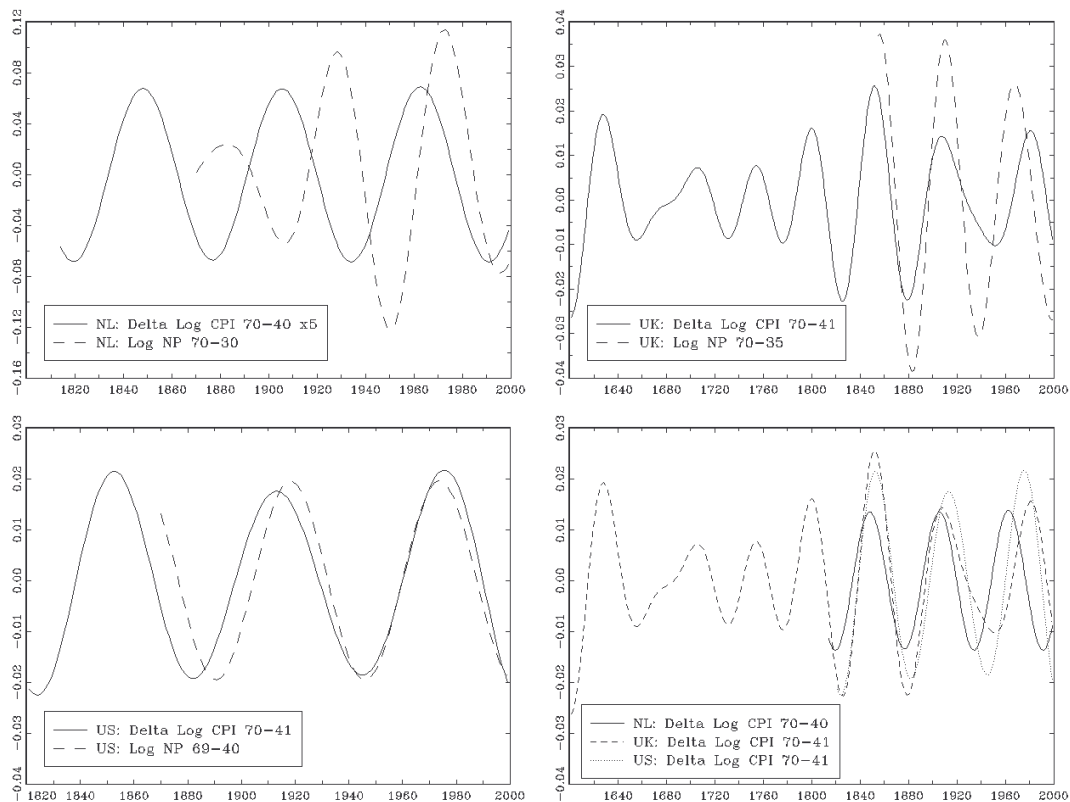
sample was not split. As usual, the stochastic component was filtered separately for the prewar, interwar and postwar period.

- The *trend component* for each of the three countries shows a piecewise linear trend of which the average growth rates are very consistent with the long term average price changes from the data analysis. For the 19<sup>th</sup> century the average inflation implied by the trend is for the US exactly the same as the actual average inflation rate while for the NL and the UK it is only 0.2% higher. With this filtering approach it is not possible to avoid that the two sections of the linear trend do not perfectly connect at the year 1900.
- For the NL and the US the *long wave components* captures the movements in the broader *periodic components* rather well, with the exception of major historical events such as the Great Depression and the two World Wars. However, for the much longer UK time series the periodic component has a much larger volatility than the long wave component. The long wave components in the three countries are of similar amplitude and also have approximately the same dating of peaks and troughs as can be seen from Table 10.4. This table also shows an average peak-to-peak and trough-trough *period length* of respectively 58 and 53 years which is about 5 to 10 years longer than for the variables studied in the previous chapters. Note that prices were the first variables in which long waves were observed. Given their great degree of international synchronization, especially for the earlier cycles as shown in Figure 10.2, it is easy to see why this is so. For the NL the *amplitude* of the long wave component is comparable to that of the national product while for the UK and the US it is respectively about five and ten times as large.
- The results in Table 10.4 for the UK and the US indicate that the long wave in prices consistently *lags* the long wave in the national product by some 10 to 20 years. One explanation for this could be that the long wave is better assumed to be present in the *changes* of prices (i.e. inflation rates) than in the original *level* of the prices. To further investigate this possibility the long wave component in the inflation rates was calculated as the first order differences of the filtered long waves in the (logarithmic) level of the prices as shown in Figure 10.2. Note that theoretically this gives the same result as the direct filtering of the inflation rates<sup>49</sup>. The results are shown in Figure 10.3. The first order differencing operator shifts the long wave about a quarter of a cycle (i.e. 15 years for a period length of 60 years) back in time while it also greatly reduces its amplitude. Because of this, the long wave component in the inflation rate for both the UK and the US is indeed much more consistent with the long wave in the national product. Unfortunately for the NL, both with respect to the dating of the peaks and troughs and with respect to the amplitude, the long wave component in the inflation rate turns out rather different than the long wave in the national product. The reason is that the long wave in the NL prices *leads* the national product by 10 to 20 years, especially for the earlier cycles. See Table 10.4. Because of the phase shift, the inflation rates will be even more out of phase than the price level. Note however that in the original inflation data for the NL the final peak seems to be dated somewhere in the middle of the 1970's rather than around 1960 as in the long wave component shown in Figure 10.3. Also as described in section 7.2, the

<sup>49</sup> Direct filtering of the inflation rates using identical pass-bands for each of the three countries indeed resulted in very similar long wave components as shown in Figure 10.3.

reported dating of the peaks and troughs for the Dutch national product seem somewhat late compared to the two other countries. Because the same transformation is applied on all series, the long waves in the inflation rates of course remain very similar across the three countries. In short, these mixed results indicate that in the NL the long wave in the prices themselves most closely matches with the long wave in (the level of) the national product while for the UK and US this clearly is the case for the long wave in the inflation rate. Given the apparent lag in the national product for the NL and the fact that the actual peak in the Dutch inflation rates is somewhere around the middle of the 1970's, the conclusion should be that the long wave in inflation rates rather than in prices moves in parallel with the long wave in the national product.

Figure 10.3 Long wave component in inflation rates for the indicated countries calculated as the first order differences of the long wave components in the (logarithmic) prices as shown in Figure 10.2 compared with the long wave component in the relevant National Product index.



- Several things can be said about the *volatility* of the *stochastic component*. First, the results in Table 10.2 now clearly show the increased stabilization of the price changes that was already (visually) noted in the previous section. For each of the three countries it is clear that the standard deviation of the stochastic component during the postwar period is approximately twice as small as before that time. Second, across the three countries the volatility for the NL and the US seems comparable while the volatility for the UK, especially for the postwar period is about 50% larger. Third, this is also visible in the volatility ratios relatively to the national product indices. For the NL and the US the postwar ratio is slightly smaller than one which is consistent with for example Kydland and Prescott (1990) who report a ratio of 0.8 for postwar US data. For the UK however it is 1.4,



which is exactly the same as reported by Blackburn and Ravn (1992). Note however that for the second half of the postwar period the UK prices also have a ratio of 1.0.

- The *correlations* of the business cycle fluctuations in prices with those in the national product for the various sub-periods again suggest the very different behavior of prices during the postwar period. All postwar correlations are negative while for the other periods positive correlations are reported. For a long time economists believed prices to behave procyclical (i.e. have a positive correlation with national product). However, since the work of for example Kydland and Prescott (1990), Cooley and Ohanian (1991) and Backus and Kehoe (1992) for various countries, it is now commonly accepted that procyclical prices were the case during the interwar period but that especially during the postwar period prices actually behaved countercyclical. There is also evidence that this also holds for the prewar period, although this is not confirmed by the correlation numbers from Table 10.2. Instead, the results in Table 10.2 indicate procyclical behavior for the prewar period. The (negative) postwar correlations for the NL and the US are very similar but for the UK these correlations are significantly smaller. This might suggest a lead / lag relation for the UK which differs from the one in the other two countries. The spectral analysis in the next section should provide more information on this matter.
- While the previous observation with respect to the correlations is still rather similar to what was found with the conventional data analysis, this is certainly not the case for the degree of *persistence* in price changes. Measured by the increased auto correlations, the data analysis suggested an increased persistence in the postwar period compared to earlier periods. However the autocorrelations of the stochastic components reported in Table 10.2 are consistently around 0.5 for all countries and all sub-periods. So the increased persistence as reported by for example Backus and Kehoe (1992) does not seem to hold if one specifically considers the business cycle type fluctuations.
- In general one could say that the statistics of the filtered stochastic components yield a more stable picture of how prices behave in the short run than the statistics of the conventional inflation rates. This is based on three observations. First, the clearly visible lower volatility of short term fluctuations is more clearly reflected in the filtered series than in the inflation rates. Second, the volatility ratios compared to the national product are more stable over the various countries and third, the degree of persistence in the filtered series turns out to be very stable across time while the inflation rates suggest marked changes in the autocorrelations. These obvious advantages of being able to adequately zoom in on the behavior of time series are probably caused by the fact that the first order differencing filter (as used to calculate the inflation rates) leaves a part of the very low frequency fluctuations intact while at the same time it enhances the high frequency fluctuations. For this also see section 5.2.1. Depending on the importance of these components in a time series, this may disturb inference about the properties of the time series under investigation.
- Judging from the results in Table 10.3, for both the prewar and interwar period the stochastic component describes virtually the entire volatility of the conventional inflation rate. However, for the postwar period the stochastic component describes only 60% of the volatility of the inflation rate. A probable



cause of these differences between the various time periods is the lower volatility of the stochastic component during the postwar period. Because of this the relative importance of the long term movements in the total price volatility increases. As noted before, these long term movements are partially maintained in the inflation rates which are obtained by applying the first order differencing operator. However, in the stochastic components these are completely suppressed by the applied band-pass filter which causes the lower volatility of the stochastic components compared to the inflation rates during the postwar period<sup>50</sup>.

- Note that the previous argument also explains the higher autocorrelations in the postwar inflation rates. Because of the lower volatility of business cycle fluctuations, the long wave or periodic fluctuations have a greater influence on the autocorrelations of the inflation rates during that period. Low frequency fluctuations have higher autocorrelations than high frequency fluctuations. So, the finding of the higher persistence in postwar inflation rates seems more a consequence of a lower business cycle volatility combined with the fact that the first order differencing filter does not completely suppresses low frequency variance than that the fundamental dynamics of the price process have changed.

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<sup>50</sup> For those working in the field of ALM and scenario analysis this finding is of special importance. If one simulates scenarios of conventional inflation rates consistent with historical statistics observed during the postwar period, then also the part of the volatility caused by the long term movements in prices is extrapolated into the future. In the past members of the Board or Investment Committee of Dutch pension funds often judged the resulting inflation scenarios as “too volatile”. If one were to use “only” the filtered stochastic component of the prices for such simulations, based on the results in Table 10.3, the resulting inflation volatility would go down by approximately 40%.

Table 10.2 Statistics of the filtered periodic, long wave and stochastic components of Consumer Price indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index. The final column for each country shows the first order auto correlations.

	Netherlands				United Kingdom				United States						
	Stdev	Ratio	Corr	Auto	Stdev	Ratio	Corr	Auto	Stdev	Ratio	Corr	Auto			
Periodic	1870-1999	17.3%	1.4	0.3	-	1855-1999	95.9%	12.2	0.4	-	1870-1999	20.7%	2.2	0.2	-
Long Wave	1870-1999	8.5%	1.3	0.6	-	1855-1999	11.5%	4.9	-0.1	-	1870-1999	13.6%	10.0	0.3	-
Stochastic	1870-1913	4.9%	1.5	0.1	0.5	1855-1913	4.6%	1.8	0.3	0.5	1870-1913	3.2%	0.9	0.4	0.6
	1919-1939	4.4%	2.4	0.2	0.4	1919-1939	8.4%	2.2	0.7	0.5	1919-1939	5.7%	1.3	0.4	0.7
	1949-1999	1.7%	0.9	-0.4	0.5	1949-1999	2.9%	1.4	-0.1	0.4	1950-1999	2.0%	0.9	-0.4	0.6
	1949-1975	2.0%	0.9	-0.4	0.4	1949-1975	3.2%	1.9	-0.1	0.3	1950-1975	1.7%	0.8	-0.4	0.5
	1976-1999	1.4%	0.8	-0.3	0.7	1976-1999	2.6%	1.0	0.0	0.6	1976-1999	2.3%	0.9	-0.4	0.7

Table 10.3 Standard deviation of growth (delta logarithm) of Consumer Price indices for various sub-periods (Hist) versus standard deviation of first differences of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States				
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio		
1813-1913	8.4%	8.6%	1.0	1600-1913	7.7%	7.9%	1.0	1820-1913	5.1%	5.8%	0.9
1919-1939	4.8%	4.9%	1.0	1919-1939	8.7%	9.1%	1.0	1919-1939	4.4%	4.5%	1.0
1949-1999	1.7%	2.7%	0.6	1949-1999	3.1%	5.0%	0.6	1949-1999	1.8%	2.9%	0.6
1949-1975	2.1%	3.0%	0.7	1949-1975	3.6%	5.7%	0.6	1949-1975	1.6%	2.8%	0.6
1976-1999	1.1%	2.1%	0.5	1976-1999	2.3%	4.1%	0.6	1976-1999	1.9%	2.9%	0.7

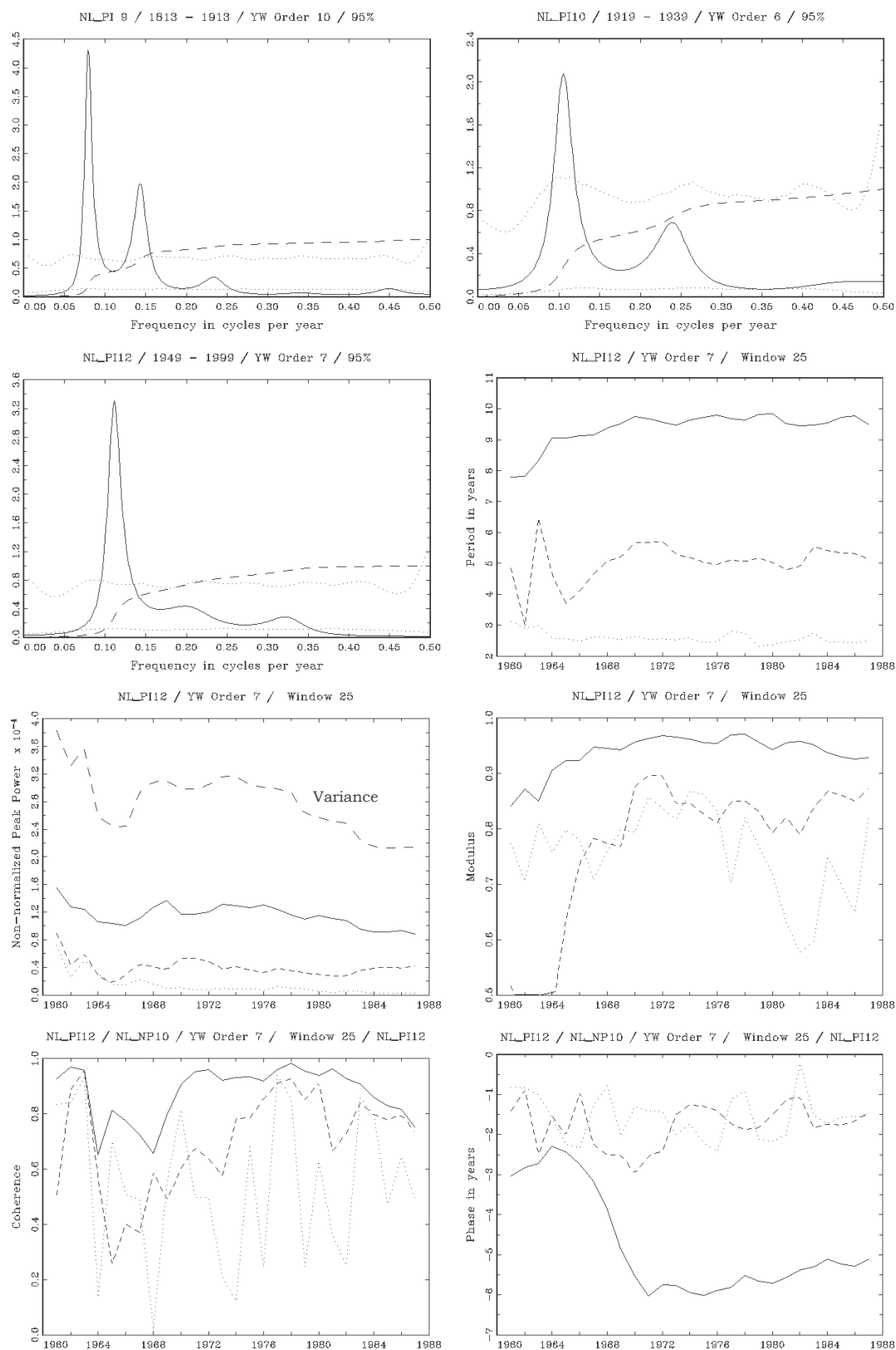


## 10.3 Spectral analysis

### 10.3.1 Static

- All the estimated spectral densities are characterized by the familiar peaks at the Juglar and Kitchin frequencies. The properties of these peaks are very similar to those in the spectral densities for the national product given in section 7.3. The Juglar peak in general has a little higher *modulus* and *peak power* than the Kitchin peak and the Juglar peak is in all cases *significant* at the 95% confidence level. A difference with the national product spectra is however that here in some cases the Kitchin peak is significant also. For the postwar period this is the case for both the UK and the US but not for the NL.
- Although the *coherence* is high most of the times, in some cases it is rather low, indicating that caution should be taken with the interpretation of the corresponding phases. Perhaps the most surprising are the low coherences of 0.6 for both the Juglar and Kitchin peak for the postwar NL spectra. The *phase* for the Juglar peak shows a lag of somewhere between two to four years making prices almost perfectly countercyclical compared to the national product. For the Kitchin peak the lag is often around one and a half year which also at this frequency implicates near perfect countercyclical behavior. Such substantial lags are also reported by for example Reiter (1995) and imply very slow adjustments of prices to changing economic circumstances. By now it is well accepted that in general price variables adjust more quickly than quantity variables. For example changes in the demand for labor show up more clearly and directly in lower employment rates than they do in a lowering of wages. For the same reasons consumer prices show slow and limited adjustments to changing ratios between supply and demand. Such phenomena are often described as *nominal rigidities*. Finally note that, as already expected from the (less negative) correlations reported in Table 10.2, the lag at the Juglar frequencies during the postwar period for the UK is much smaller than for the NL and the US.
- The most important *change* in the various spectral measures across the three sub-periods is that both during the prewar and postwar periods prices are lagging the national product by several years while during the interwar period they show a tendency of moving in phase with the national product. This is consistent with the literature mentioned in the previous section which reports prices to have behaved countercyclical during the prewar and postwar periods whereas they behaved in a procyclical manner only during the interwar period.

Figure 10.4 Static and dynamic estimates of the spectral density of the stochastic component of a Consumer Price index for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



### 10.3.2 Dynamic

- In general the *average measures* of the rolling window spectral estimates from Table 10.6 are very consistent with the static estimates from Table 10.5 for the postwar period. For the NL however two important differences can be observed as well. First, the peak power of the Juglar peak has increased up to 40% while the *peak power* of the Kitchin peak has decreased to only 10%. Combined with the result from the previous section (only in the NL the Kitchin peak was not significant), this suggests that the Kitchin type fluctuations play a less important role in the evolution of consumer prices in the NL than they do in the UK and the US. Second, the average *coherence* of especially the Juglar peak is 0.9 instead of only 0.6 in the static estimates while the corresponding phase is approximately the same. This gives extra confidence in the reported lag of four to five years at these frequencies, despite the low coherence in the static estimates.
- For the NL the *period length* of both the Juglar and the Kitchin peak have lengthened by one or two years. For the other two countries this only has been the case for the Kitchin peak in the US. All other period lengths remained rather constant during the entire postwar sample.
- For both the UK and the US the *variance* has increased during most of the years with a slight decrease at the end of the sample. Especially in the US this decrease has mostly come from a lower peak power of the Kitchin peak. In the NL both the variance and the peak power of the two spectral peaks have decreased during most of the years. The in general decreasing variance at the end of the sample may be a result of the changing monetary policy towards a proactive control of inflation as mentioned in section 10.1.
- The *modulus* of the two spectral peaks have been rather constant in the UK. In the NL and the US increases in the modulus can be observed, mostly at the beginning of the sample.
- In the NL the *coherence* of both peaks have been very low at the start of the sample, possibly because this was a rather turbulent period. During most of the sample the coherence has increased from these low levels towards “normal” levels of above 0.9. In recent decades however a decrease seems to have set in again. Note that the initially low values of the coherence could be the cause of the low overall coherence found in the static estimates which obviously are not very representative for most of the sample. The decreasing coherence at the end of the sample can also be observed in the UK. The coherence in the US has shown a very different pattern than in the NL and the UK. Throughout almost the entire sample the coherence of both the Juglar and Kitchin peak has been close to 1.0 indicating a very strong (phase corrected) correlation between business cycle fluctuations in prices and national product. This is confirmed by the results in Table 10.5 and 10.6.
- Because of the almost perfect countercyclical behavior the rolling window phase often alternates between a lag of half a period and a lead of half a period. Because in this way it is hard to see how the phase has changed through time and the average phase as reported in Table 10.6 would be approximately zero, which is clearly very misleading information, the phases of all the complex roots are

reported as the smallest possible *negative phase* (i.e. lag). Remember that a phase is only unique up to an integer times the corresponding period length.

- The *phase* of the Kitchin peak in each of the three countries has very constantly been around a lag of two years. In the UK this also holds for the phase of the Juglar peak. In the US however the phase of the Juglar peak has steadily increased from a lag of about six years towards a lag of only three years. After a strong initial increase, also in the NL the lag has decreased by about a year during the second half of the sample. Note that these indications of a less lagging behavior of prices may also come from the changes in macroeconomic policies towards a more proactive prevention of inflation as described in section 10.1.



Table 10.5 Statistics of static spectral estimates of the filtered stochastic components of Consumer Price indices for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
	Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase
1813-1913	12.5#	0.2	1.0	0.8	-4.9	1600-1913*	11.4#	0.3	0.9	0.6	-1.9	1820-1913**	8.9#	0.4	1.0	0.9	-0.6
	6.9#	0.2	0.9	0.8	-0.2		5.6#	0.3	0.8	0.6	-1.9		4.9	0.1	0.8	0.5	-0.4
	4.2	0.1	0.9	0.4	1.2		3.6	0.1	0.8	0.7	-0.1		2.8	0.1	0.8	0.2	-0.9
	2.2	0.0	0.9	0.4	0.0		2.3	0.0	0.8	0.8	-0.5						
1919-1939	9.5#	0.3	0.9	1.0	0.2	1919-1939	10.2#	0.3	0.9	1.0	0.0	1919-1939	10.7#	0.4	0.9	0.9	1.8
	4.1	0.2	0.9	0.7	1.2		4.2	0.2	0.8	0.7	-0.7		4.7	0.1	0.7	0.3	-1.0
	2.3	0.1	0.7	0.6	-0.4		2.5	0.1	0.7	0.7	-0.9		2.4	0.1	0.7	0.4	0.3
1949-1999	9.0#	0.3	0.9	0.6	-4.2	1949-1999	9.7#	0.2	0.9	0.8	-2.4	1949-1999	10.1#	0.3	1.0	0.9	-3.9
	4.8	0.2	0.8	0.6	-1.4		4.9#	0.3	0.9	0.8	-1.7		5.3#	0.2	0.9	0.9	-1.6
	3.0	0.1	0.8	0.7	-0.9		3.1	0.1	0.8	0.6	-0.9		3.0	0.0	0.8	0.6	-0.6

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1855-1913 estimates of the co-spectrum for the indicated 1600-1913 frequencies / period lengths.

\*\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1820-1913 frequencies / period lengths.

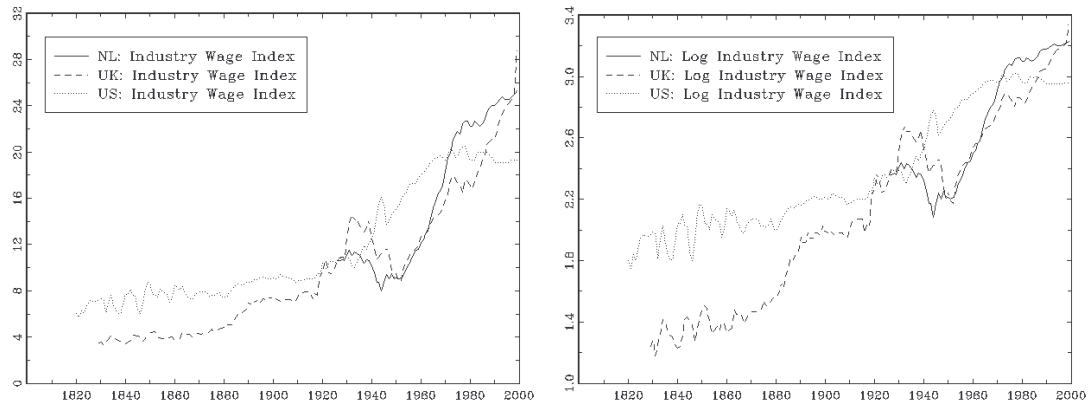
Table 10.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of Consumer Price indices for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 10.5.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
9.4	0.4	0.9	0.9	-4.8	9.7	0.3	0.9	0.8	-2.3	10.5	0.3	0.9	1.0	-3.8
5.0	0.1	0.8	0.7	-1.8	4.9	0.3	0.9	0.9	-1.8	5.3	0.3	0.9	1.0	-1.7
2.6	0.0	0.8	0.5	-1.6	3.0	0.1	0.6	0.6	-0.4	3.1	0.0	0.7	0.7	-0.8

# 11 Industry Wages

## 11.1 Data analysis

Figure 11.1 Real Industry Wage indices (i.e. corrected for Price Inflation) and their (natural) logarithm. The series pass through a common point in 1926 (start of NL series).



- The wage indices analyzed here aim to show how the relative earnings of people working in the industrial sector of an economy have evolved over time. The indices hold in real terms because, as indicated in Appendix F, they are constructed from nominal wage indices divided by the relevant consumer price indices analyzed in the previous chapter.
- For each of the three countries the series show a *slight exponential trending* behavior. Table 11.1 shows that the average long term real increase in wages is approximately 1.0%. This rather small percentage illustrates the large effects price inflation has on salary increases which by themselves may seem substantial. For example in 1999 an employee working in some UK industry in real terms earned only about seven times as much as he would have earned almost two centuries back in 1829.
- Substantial deviations from this long term *average wage inflation* of about 1.0% have occurred. Especially noteworthy are the high averages of around 2.0% for the postwar period in the NL and the UK. Also in each of the three countries during the first half of the postwar period the increase in wages has been much higher than during the second half. Exceptional is the average real wage inflation of  $-0.1\%$  in the second half of the postwar period in the US. This means that during this period real wages in the US did not increase at all.
- The large peaks in the indices for the NL and the UK during the 1930's are probably caused by the large decrease in prices during that period (see section 10.1).
- With respect to the *shorter term fluctuations*, judging from the results in Table 11.1, the *volatility* in the NL and the US are comparable while the volatility for the UK is the highest of the three, just as was the case for the consumer prices<sup>51</sup>.

<sup>51</sup> Because the standard deviation of also the *nominal* wage inflation has in general been the largest for the UK, the possibility of the higher volatility of the price inflation being the only “artificial” cause of this

This is also the case for the relative *volatility ratios* compared to the growth of the national product. For both the NL and the US, real wages are approximately just as volatile as the national product (i.e. a ratio of one) while for the UK the ratio is pretty steady at around one and a half.

- For the postwar period, real wage inflation is in general positively *correlated* with the growth of the national product indicating wages to behave in a procyclical manner. The only (confusing) exception is the second half of this period in the NL with a slightly negative correlation. For the prewar and interwar periods negative correlations occur more often. So there might have been a change in the lead / lag relations between the three sub-periods analyzed. In any case, the severe changes in the correlations after World War II that were observed for the consumer prices (see Table 10.1) did not occur for the real wages.
- Contrary to what was the case for the consumer prices, also during the postwar period actual decreases in real wages frequently occurred.

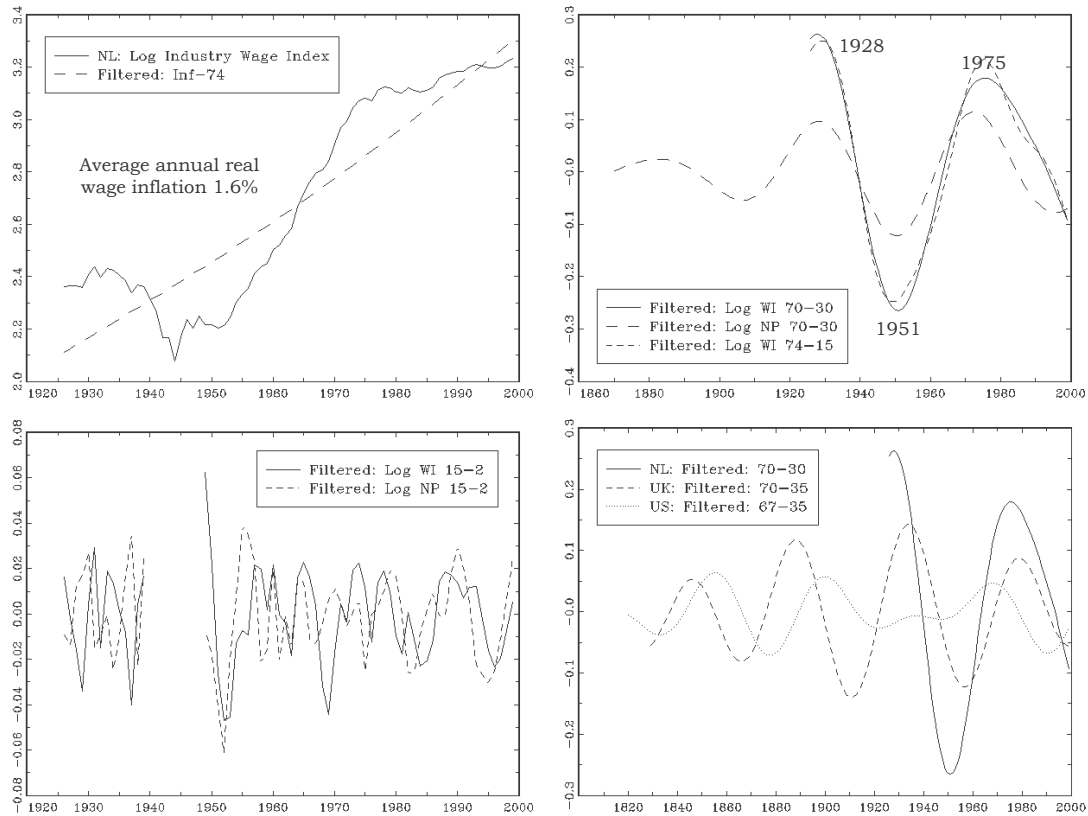
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higher volatility of the real wage inflation in the UK is ruled out. The nominal volatilities for the postwar period are 3.7% for the NL, 4.1% for the UK and 2.5% for the US.



## 11.2 Filtering results

Figure 11.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (real) Industry Wage index for the NL and long wave component compared to UK and US data (bottom right).



- For each of the three countries the *trend component* shows a linear trend of which the average growth rate is almost identical to the long term average real wage inflations reported in Table 11.1.
- For the NL and the UK the *long wave component* describes the broader *periodic component* rather well. For the US however in the periodic component a similar pattern can be observed as for example for the national product. Roughly speaking during the 19<sup>th</sup> century real wages were above trend, during the first half of the 20<sup>th</sup> century they were below trend while during the postwar period they were again above trend. Although the long wave components in the three countries seem to run rather parallel, it is hard to draw a general conclusion about the *lead / lag* relation of the real wage index when compared to the long wave component in the national product or other variables analyzed thus far. Table 11.4 shows for the NL no significant lead or lag, for the UK a lag of approximately 20 years and for the US a lead or around 10 years. The average long wave in the real wage indices therefore moves in parallel with the national product. The only careful conclusion about the lead / lag relation is that the postwar long wave cycle has run approximately in parallel with the national product. Although for the NL this is very clear from Figure 11.2, the dating of the peaks and troughs is not that consistent for the other two countries. Both the average peak-to-peak and trough-to-trough *period lengths* are 43 years. The

*amplitude* of the long wave component is two to three times as large as that in the national product.

- The *volatility ratio* of the *stochastic component*, is close to one for both the NL and the US while for the UK it is a little higher. This finding is consistent with for example the summary of results from other research as given in Englund et al. (1992) which in general report a ratio a little below one. Blackburn and Ravn (1992) report a ratio of 1.1 for the UK. Kydland and Prescott (1990) however report a much lower ratio of only 0.5 for the US.
- All these references agree on business cycle fluctuations in real wages behaving procyclical when compared to the national product. Just as Canova (1998), here we find a *correlation* with the stochastic component in the national product of approximately 0.4. Note that contrary to the conventional statistics reported in Table 11.1, the second half of the postwar period in the NL is now no longer an exception to this finding. This procyclical behavior of real wages is by now well accepted as a stylized fact of macroeconomic behavior. For a long time however it was believed that real wages behaved acyclical. For this, Kydland and Prescott (1990) refer to Summers (1986, p. 25) while also Blanchard and Fisher (1989) describe the “conventional wisdom” of real wages behaving acyclical, or perhaps weakly procyclical.
- A visual inspection of the stochastic component already reveals some information about the *lead / lag relations* of the real wages compared to the national product in the various countries. For both the UK and the US real wages seem to lead the national product by approximately one or two years during the postwar period, especially towards the end of the sample. For the NL however a lag of one or two years seems to have been the case. The results of the spectral analysis will have to confirm these findings.
- Contrary to what was the case for the consumer prices, the stochastic component of the real wages describes virtually the entire *volatility* of the conventional real wage inflation. For this see Table 11.3.

Table 11.2 Statistics of the filtered periodic, long wave and stochastic components of (real) Industry Wage indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands			United Kingdom			United States					
	Stdev	Ratio	Corr	Stdev	Ratio	Corr	Stdev	Ratio	Corr			
Periodic	1926-1999	15.8%	1.1	0.7	1855-1999	12.7%	1.6	-0.1	1870-1999	17.2%	1.8	0.2
Long Wave	1926-1999	15.8%	2.0	0.9	1855-1999	8.2%	3.1	-0.5	1870-1999	3.7%	2.7	0.1
Stochastic					1855-1913	3.1%	1.2	0.1	1870-1913	3.1%	0.9	-0.2
					1919-1939	6.8%	1.8	-0.6	1919-1939	4.3%	0.9	0.7
					1949-1999	2.6%	1.2	0.2	1950-1999	2.3%	1.0	0.4
					1949-1975	3.0%	1.8	0.4	1950-1975	2.0%	1.0	0.4
					1976-1999	2.0%	0.8	0.1	1976-1999	2.6%	1.1	0.5

Table 11.3 Standard deviation of growth (delta logarithm) of (real) Industry Wage indices for various sub-periods (Hist) versus standard deviation of first differences of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States				
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio		
1926-1939	2.7%	2.8%	1.0	1829-1913	4.8%	5.0%	1.0	1820-1913	8.4%	8.5%	1.0
1950-1999	1.8%	2.3%	0.8	1919-1939	5.8%	5.9%	1.0	1919-1939	5.1%	5.4%	0.9
1950-1975	2.1%	2.1%	1.0	1949-1999	2.8%	3.0%	0.9	1950-1999	2.2%	2.4%	0.9
1976-1999	1.3%	1.3%	1.0	1949-1975	3.3%	3.5%	1.0	1950-1975	2.2%	2.5%	0.9
				1976-1999	2.1%	2.3%	0.9	1976-1999	2.1%	2.1%	1.0



Table 11.4 Peaks and troughs of the long wave component in (real) Industry Wage indices. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

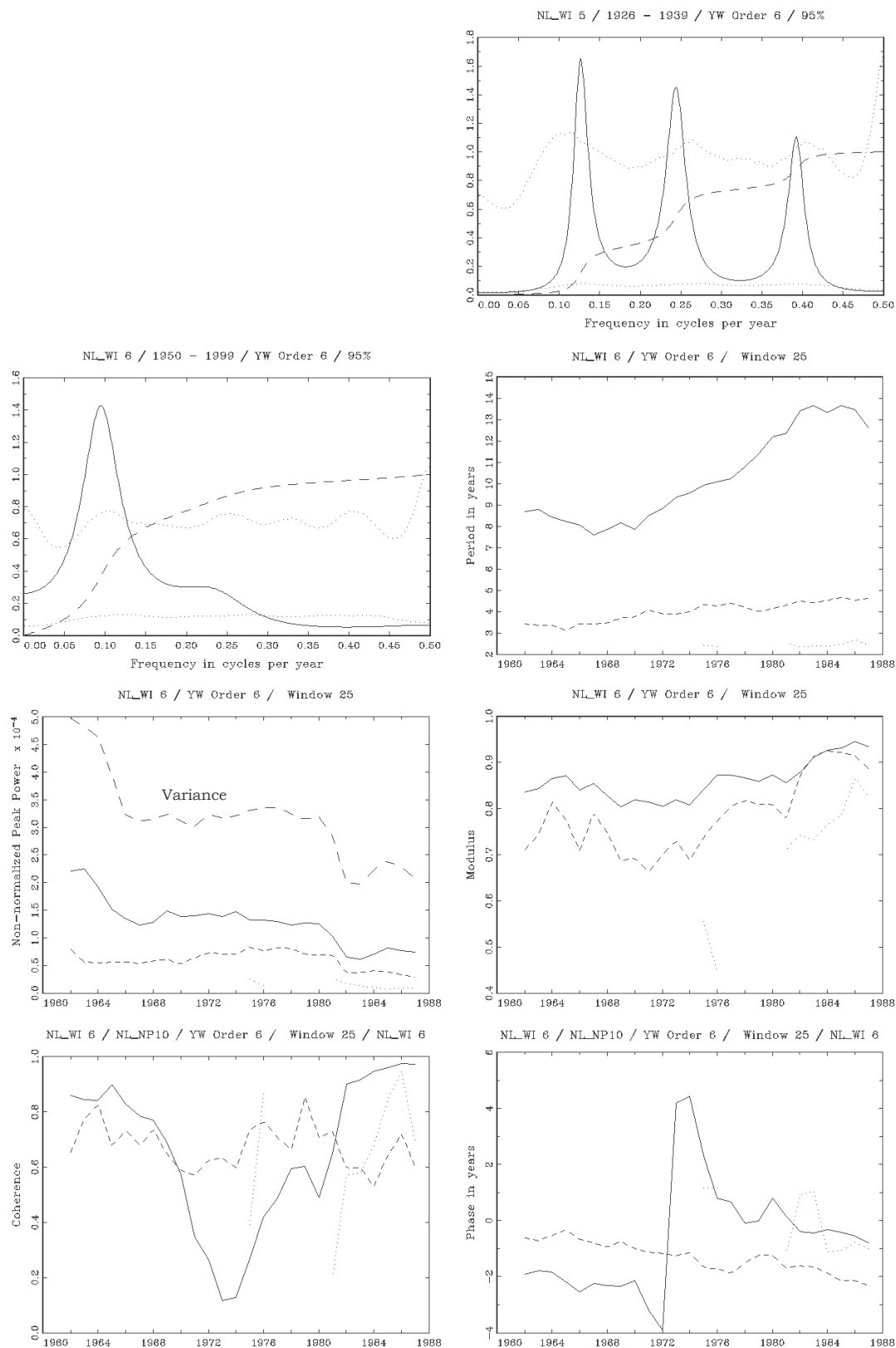
	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>Peak-Peak Trough-Trough Lead</i>
Netherlands			1928	47 1975		47
Peak				1951	1999	48
Trough			0	-1	-3	-2
Lead				45		44
Peak	1846	42 1888	1934	45 1979		44
Trough		1866	46 1911	42 1957	1999	44
Lead		-32	-28	-19	0	-19
Peak		1856	44 1900	69 1969		57
Trough			53 1877	61 1930	1991	57
Lead			13	16	5 8	12
Average						
Peak	1846	26 1872	49 1921	54 1974		43
Trough		1866	52 1894	50 1946	1996	43
Lead		-3	-1	-1	-3	-1

## 11.3 Spectral analysis

### 11.3.1 Static

- The by now familiar spectral peaks around the Juglar and Kitchin frequencies are present in most of the estimated spectral densities for the stochastic component in the real wages. The major exception concerns the prewar periods in the UK and in the US which basically only show a single peak with a *period length* of around eight years, which is somewhere in between the ten year period length of the Juglar peak and the five year period length of the Kitchin peak. Also, during the interwar period the spectral density for the NL shows a third (significant) peak at a very high frequency (period length 2.5 years), which is probably caused by the turbulence during that period. Without exception the Juglar peaks are *significant* at the 95% confidence level. Just as was the case for the consumer prices, during the postwar period also the Kitchin peaks in the spectral densities for the UK and the US are significant while this clearly does not hold for the NL. In general the Juglar peak describes 30% to 40% of the total variance while for the Kitchin peak this *peak power* is mostly around 20%. The *modulus* of both peaks are similar and around 0.9 most of the time.
- With high values of the *coherence*, except for the prewar period, the reported *phases* can be safely interpreted. For most of the periods and countries the phases indicate a substantial *lag* of several years of the real wages over the national product. In the light of the *nominal rigidities*, as mentioned in the discussion of the results for the consumer prices, this seems fairly logical. Wages need time to react to changing economic circumstances. Note that in chapter nine also employment was found to lag the national product, though in general to a lesser extent than found here for the real wages. Two exceptions hold for the UK and US. As already noted from a graphical inspection of the filtered stochastic components, real wages for the postwar period *lead* the national product by one to two years in those countries, both at the Juglar and the Kitchin frequencies. These last findings are consistent with for example Englund et al. (1992) and Kydland and Prescott (1990) who report a lead for the real wages during the postwar period. Blackburn and Ravn (1992) find no significant lead or lag for the UK and therefore conclude real wages to move contemporaneously to the national product. The results here indicate however, that real wages were actually also lagging the national product during earlier periods in the UK and the US. Looking ahead at the dynamic results in the next section, we also see that during the postwar period wages initially started out with a lag over the national product in the UK and the US, while they steadily developed a leading behavior during that period.
- Two important *differences* can be observed when comparing the three sub-periods for the static estimates. The first is the fact that it is not possible to find separate spectral peaks at the Juglar and the Kitchin frequencies during the prewar period. The second is the observed lead of the real wages for the UK and the US during the postwar period as just discussed.

Figure 11.3 Static and dynamic estimates of the spectral density of the stochastic component of a (real) Industry Wage index for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



### 11.3.2 Dynamic

- The *average measures* of the rolling window spectral estimates for the postwar period do not give very different information than the static estimates for that period. The most striking difference is the lower coherence at the Juglar and Kitchin frequencies for the NL while for the UK this coherence is higher than in the static estimates.
- In the NL the *period length* of both the Juglar and the Kitchin peak has increased throughout the postwar sample while for the UK this only has been the case for the Juglar peak. The period lengths for the US remained relatively constant.
- While in the NL and the UK the *variance* seems to have decreased in a stepwise fashion only because of one or two exceptional observations, in the US a genuine increasing variance can be observed. This seems mostly caused by an increased peak power at the Juglar frequencies.
- In the NL the *modulus* has been rather low for a long period of time before it started to increase towards the end of the sample. For the UK and the US the modulus has not changed very much with perhaps the exception of a slight increase in the modulus of the US Juglar peak.
- Just as with the consumer prices, the *coherences* for the US have been extremely high throughout the entire postwar period. For the UK this also holds for the coherence at the Juglar frequencies while there the coherence at the Kitchin frequencies has decreased. The coherence for the Juglar peak in the NL shows very big fluctuations. Both at the beginning and the end of the sample it is close or above 0.9, while in the middle it is as low as 0.2. These big changes may be the reason for the mixed results from the static and average rolling window estimates.
- Because of the low coherence, no meaning should be attached to the *phase* at the Juglar frequencies for the NL in the middle of the sample. In the beginning of the sample there is a lag of about two years while this has turned into no significant lead or lag at all at the end of the sample. This is consistent with the pattern of the Juglar phase for the UK and the US which show a steady increase from a lag towards a substantial lead. Note therefore that in each of the three countries, also for the NL, there seems to have been a transformation of *lagging* real wages towards *leading* real wages during the postwar period. The phase at the Kitchin frequencies in general has not changed very much.

Table 11.5 Statistics of static spectral estimates of the filtered stochastic components of (real) Industry Wage indices for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase			
1926-1939	8.0#	0.2	0.9	1.0	-3.3	1829-1913*	7.5#	0.4	0.9	0.6	-1.1	1820-1913**	7.8#	0.5	0.9	0.6	-3.1
	4.1#	0.2	0.9	0.9	-2.0	1919-1939	2.7	0.2	0.6	0.2	0.4	1919-1939	2.9	0.1	0.6	0.5	-0.8
	2.5#	0.1	0.9	0.6	-1.1		12.1#	0.4	0.9	0.9	-4.6		8.3#	0.3	0.9	1.0	0.1
1950-1999	10.6#	0.4	0.8	0.9	-1.6	1949-1999	10.0#	0.3	0.9	0.7	1.9	1950-1999	10.9#	0.3	0.9	0.9	1.2
	4.2	0.2	0.7	0.5	-0.9	1949-1999	5.1#	0.4	0.8	0.7	0.6	1950-1999	5.6#	0.2	0.9	0.9	1.0
							3.2	0.1	0.8	0.9	0.7		3.4	0.1	0.8	0.8	0.3
													2.1	0.0	0.6	0.9	0.0

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1855-1913 estimates of the co-spectrum for the indicated 1829-1913 frequencies / period lengths.

\*\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1820-1913 frequencies / period lengths.

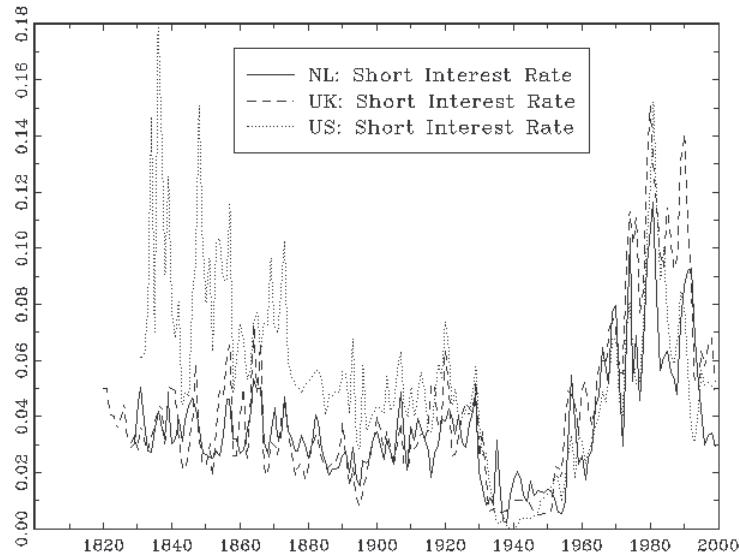
Table 11.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of (real) Industry Wage indices for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 11.5.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
10.3	0.4	0.9	0.7	-0.6	10.2	0.3	0.9	0.9	1.9	10.8	0.3	0.9	0.9	1.1
4.0	0.2	0.8	0.7	-1.3	5.1	0.3	0.9	0.9	0.1	5.5	0.3	0.9	0.9	0.9
2.5	0.1	0.7	0.6	-0.1	2.8	0.1	0.8	0.9	0.6	3.5	0.1	0.8	0.8	0.5

## 12 Short Term Interest Rates

### 12.1 Data analysis

Figure 12.1 (nominal) Short Term Interest Rates.



- Studying the empirical behavior of the *term structure of interest rates* is a complex exercise. A first reason for this is that interest rates are the equilibrium price agreed on between agents for shifting wealth forward or backwards in time and thereby fulfill a crucial role in the processes of *inter-temporal substitution* within the economic process. A second reason is that in a sense interest rates have four dimensions. The first two are the usual dimensions of *time* and (interest rate) *value* as shown in Figure 12.1. The third dimension is the *term* of the interest rate. That is, the period of time for which lender and borrower agreed to exchange wealth. The fourth dimension is the *credit worthiness* (rating) of the borrower. In this chapter we start the study of these various aspects of interest rates by analyzing the behavior of *short term interest rates* (with a maximum term of only a few months) with no or very low credit risk. In the next chapter *long term* (government) *interest rates* are analyzed, also with a very low level of credit or default risk. In Chapter 14 separate attention is paid to the behavior of the difference between these long and short term interest rates. This is also called the *term spread* of interest rates. The analysis of interest rates is closed in Chapter 15 with an analysis of a *default spread*. That is, the difference between (long term) interest rates paid by borrowers with a substantial level of credit risk (for example companies active in some risky business or private persons raising money for a mortgage) and by borrowers for which there is almost no risk that they will not be able to pay off their debt (for example the government of a western country).

- It is important to note that the interest rates and spreads analyzed here and in the next chapter hold in *nominal* rather than in *real* terms. Although it would be the most ideal to analyze real interest rates, this is not possible because (long) time series of real interest rates are not available. In section 13.2.1 we do perform a simple analysis on the long term behavior of real interest rates.
- Figure 12.1 suggests very large *similarities* between the development of short term interest rates in the three countries. This holds both with respect to the long term movements and the short term fluctuations. A striking exception is the 19<sup>th</sup> century US data. During this period short term interest rates in the US have been much higher than in the NL and the UK. The probable cause is that for the 19<sup>th</sup> century in the US only *commercial paper rates* are available (and used here) which contain significant (credit) risk premiums over true default free interest rates. Siegel (1992) also indicates that capital markets in the UK were far more developed during the 19<sup>th</sup> century than they were in the US. London was the world financial center during that time which is why the UK data probably give the best information about the early development of default free short term interest rates. Note that the NL data are very much in line with the UK data for the early part of the sample as well.
- As was to be expected, interest rates do not show any specific *trending behavior*. Instead, with the exception of the US as just explained, the long term pattern starts with short rates averaging somewhere around 3 to 4% for a long time until the start of the Great Depression in the 1930's. This was followed by a period of very low interest rates of around 1% until the middle of the 1950's. Also the postwar period shows a rather exceptional pattern of very strong increases in interest rates during the first half of the period followed again by a decreasing pattern towards prewar levels of around 4%. Besides the postwar period other periods of very *long term swings* in the interest rates can be observed as well. At the end of the 19<sup>th</sup> century rates were for instance relatively low, while they were again relatively high during the next thirty years until the time of the Great Depression. Note that such a simple visual inspection already reveals a dating of peaks and troughs of possible long wave movements in interest rates which is consistent with the approximate dating of long waves given in Table 2.1.
- Figure 13.1 also (again) points out the dangers of inference about the long term behavior of economic variables based on relatively short samples. If for example only the postwar period is considered, researchers are often led to the conclusion that interest rate processes contain unit roots and are driven by *stochastic trends*. Also see sections 2.2.6, 3.2.1 and 5.3.2 for more information on stochastic trends and unit roots. However, based on the longest view possible, slow mean reversion towards a long term mean seems a more appropriate way to model interest rates. Also, in the light of the long term evidence, the high level of interest rates during the postwar period does not seem very representative for the long term level of interest rates. Note that both these two examples may be considered as variants of the *perspective distortion* as described by Reijnders (1990).
- Around the long term movements, the short rates also clearly show *shorter term fluctuations*. At this stage of the analysis, these fluctuations can best be analyzed in terms of the annual *changes in interest rates*. The results in Table 12.1 indicate that the standard deviation of these changes is somewhere around 1.5% for all sub-periods and countries. During the postwar period the short rate volatility



ratio indicates a *volatility* of about 80% of that of the national product while this is only approximately 30% during the prewar period. The latter is more a result of the higher national product volatility than of a lower interest rate volatility during that time. Despite the relatively stable standard deviation of interest changes, in general the volatility does appear somewhat higher during the postwar period which indicates a positive relation between the level of the short rates on the one hand and their volatility on the other. This so called *level effect*, see for example Chowdhury (1991) and Bams (1999), is also especially clear from simply looking at the behavior of the series in Figure 12.1. In the next section specific attention is paid to this property of (nominal) interest rates.

- The *correlation* numbers in Table 12.1 indicate a consistent positive relation between changes in short term interest rates and the growth of the national product. This procyclical behavior seems stronger during the postwar period than before that time. This might very well be policy induced. In periods of high growth the Central Banks increase the short term interest rates to temper the economy while in periods of low growth the short term interest rates are lowered to stimulate the economy.

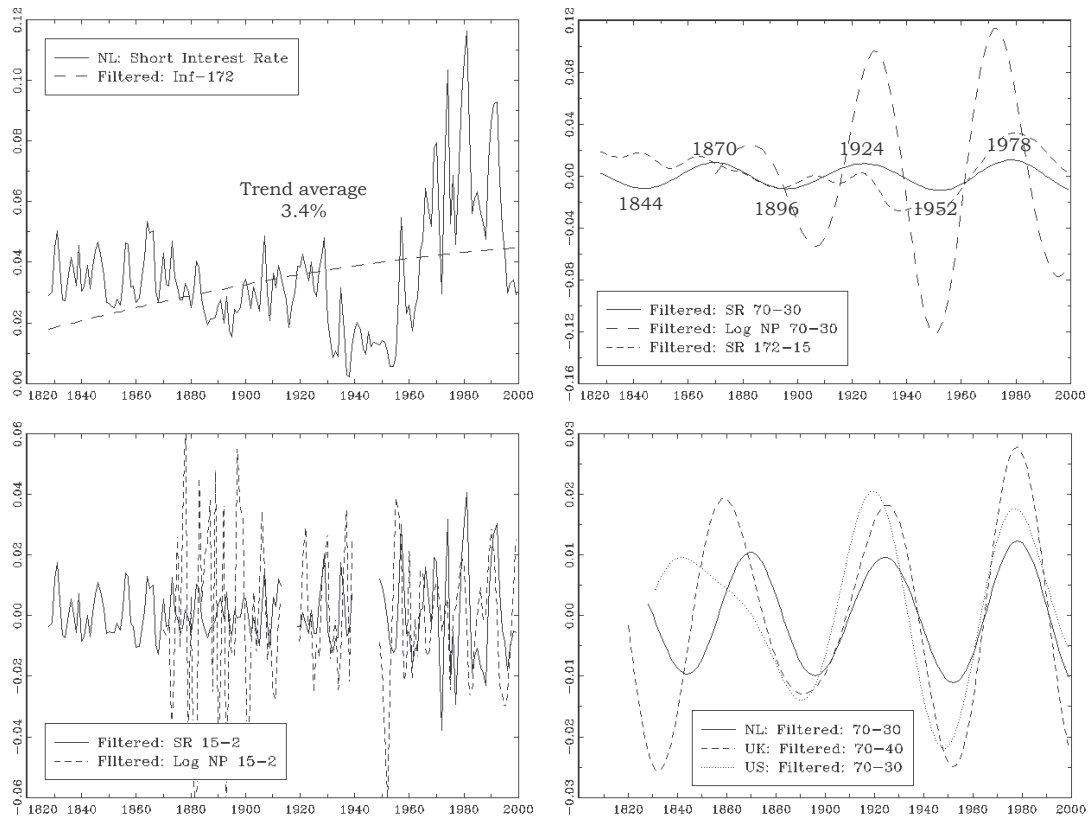
Table 12.1 Statistics of (nominal) Short Term Interest Rates for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

Netherlands				United Kingdom				United States			
	<i>Avg</i>	<i>Stdev1*</i>	<i>Stdev2**</i>	<i>Ratio</i>	<i>Corr</i>		<i>Avg</i>	<i>Stdev1</i>	<i>Stdev2</i>	<i>Ratio</i>	<i>Corr</i>
1828-1999	3.6%	2.0%	1.2%	-	-	1820-1999	4.2%	2.9%	1.3%	-	-
1870-1999	3.6%	2.3%	1.3%	0.2	0.0	1855-1999	4.4%	3.2%	1.4%	0.4	0.1
1870-1913	3.0%	0.7%	0.8%	0.2	-0.2	1855-1913	3.1%	1.3%	1.2%	0.4	0.1
1919-1939	2.6%	1.4%	1.0%	0.2	0.3	1919-1939	2.8%	1.8%	1.2%	0.3	-0.1
1950-1999	5.2%	2.8%	1.8%	0.8	0.4	1950-1999	7.2%	3.5%	1.7%	0.9	0.3
1950-1975	4.0%	2.5%	1.9%	0.8	0.3	1950-1975	5.1%	2.6%	1.2%	0.6	0.3
1976-1999	6.5%	2.4%	1.6%	1.1	0.5	1976-1999	9.5%	2.9%	2.1%	1.2	0.3

\* Standard deviation of *level* of interest rates.\*\* Standard deviation of annual *changes* of interest rates which is also used in the (standard deviation) ratio and correlation numbers.

## 12.2 Filtering results

Figure 12.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Short Term Interest Rate for the NL and long wave component compared to UK and US data (bottom right).



- The estimated *trend* for each of the three countries captures the long term averages as presented in Table 12.1 rather well. For the UK and the NL the trend is upwards sloping because of the relatively high interest rates during the postwar period. For the US the trend also shows a relatively high level at the beginning of the sample because of the (credit) risk premiums probably present in the US 19<sup>th</sup> century data. Despite these non-constant levels of the filtered trends, their averages are virtually identical to the long term averages as reported in Table 12.1.
- In general the *long wave component* captures the underlying long term fluctuations in the *periodic component* rather well. As expected from the data analysis, the long waves in the three countries look very similar, especially with respect to the dating of the peaks and troughs. See Table 12.4. As a rough indication one could say that the *amplitude* of the long waves in the short rates is about 2%. Given an ultra long average rate of say 4%, this means that the average short rates can be between 2% and 6% because of these long term fluctuations. For the UK and the US this amplitude is almost exactly the same as for the national product. The long wave in the short term interest rates moves pretty much in parallel with the long wave in the national product. No consistent *leads* or *lags* can be detected. The average peak-to-peak and trough-to-trough *period lengths* are respectively 60 and 54 years.

- As can clearly be seen from Table 12.2, the *volatility* of the *stochastic component* is substantially higher during the postwar period than it has been before that time. The next subsection further explores the relation between this changing volatility and the general level of the interest rates. Just as in the previous section, the volatility ratios indicate that for the postwar period the volatility of business cycle fluctuations in short term interest rates is about 80% of the national product volatility. For both the prewar and interwar period this ratio is only about 30%<sup>52</sup>. Furthermore, a close inspection of the filtered series in Figure 12.2 indicate for all countries and most periods a lag of one or two years of the short rates over the national product. The spectral analysis in the next section will have to confirm this finding. Apart from this lagging behavior, the short rates show very similar type business cycle fluctuations as the national product. The reported *correlations* are fairly constant around 0.3 and thereby confirm the procyclical behavior of short term interest rates, also for the prewar and interwar periods.
- The results in Table 12.3 show that, without exception, the stochastic component describes the entire volatility of the conventional annual changes in the short term interest rates. Apparently the very low frequency fluctuations play only a small role in the process of interest rate *changes* (but a large role in the process of the *level* of interest rates).

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<sup>52</sup> With the production indices, employment rates, consumer prices and industry wages it was possible to compare some of the (filtering) results, for example the volatility ratios, with the results of other researchers. Most of these sources were from the literature on Real Business Cycles (see section 2.2.10). Unfortunately (and surprisingly) this line of research is almost never concerned with financial variables such as interest rates, credit spreads and equity indices that are analyzed in this and subsequent chapters. The only article from this line of research we are aware of in which interest rates are analyzed is Blackburn and Ravn (1992). For the (ex post) *real* long term interest rate for the UK they find a volatility ratio of approximately 0.7 compared to the national product. Furthermore their results show an approximate zero contemporaneous correlation and a *lead* of the real interest rate over the national product.

Table 12.2 Statistics of the filtered periodic, long wave and stochastic components of (nominal) Short Term Interest Rates for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands			United Kingdom			United States					
	Stdev	Ratio	Corr	Stdev	Ratio	Corr	Stdev	Ratio	Corr			
Periodic	1870-1999	1.7%	0.1	0.8	1855-1999	2.7%	0.3	0.4	1870-1999	1.9%	0.2	0.6
Long Wave	1870-1999	0.8%	0.1	0.8	1855-1999	1.5%	0.6	0.6	1870-1999	1.3%	0.9	0.9
Stochastic	1870-1913	0.6%	0.2	-0.2	1855-1913	1.0%	0.4	0.2	1870-1913	1.5%	0.4	0.2
	1919-1939	0.9%	0.5	0.3	1919-1939	1.1%	0.3	0.4	1919-1939	0.8%	0.2	0.3
	1949-1999	1.7%	0.9	0.2	1949-1999	1.6%	0.7	0.3	1950-1999	1.6%	0.7	0.2
	1949-1975	1.5%	0.7	0.2	1949-1975	1.0%	0.6	0.4	1950-1975	1.3%	0.6	0.5
	1976-1999	1.9%	1.1	0.4	1976-1999	2.0%	0.8	0.3	1976-1999	1.9%	0.8	0.0

Table 12.3 Standard deviation of annual change in (nominal) Short Term Interest Rates for various sub-periods (Hist) versus standard deviation of first differences of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States				
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio		
1829-1913	0.8%	0.8%	1.0	1821-1913	1.2%	1.2%	1.0	1832-1913	2.7%	2.7%	1.0
1919-1939	1.0%	1.0%	1.0	1920-1939	1.1%	1.1%	1.0	1919-1939	1.0%	1.0%	1.0
1949-1999	1.8%	1.8%	1.0	1950-1999	1.7%	1.6%	1.0	1950-1999	1.5%	1.5%	1.0
1949-1975	1.9%	1.9%	1.0	1950-1975	1.2%	1.2%	1.0	1950-1975	1.2%	1.1%	1.0
1976-1999	1.6%	1.6%	1.0	1976-1999	2.1%	2.0%	1.0	1976-1999	1.7%	1.8%	1.0

Table 12.4 Peaks and troughs of the long wave component in (nominal) Short Term Interest Rates. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

	1	2	3	4	5	Peak-Peak Trough-Trough Lead			
Netherlands	Peak	1870	54	1924	54	1978	54		
	Trough	1844	52	1896	56	1952	47	1999	
	Lead	13	11	4	-2	-6	-3	3	
United Kingdom	Peak	1859	66	1925	53	1978	60		
	Trough	1832	59	1891	61	1952	47	1999	
	Lead	-3	-8	-15	-14	-10	0	-8	
United States	Peak	1842	77	1919	58	1977	68		
	Trough		1890	58	1948	51	1999	55	
	Lead		0	-1	-2	-3	0	-1	
Average	Peak	1857	66	1923	55	1978	60		
	Trough	1838	54	1892	58	1951	48	1999	54
	Lead	13	1	-4	-6	-6	-1	-1	

### 12.2.1 Volatility versus level

- In the previous sections a possible relation between the level of the interest rates and their volatility (i.e. the level effect) was noted. In this section the filtered trend, periodic and stochastic components are used to further explore this relation. The *sum* of the trend and periodic component (i.e. all fluctuations with a period length longer than fifteen years) is used as an indicator of the *level* of the interest rates. We are then interested in how the *volatility* of the stochastic component (i.e. of all fluctuations with a period length shorter than fifteen years) relates to this level of the interest rates. Such a split of the interest rate series is illustrated in Figure 12.3 for the Dutch short term interest rate.
- All available pairs of the level and the stochastic component are then ordered (per country) based on the level of the interest rate. The two World Wars are excluded because for the stochastic component no data are available from the filtering procedure. Based on this ordering, first a *frequency diagram* is calculated for the level component with 1% intervals. For the NL this is shown in the right hand side picture of Figure 12.4. Especially for the NL and the UK (see Appendix D) this diagram shows a distribution that is *skewed* to the left. In about forty percent of the cases the level of the short rates is around the long term average of between 3 and 4%. Very high levels of the interest rates occur only with small probability.
- In principle then the standard deviation of the stochastic component is calculated for each of the 1% (level) intervals. However in case there are only a few observations in an interval, adjacent intervals are joined to obtain more robust standard deviation estimates. The left hand side picture in Figure 12.4 plots the average level of the short term interest rate versus the standard deviation of the stochastic component for each of the resulting (level) intervals. This picture also shows (OLS) estimates of first and second order polynomials to describe the relation between level and volatility. For each of the three countries these results clearly show a *positive relation* between the level and the volatility of the short term interest rate. Focussing on the estimated linear relations, for the NL the volatility shows an increase of about 0.4% for each 1% increase in the underlying level. For the UK and the US these percentages are respectively 0.2% and 0.3%. Furthermore, especially for the NL, the results suggest that for extremely low levels of interest rates (around 1%) the volatility does not decrease any further or even increases. Note however that these very low levels stem from the turbulent interwar period making it difficult to associate this observation with a true economic phenomenon<sup>53</sup>. For the UK and the NL the volatility is about 1% at the most common level of short term interest rates of around 3 to 4%. For the US this is a little higher at about 1.3%.
- Section 13.2.1, 15.2.1 and 20.4.2 will reveal the presence of a similar level effect for respectively the nominal short term interest rates, default spreads and inflation rates. Because such a level effect is not found for the other variables that all hold in real (volume) terms and hence are all corrected for changes in the price level, this raises the suspicion that the price inflation process is in fact the fundamental cause of the level effects found here.

<sup>53</sup> Note that for the estimation of the linear relations the very low levels of 1 and 2% were excluded.



Figure 12.3 A (nominal) Short Term Interest Rate for the NL together with its “level” represented by the sum of the filtered trend and periodic component and its filtered stochastic component.

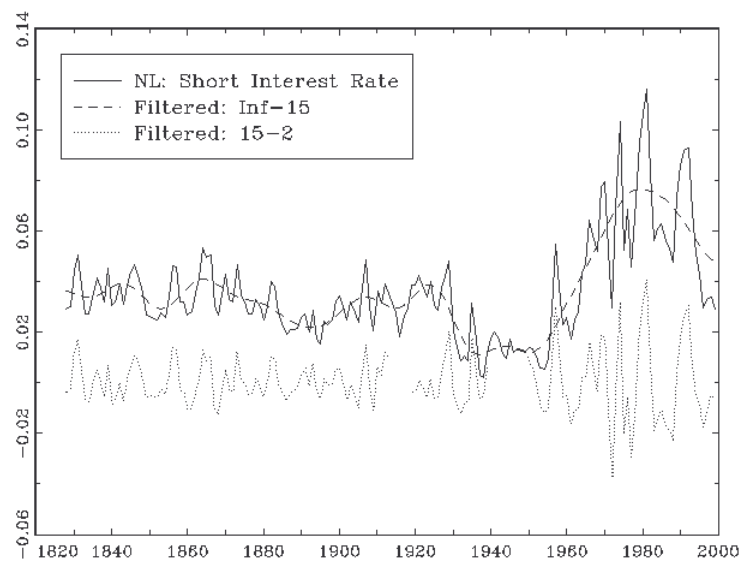
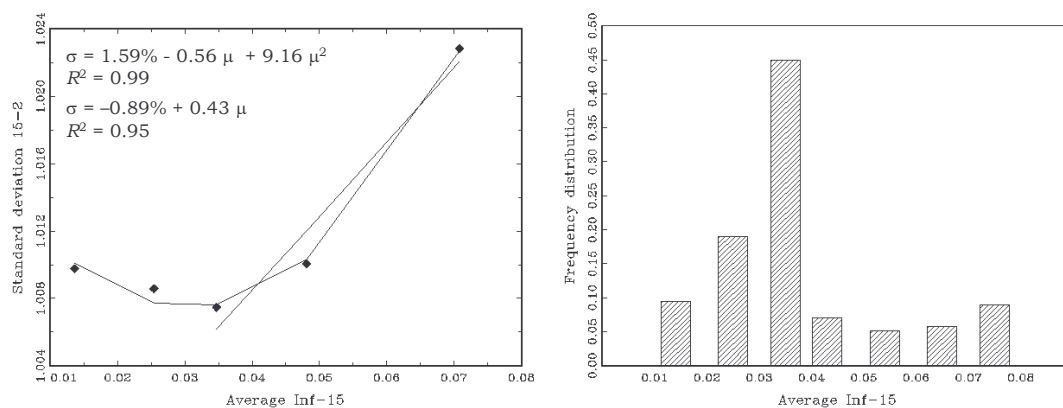


Figure 12.4 For a (nominal) Short Term Interest Rate for the NL, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).

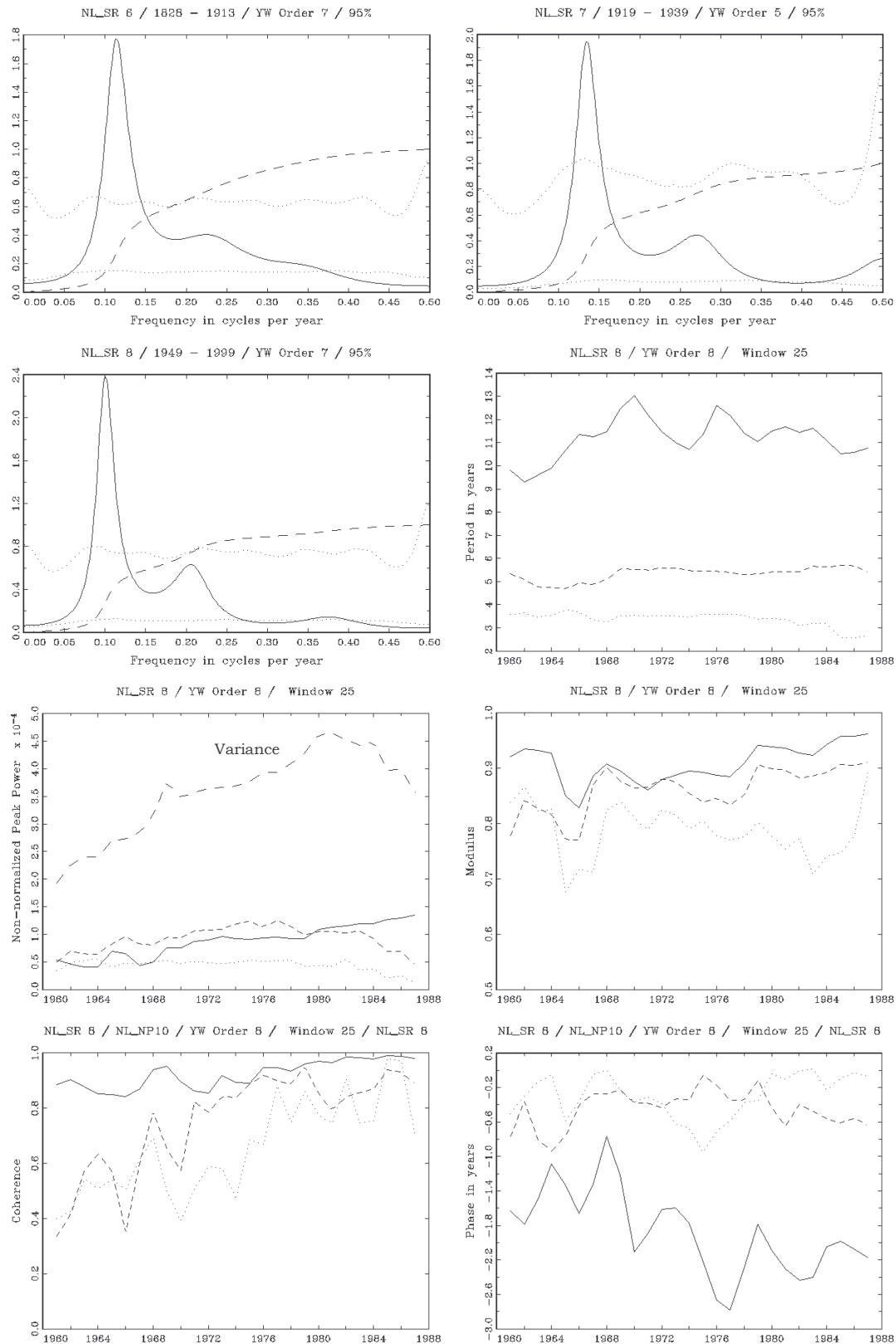


## 12.3 Spectral analysis

### 12.3.1 Static

- Also in short term interest rates Juglar type fluctuations are found for every country and every sub-period. They have the familiar properties of a *period length* of nine to ten years, are *significant* in all cases, have the highest *peak power* of around 30% and a high *modulus*. Note that the period length for the prewar US period is exceptionally long (around thirteen years) which may be a result of the poor quality of the 19<sup>th</sup> century US data as noted before. In general the *coherence* with the national product at these frequencies is high. As already noted in the previous section it is clear that in general the short term interest rates tend to *lag* the national product with about one to two years. Also the Kitchin type fluctuations are present in a lot of the sub-samples, although it less prominent than the Juglar fluctuations. The Kitchin peak is *significant* only for the postwar UK sample. At about 20% it has a lower *peak power* than the Juglar peak while also its *modulus* is in general somewhat lower.
- Just as with most of the variables analyzed in the previous chapters, the postwar period shows very similar spectral properties *across* the three *countries*. Also, again the consistency of the UK results *across* the three *sub-periods* is striking. In general this can both be because of the good quality of the UK data and because of the fact that the UK have been a very developed economy throughout the entire sample analyzed.

Figure 12.5 Static and dynamic estimates of the spectral density of the stochastic component of a (nominal) Short Term Interest Rate for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures here.



### 12.3.2 Dynamic

- The *averages* of the rolling window measures for the postwar period, as given in Table 12.6, show that the static estimates adequately reflect the average shape of the spectral density during that period.
- The *period length* of both the Juglar and Kitchin peak seems to have lengthened throughout the postwar period.
- In each of the three countries the *variance* increased during most of the sample while a decrease set in again at the end of the sample. Note that this is consistent with the positive relation between the level and the volatility of the short rates as found in section 12.2.1. Striking is that in each of the three countries the increasing variance seems to have been caused by both spectral peaks while the decrease at the end of the sample is only visible in the decreasing *peak power* at the Kitchin frequencies.
- Especially in the first half of the sample, the *modulus* of both the Juglar and the Kitchin peak have occasionally been very low indicating not very clear pseudo periodic behavior in those times. Towards the end of the sample the modulus has increased. At the end of the sample the modulus of the Juglar peak is approximately 0.95 in each of the three countries. For the Kitchin peak in the UK and the NL this is 0.9 while in the US it seems to have stayed behind and ends at only 0.8.
- The *coherence* of both peaks increased to levels in general close to one. Only the coherence at the Kitchin frequencies in the NL started of very low (0.4).
- The rolling window *phase* estimates indicate a very constant lag of approximately two years at the Juglar frequencies and a lag of one year at the Kitchin frequencies. Also see the averages in Table 12.6.

Table 12.5 Statistics of static spectral estimates of the filtered stochastic components of (nominal) Short Term Interest Rates for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
	Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase
1828-1913*	8.9#	0.3	0.9	0.6	-4.0	1820-1913**	8.9#	0.3	0.9	0.9	-0.4	1831-1913***	13.3#	0.5	0.8	0.7	1.2
	4.3	0.3	0.7	0.4	0.3		4.3	0.2	0.8	0.3	0.9		3.2	0.2	0.8	0.7	-1.0
	2.8	0.1	0.7	0.7	-1.3		2.9	0.2	0.8	0.3	0.8						
1919-1939	7.5#	0.3	0.9	0.9	1.1	1919-1939	10.4#	0.3	0.9	1.0	-0.8	1919-1939	8.4#	0.4	0.9	0.9	0.1
	3.6	0.2	0.8	1.0	0.9		4.5	0.2	0.8	0.9	-1.0		3.0	0.2	0.8	0.6	-0.2
							2.7	0.1	0.8	0.9	-1.2						
1949-1999	10.1#	0.3	0.9	0.9	-1.9	1949-1999	9.3#	0.3	0.9	1.0	-1.6	1950-1999	8.7#	0.4	0.9	0.7	-2.4
	4.8	0.2	0.8	0.6	-0.6		5.2#	0.3	0.9	0.9	-1.3		4.6	0.2	0.8	0.8	-0.8
	2.6	0.1	0.8	0.5	0.0		2.9	0.1	0.8	0.3	-0.3		2.7	0.1	0.8	0.7	-0.1

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1828-1913 frequencies / period lengths.

\*\* Coherence and phase come from the 1855-1913 estimates of the co-spectrum for the indicated 1820-1913 frequencies / period lengths.

\*\*\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1831-1913 frequencies / period lengths.

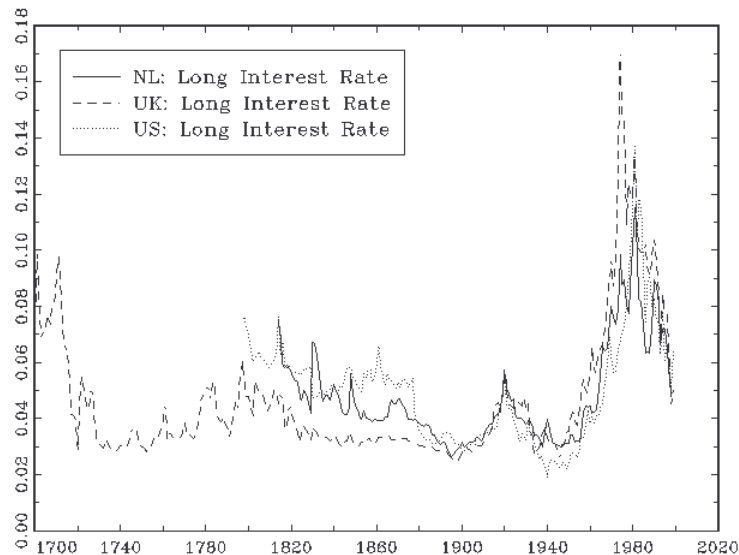
Table 12.6 Average statistics of dynamic spectral estimates of the filtered stochastic components of (nominal) Short Term Interest Rates for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 12.5.

Netherlands				United Kingdom				United States						
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
11.2	0.2	0.9	0.9	-1.9	9.1	0.3	0.9	0.9	-2.1	8.0	0.4	0.8	0.9	-2.1
5.3	0.3	0.9	0.8	-0.5	5.0	0.3	0.9	0.9	-1.0	4.6	0.3	0.8	0.9	-0.9
3.4	0.1	0.8	0.7	-0.3	2.9	0.1	0.8	0.8	-0.2	2.7	0.1	0.9	0.9	-0.1

## 13 Long Term Interest Rates

### 13.1 Data analysis

Figure 13.1 (nominal) Long Term Interest Rates.



- In general the long term interest rates analyzed in this chapter are yields on long term government bonds with a remaining time to maturity of say ten years on average (a wild guess).
- Especially for the 20<sup>th</sup> century there is a great deal of *similarity* in the development of the long term interest rates in the various countries. This is not the case for earlier periods which could be caused by differences in the underlying bond data, for example with respect to the level of credit risk involved, the increasing internationalization of the world economy and the opening up of capital markets.
- From a comparison of Figure 12.1 and 13.1 it is immediately clear that short and long term interest rates follow a very similar pattern, both with respect to the long and the short term fluctuations. As an illustration of the latter, the correlation between the changes of the short and long term interest rates for the entire 20<sup>th</sup> century are respectively 0.7, 0.5 and 0.4 for the NL, UK and US. These similarities are a consequence of the fact that interest rate term structures are in general rather smooth curves which cannot display very different interest rates for adjacent terms to maturity<sup>54</sup>. Long term interest rates behave very similar to the short term interest rates which is why the discussion of the results here will focus on the *main differences* with the short rate behavior as analyzed in the previous chapter.

<sup>54</sup> If a term structure would show very large differences across the various terms, this would offer arbitrage possibilities by some smart combination of borrowing and lending at different terms. In reality such possibilities can of course not exist for a long time.

- From the statistics in Table 13.1 we see that in each of the three countries the *long term average* level of the long term interest rate is approximately 5% which is about 1% higher than the long term average short term interest rates. This *term spread* in general is a *risk premium* for the extra risk incurred by lending money for a long period of time. Sources of this risk are for example (averse) changes in the interest rates during the lending period, liquidity risk and inflation risk. In the next chapter a separate analysis for the term spread is performed.
- Although not entirely constant over countries and sub-periods, the *volatility* of the changes in the long term interest rates is approximately half the size as for the short term interest rates. That is, long term interest rates behave about twice as stable as short term interest rates which, apart from the exact proportion, is a well known property of interest rates.
- The *correlation* of long term interest rate changes with the growth of national product is approximately 0.1 to 0.2 lower than it was for the short term interest rates. Short term interest rates therefore seem to adjust more quickly to changing economic circumstances than long term interest rates. Different results from the multivariate spectral analysis (coherence and phase) in the next section are therefore to be expected.



Table 13.1 Statistics of (nominal) Long Term Interest Rates for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

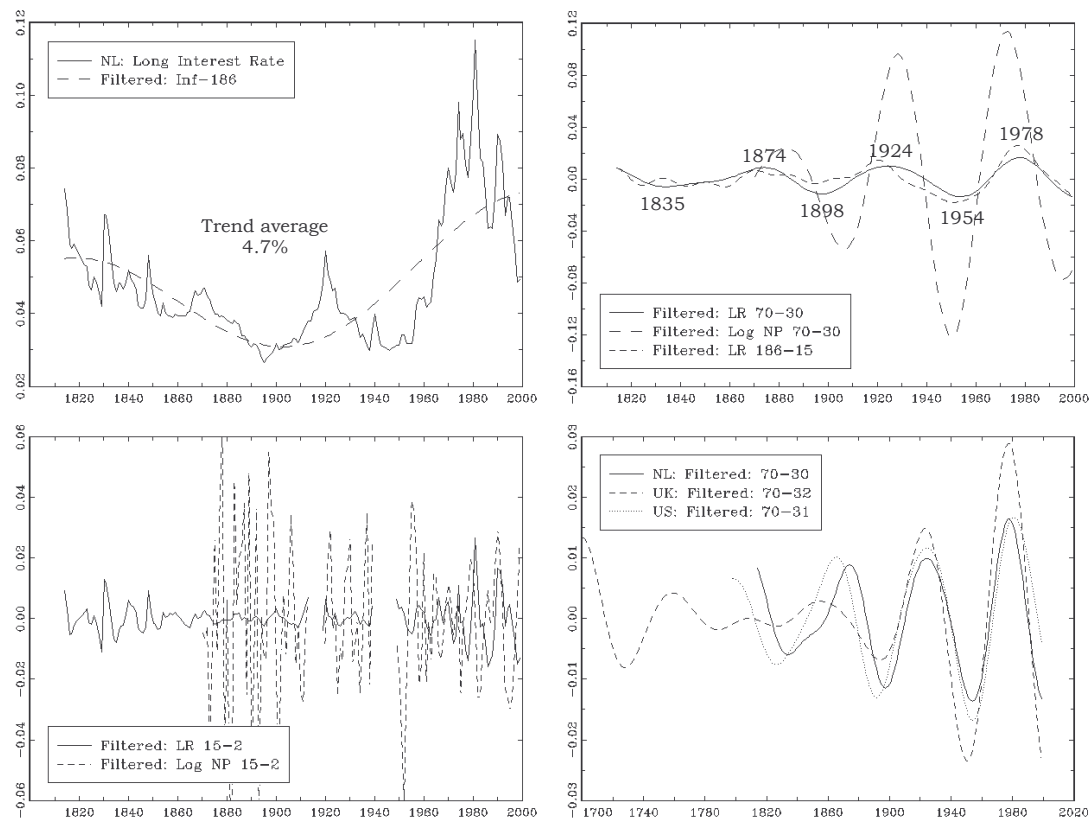
Netherlands						United Kingdom						United States					
	Avg	Stdev1**	Stdev2**	Ratio	Corr		Avg	Stdev1	Stdev2	Ratio	Corr		Avg	Stdev1	Stdev2	Ratio	Corr
1814-1999	4.8%	1.7%	0.5%	-	-	1700-1999	4.6%	2.4%	0.7%	-	-	1798-1999	5.0%	2.0%	0.6%	-	-
1870-1999	4.7%	2.0%	0.5%	0.1	0.1	1855-1999	5.0%	3.0%	0.8%	0.2	0.0	1870-1999	4.6%	2.3%	0.7%	0.1	0.1
1870-1913	3.5%	0.5%	0.1%	0.0	0.1	1855-1913	3.1%	0.2%	0.1%	0.0	0.2	1870-1913	3.7%	0.8%	0.3%	0.1	-0.1
1919-1939	4.1%	0.7%	0.3%	0.1	0.2	1919-1939	4.1%	0.8%	0.5%	0.1	-0.3	1919-1939	3.6%	0.8%	0.4%	0.1	-0.1
1950-1999	6.5%	2.1%	0.8%	0.3	0.3	1950-1999	8.3%	3.2%	1.3%	0.7	0.2	1950-1999	6.4%	2.7%	1.0%	0.4	0.2
1950-1975	5.4%	2.0%	0.6%	0.2	0.2	1950-1975	7.1%	3.4%	1.2%	0.6	0.1	1950-1975	4.5%	1.4%	0.4%	0.2	0.1
1976-1999	7.7%	1.6%	0.9%	0.6	0.2	1976-1999	9.6%	2.4%	1.3%	0.8	0.1	1976-1999	8.5%	2.1%	1.4%	0.7	0.3

\* Standard deviation of *level* of interest rates.

\*\* Standard deviation of annual *changes* of interest rates which is also used in the (standard deviation) ratio and correlation numbers.

## 13.2 Filtering results

Figure 13.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Long Term Interest Rate for the NL and long wave component compared to UK and US data (bottom right).



- As usual the average of the *trend component* nicely match with the long term averages as reported in Table 13.1. For both the NL and the UK the trend shows a u-shaped pattern while for the US it is an almost constant average with the u-shaped pattern turning up in the periodic component.
- The dating of the peaks and troughs of the highly synchronized *long wave component* in the long term interest rates are very similar to those in the national product and in the short term interest rates. Though theoretically correct, it is hard to say something about a possible long wave in the spread between the long and short term interest rates based only on the differences in amplitudes of the long wave in the short and long term interest rates. The term spread is analyzed separately in the next chapter.
- As was already clear from the data analysis, the *volatility* of the *stochastic component* of the long term interest rates is about twice as small as that of the short term interest rates.
- The *correlation* of the business cycle fluctuations in the long term interest rates with those in the national product is in general much lower than for the short term interest rates. The correlations are about 0.3 lower rendering long term interest rates sometimes countercyclical (i.e. negatively correlated with national product).

- Just as was the case for the short term interest rates, the stochastic component of the long term interest rates clearly shows a *lagging* behavior compared to the national product. Because of the lower correlations it is very plausible that long term interest rates are more out of phase than short term interest rates. This would mean that long term interest rates are lagging further behind than short term interest rates. The spectral analysis in the next section should give us more information on this issue.
- Just as for the short term interest rates the stochastic component describes virtually the entire *volatility* of the long term interest rate changes. For this, see Table 13.3.

Table 13.2 Statistics of the filtered periodic, long wave and stochastic components of (nominal) Long Term Interest Rates for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands				United Kingdom				United States			
	Filt	Hist	Ratio	Corr	Filt	Hist	Ratio	Corr	Filt	Hist	Ratio	Corr
Periodic	1870-1999	1.1%	0.1	0.6	1855-1999	2.1%	0.3	0.5	1870-1999	2.2%	0.2	0.3
Long Wave	1870-1999	0.9%	0.1	0.8	1855-1999	1.3%	0.6	0.4	1870-1999	1.0%	0.7	0.7
Stochastic	1870-1913	0.2%	0.1	0.0	1855-1913	0.1%	0.0	0.4	1870-1913	0.2%	0.1	-0.1
	1919-1939	0.2%	0.1	0.2	1919-1939	0.4%	0.1	0.4	1919-1939	0.3%	0.1	0.1
	1949-1999	0.8%	0.4	-0.1	1949-1999	1.0%	0.5	0.1	1950-1999	0.8%	0.3	-0.1
	1949-1975	0.5%	0.2	-0.2	1949-1975	1.0%	0.6	0.3	1950-1975	0.5%	0.2	-0.2
	1976-1999	1.1%	0.6	0.0	1976-1999	1.0%	0.4	-0.1	1976-1999	1.0%	0.4	0.0

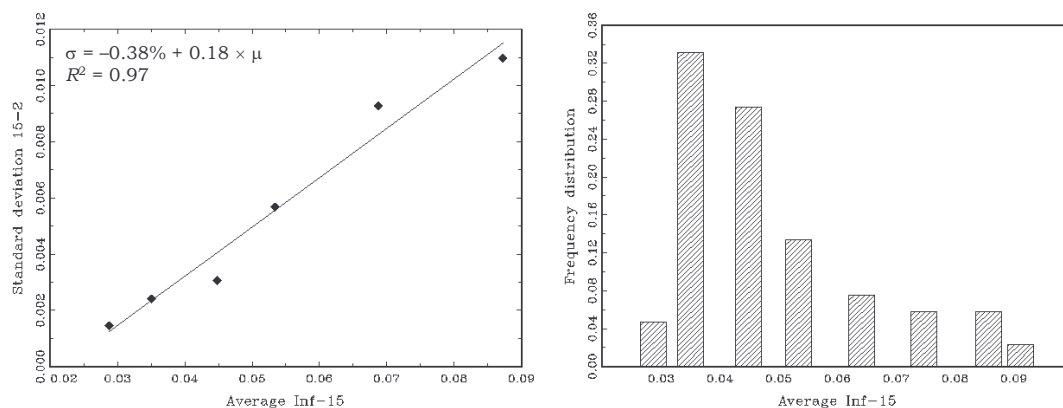
Table 13.3 Standard deviation of annual change in (nominal) Long Term Interest Rates for various sub-periods (Hist) versus standard deviation of first differences of filtered stochastic components (Filt).

	Netherlands			United Kingdom			United States		
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio
1814-1913	0.4%	0.4%	1.0	1701-1913	0.4%	1.0	1798-1913	0.3%	0.3%
1919-1939	0.3%	0.3%	1.0	1920-1939	0.4%	1.0	1919-1939	0.4%	0.4%
1949-1999	0.8%	0.8%	1.0	1950-1999	1.3%	0.9	1950-1999	1.0%	1.0%
1949-1975	0.6%	0.6%	1.0	1950-1975	1.2%	0.9	1950-1975	0.5%	0.4%
1976-1999	0.9%	0.9%	1.0	1976-1999	1.4%	1.0	1976-1999	1.3%	1.4%



## 13.2.1 Volatility versus level

Figure 13.3 For a (nominal) Long Term Interest Rate for the NL, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).



- Also for the long term interest rates there seems to be a relation between the general level of the interest rates and their volatility. Based on a similar analysis as for the short rates we find two things.
- The first is that also for the long term interest rates the *frequency distribution* is skewed to the left. Normal levels of the long term interest rate occurred some 50 or 60% of the times while very high interest rates occurred with much smaller probability.
- The second finding is that also for the long term interest rates there is a *positive relation* between the volatility and the underlying level of the interest rates. The relation is clearly linear in each of the three countries, indicating that the volatility increases by approximately 0.1 to 0.2% per 1% increase in the interest rate level. For the short term interest this was approximately 0.3% which indicates that the volatility of long term interest rates is less responsive to this level effect. At the most common level of long term interest rates of say 5%, the volatility is approximately 0.5% in each of the three countries which is half the volatility of the short term interest rates at their most common level (3 to 4%). This volatility ratio was also found in the previous sections.

### 13.2.2 Real Interest Rates

- As already indicated in Chapter 6 it would be the most ideal to analyze *real* instead of *nominal* interest rates, just as for the other relevant variables (national product, industrial production, industry wages and equities). Because long term data on real interest rates are not available, a full scale analysis (including business cycle fluctuations) of real interest rates is impossible. We resorted to the analysis of nominal interest rates instead. However in this sub-section some information on the *long term behavior* of real interest rates<sup>55</sup> is extracted from the nominal series.
- In the absence of real interest rate data the only alternative is to try and construct real interest rates from other variables. Following the famous work of Fisher (1930) nominal interest rates are often thought of as the sum of the real interest rate, the expected inflation and an inflation risk premium<sup>56</sup>. The real interest rate represents the true time value of money. Besides this compensation, lenders also want to be compensated for the expected loss of purchasing power in the form of the expected inflation rate during the lending period. Finally, they require a compensation for the risk that this expectation may not come true. This *Fisher hypothesis* suggests two possibilities for constructing real interest rates from the available data.
- If we abstract from the inflation risk premium, possibility number one is to first in some way estimate the expected inflation rate from price index data and then to subtract the resulting expected inflation rates from the nominal long term interest rates. This approach requires a rather subjective choice of a model for the expected inflation. Developing such a model and testing it on actual (limited) real interest rate data would also require too much work in the current context.
- The second possibility is to settle for the *ex post* real interest rates instead of trying to estimate the actual *ex ante* real interest rates. Ex post real interest rates can easily be constructed by subtracting the realized (instead of the expected) inflation rates from the nominal interest rates<sup>57</sup>. Such ex post series are not well suited for the analysis of the short term (business cycle) properties of real interest rates because the (large) unexpected inflation shocks dominate the rather stable evolution of the long term interest rates. However, if one only considers the *long term* behavior of these ex post real interest rates, a plausible assumption is that these unexpected inflation shocks level out. In fact we then assume that the unexpected inflation shocks are 0% on average. That is, inflation is not

<sup>55</sup> Filtering and spectral analysis results of a true (ex ante) UK real interest rate series for the 1981-2001 period not shown here, indicate that the short term fluctuations in real interest rates are especially dominated by Kitchin type fluctuations. Juglar type fluctuations are also visible but less dominant, probably because of the short sample of the series.

<sup>56</sup> Whether this relation is defined in an *additive* or *multiplicative* fashion is not very relevant here given the huge uncertainties involved in the analysis anyway. We therefore chose to work with the simple additive definition. That is, *nominal interest rate* = *real interest rate* + *expected inflation* + *risk premium*.

<sup>57</sup> Here the approach of simply subtracting the realized inflation rate of year  $t$  from the, say ten year, interest rate of year  $t$  was used to obtain the ex post real interest rate for year  $t$ . A theoretically more correct alternative would be to in some way subtract the realized inflation rates of the ten years following year  $t$ . After all these determine the true realized real interest rate earned by lending money for ten years at the nominal interest rate of year  $t$ . It is however not to be expected that this slightly more complicated approach would lead to (long wave) results very different from those described in this section while at the same time it would shorten the sample. After all, following this approach it is not possible to calculate the ex post real interest rate for the last ten years of the sample.



structurally under or overestimated by economic agents. In that case for example the long term average nominal interest rate minus the long term average inflation rate does give some useful information on the behavior of real interest rates. Table 13.5 therefore reports the averages of the ex post long term real interest rate for the various sub-periods and countries. These results lead to two important observations.

- The first is that the (arithmetic) long term real interest rate is very consistent around 3% in each of the three countries. This finding is similar to Siegel (1992) who reports real long term interest rates of 3.2% and 3.7% for respectively the UK and the US for the period 1800-1990. Note that these levels of real interest rates are comparable to the long term real growth of national product as reported in Table 7.1 (1871-1999 NL 2.7%, UK 1.9% and US 3.3%). One hypothesis for this similarity, coming from long term growth theory, is that the real interest rate is an equilibrium lender compensation for the *opportunity costs* incurred by not participating in the economic growth because of having temporally transferred wealth onto the borrower. If the real interest rate is structurally higher than economic growth, more agents would want to lend money which would decrease the real interest rate. If, the other way around, real interest would be structurally lower than economic growth, few agents would be willing to transfer wealth into the future and the real interest rate would have to increase to persuade more of them to do so<sup>58</sup>.
- The second important observation from Table 13.5 is that real interest rates can average around both significantly lower and higher levels than the ultra long average of say 3% for several decades in a row. For example during the first half of the postwar period, real interest rates where around 2% lower than the long term average while during the second half they were again substantially higher.

Table 13.5 Average ex post real Long Term Interest Rate for various sub-periods calculated as the nominal long term interest rate minus the realized inflation rates.

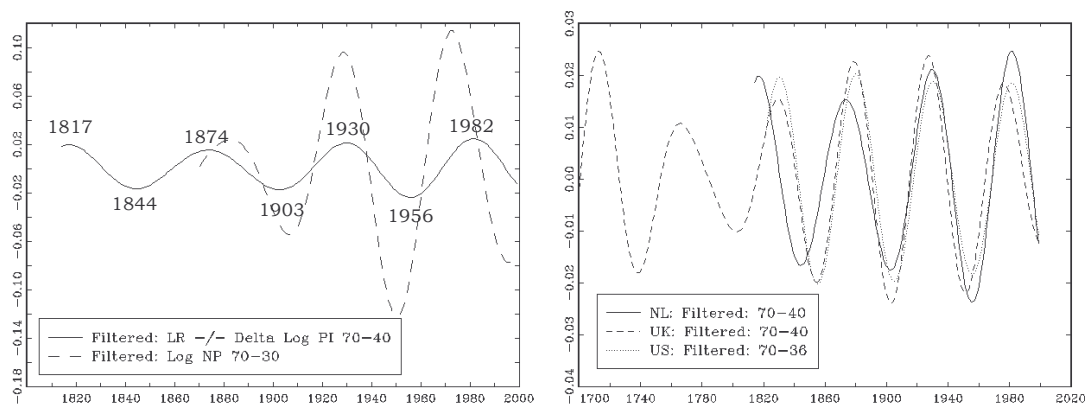
<i>Netherlands</i>		<i>United Kingdom</i>		<i>United States</i>	
	<i>Avg</i>		<i>Avg</i>		<i>Avg</i>
1814-1999	3.3%	1700-1999	3.2%	1821-1999	3.0%
1870-1999	2.3%	1870-1999	2.3%	1870-1999	2.4%
1870-1913	3.0%	1700-1913	3.8%	1821-1913	4.1%
1919-1939	5.2%	1920-1939	9.8%	1919-1939	4.3%
1950-1999	2.6%	1950-1999	2.4%	1950-1999	2.5%
1950-1975	0.8%	1950-1975	1.6%	1950-1975	1.2%
1976-1999	4.6%	1976-1999	3.3%	1976-1999	3.9%

- To further explore the possible long term fluctuations in real interest rates, a simple filtering exercise is performed as well. This assumes that the period lengths of the *long wave* frequency range (between approximately thirty and

<sup>58</sup> Strictly speaking such an argumentation concerns the *short term* real interest rates rather than the long term real interest rates analyzed here. From earlier sections we already know that in the long run short term interest rates are approximately 1% lower than long term interest rates. Combining this with the first line of Table 13.5 this yields that the long term average real short term interest rates are 2.3%, 2.2% and 2.0% for respectively the NL, UK and US. Also these are comparable to the long term growth rates of national product as reported in Table 7.1. Considering the results for each of the three countries it is hard to say whether it is the real long term or short term interest rate that corresponds most closely to the long term economic growth rate.

seventy years ) as used in the other results are long enough for the argument of unexpected inflation shocks that level out to hold. Theoretically speaking the long wave in the real interest rates can be obtained without extra filtering by simply subtracting the first order differences of the long wave in the (logarithmic) price index series from the long wave in the long term interest rate as shown in Figure 13.2. However, in order to minimize the effect of possible filter errors, we chose to directly filter the ex post real interest rate series for the long wave type fluctuations<sup>59</sup>. Figure 13.4 shows the resulting long wave component for the real long term interest rate for the NL and compares it with that the other two countries. The long waves in the three countries are very similar and their *amplitude* of around 2% is consistent with the different levels observed from Table 13.5. For the UK and the US this amplitude is approximately the same as for the long wave in the national product. For the NL it is relatively much smaller. The average peak-to-peak and trough-to-trough *period lengths* are respectively 51 and 49 years. For the UK and the US the long wave in the real interest rate *lags* the national product by approximately 10 to 15 years. For the NL this is only so for the postwar period. The fact that for the early cycles in the NL the real interest rate moves practically in phase with national product is probably caused by the lagging behavior of the long wave in the Dutch national product. Also see section 7.2. To be complete, Table 13.6 gives the usual statistics for both these long wave components and also the *periodic components* in the real interest rates.

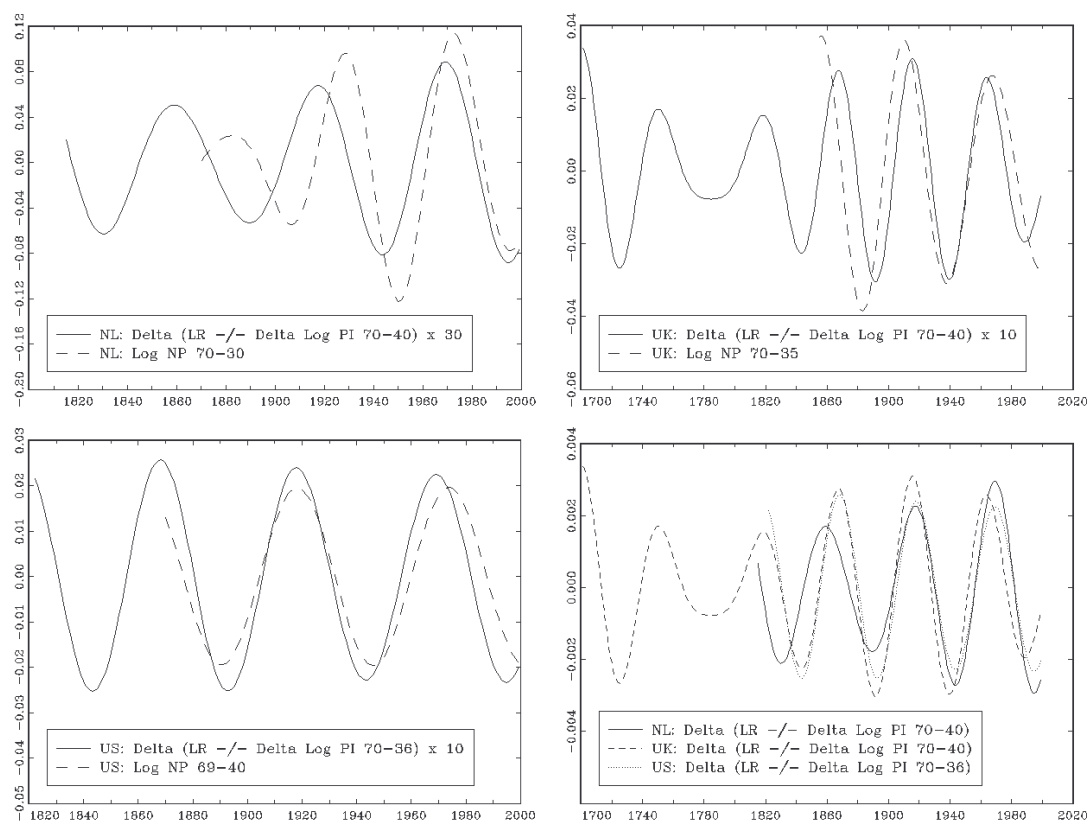
Figure 13.4 Long wave component in (ex post) real Long Term Interest Rate versus long wave in the National Product index for the NL (left) and a comparison with the long wave component in (ex post) real long term interest rates for the UK and the US (right).



<sup>59</sup> Comparing the long waves in real interest rates resulting from the indirect and direct filtering approach did not yield very substantial differences.

- Because of the apparent lag of approximately one decade of the long wave in the real interest rates compared to the national product it seems that the *changes* rather than the level of the real interest rates move in parallel with the national product. To further investigate this possibility, the long wave in real interest rate changes was calculated as the first order differences of the filtered long wave components in the levels of the real interest rates as shown in Figure 13.4. The results are shown in Figure 13.5. From this we see that the long wave in real interest *changes* indeed moves practically in parallel with the *level* of the national product, especially if we remember the lagging behavior of the Dutch national product for the earlier cycles. So, when the national product is at its (relatively) highest level, the real interest rate shows the fastest increase and vice versa. Note that this is exactly the same as was found for the long wave in consumer prices and inflation rates in section 10.2 and that nominal interest showed no significant lead or lag over the national product as shown in section 13.2.

Figure 13.5 Long wave component in *changes* of (ex post) real Long Term Interest Rate for the indicated countries calculated as the first order differences of Long wave component in *levels* of (ex post) real Long Term Interest Rate as shown in Figure 13.4 compared with the long wave component in the relevant National Product index.



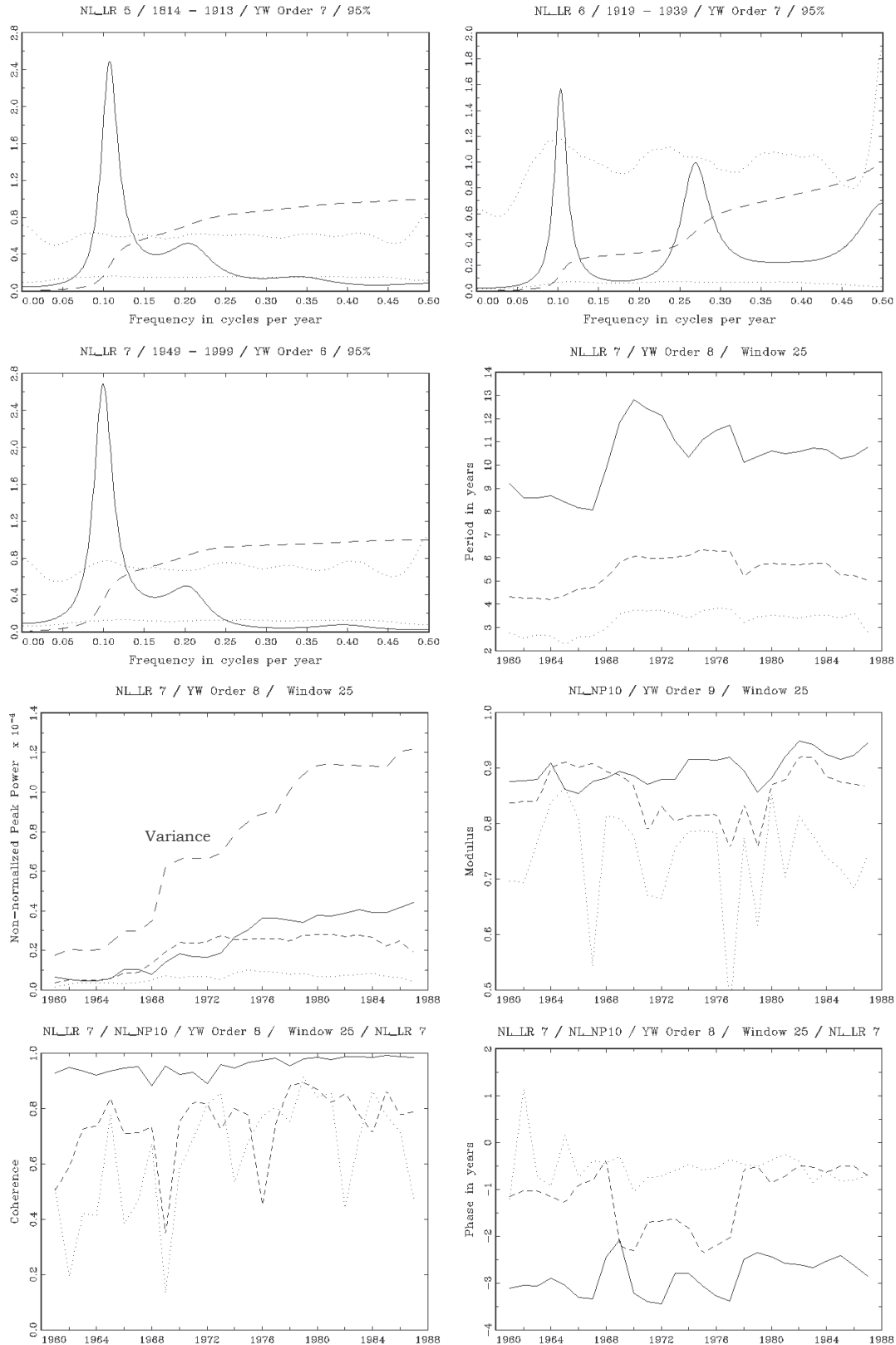


## 13.3 Spectral analysis

### 13.3.1 Static

- The estimated static spectral densities of the stochastic components of the long (nominal) term interest rates are in general very similar to those of the short term interest rates in the previous chapter. For both the Juglar and the Kitchin peak, period length, peak power, modulus and coherence are comparable with the spectral properties of the short rates.
- Apart from the lower volatility as noted before, the only major difference is the *phase* at the relevant frequencies. It seems that at the business cycle frequencies the very consistent *lag* of the long term interest rates over the national product is in general a half to a whole year longer than for the short term interest rates. This means that the short term interest rates *lead* the long term interest rates by approximately the same amount. From the perspective of the static estimates the postwar UK period seems to be an exception in the sense that the long and short term interest rates seem almost in phase. However, with the dynamic analysis in the next sub-section we will see that nowadays the short term interest rates lead the long term interest rates by approximately one year in the UK as well. In general the lead of the short rates over the long rates reflects a greater flexibility of short term interest rates in responding to changing economic circumstances. One of the reasons for this may be the fact that besides the laws of supply and demand, short term interest rates are driven by monetary policies. Finally note that, as already suggested in section 13.2, long term interest rates are apparently more out of phase with national product than short term interest rates which explains the lower national product correlations in Table 13.2 for the long rates compared to Table 12.2 for the short rates.

Figure 13.5 Static and dynamic estimates of the spectral density of the stochastic component of a (nominal) Long Term Interest Rate for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures here.



## 13.3.2 Dynamic

- The *averages* of the rolling window spectral measures for the postwar period from Table 13.9 are very similar to the static estimates for the same period from Table 13.8. Based on the averages for the short term interest rates in Table 12.5, at the Juglar frequencies the short term interest rate in the NL and the US lead the long term interest rates by about one year. For the UK, on average, they seem to move in phase.
- Apart from some incidental shocks, no clear pattern of a lengthening or shortening of the *period lengths* can be detected.
- The *variance* and peak powers have increased during most of the sample with a stabilization or decrease at the very end of the sample. Just as for the short term interest rates, this is probably due to the level effect as discussed in section 13.2.1.
- In each of the three countries the *modulus* of the Kitchin peak has been relatively low in the middle of the sample. Otherwise no clear changes in the amount of periodicity can be detected for the long term interest rates.
- For both the NL and the US the *coherence* at the Juglar frequencies has increased to levels close to one while for the Kitchin peak it has consequently been much lower around 0.7. Also see Table 13.9. For the UK, the other way around, the Kitchin coherence has been around 0.9 most of the times while the Juglar coherence has been lower and also markedly decreased in the second half of the sample towards very low levels of around 0.4. In general these deviating findings for the UK may be caused by the relatively large fluctuations in the stochastic component of the long term interest rates during the 1970's which may disturb the spectral estimation. For this see Figure D.14 in Appendix D.
- From the rolling window *phase* of the Juglar peak we see that during the second half of the postwar period this has constantly been around a lag of three years. The fact that this also holds for the UK shows that nowadays there is apparently no difference in the lead / lag relations of long term interest rates between the UK and the other countries as the results from Table 13.8 and 13.9 suggested. In each of the countries at the Juglar frequencies the short term interest rates therefore lead the long term interest rates by approximately one year. At the Kitchin frequencies the NL and UK results show a constant lag of about one year over the national product which means that long and short term interest rates move in parallel at these frequencies. The US seems an exception here because the rolling window phase at the "Kitchin" peak is constantly around zero. This is probably because with a period length of around three years (see Table 13.8 and 13.9) true Kitchin type fluctuations are not present in postwar US long term interest rates.



Table 13.8 Statistics of static spectral estimates of the filtered stochastic components of (nominal) Long Term Interest Rates for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands						United Kingdom						United States					
Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase	
1814-1913*	9.4#	0.3	0.9	0.5	-3.9	1700-1913**	11.8#	0.3	1.0	0.9	-1.4	1798-1913***	9.9#	0.2	0.9	0.9	-3.4
	4.8	0.2	0.8	0.5	-0.3		5.2#	0.2	0.9	0.5	-0.5		5.3	0.2	0.9	0.5	-1.6
	2.9	0.1	0.7	0.3	1.3		3.3	0.1	0.8	0.4	-0.1		3.1	0.1	0.8	0.4	-0.5
1919-1939	9.7#	0.1	0.9	1.0	-1.6	1919-1939	10.4#	0.4	1.0	1.0	-1.3	1919-1939	12.4#	0.3	0.9	0.8	0.0
	3.7	0.2	0.9	1.0	1.0		5.2	0.1	0.8	0.8	-1.4		4.8	0.1	0.7	0.6	0.0
	2.7	0.4	0.6	0.6	0.6		2.7	0.2	0.8	0.9	-0.9		2.8	0.1	0.8	0.8	0.3
1949-1999	10.1#	0.4	0.9	0.9	-2.7	1949-1999	7.7#	0.3	0.9	0.7	-1.8	1950-1999	9.3#	0.3	0.9	1.0	-3.0
	4.8	0.2	0.8	0.7	-0.6		5.0#	0.3	0.9	0.8	-1.1		3.3	0.2	0.8	0.5	0.2
	2.5	0.1	0.8	0.4	-0.2		2.5	0.2	0.8	0.7	-0.3						

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1814-1913 frequencies / period lengths.

\*\* Coherence and phase come from the 1855-1913 estimates of the co-spectrum for the indicated 1700-1913 frequencies / period lengths.

\*\*\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1798-1913 frequencies / period lengths.

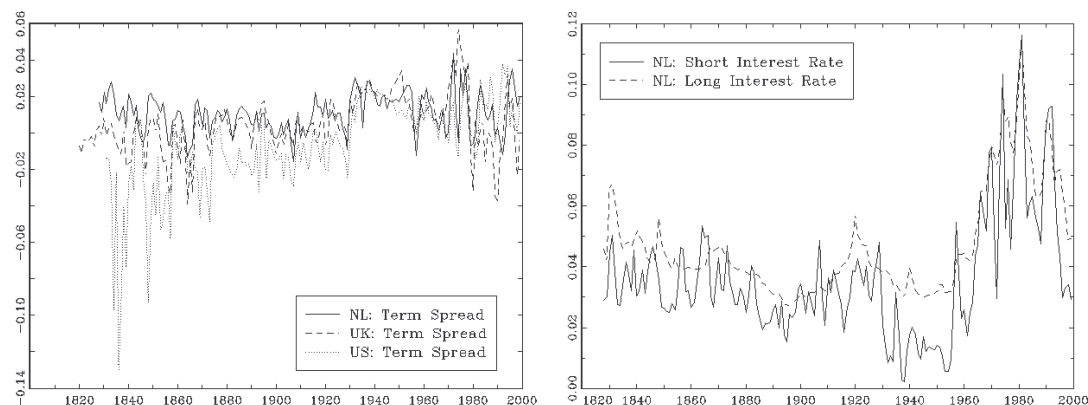
Table 13.9 Average statistics of dynamic spectral estimates of the filtered stochastic components of (nominal) Long Term Interest Rates for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 13.9.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
10.4	0.3	0.9	1.0	-2.9	8.1	0.3	0.9	0.7	-2.2	9.7	0.4	0.9	1.0	-3.1
5.4	0.3	0.9	0.7	-1.2	4.9	0.3	0.8	0.9	-1.0	3.6	0.2	0.8	0.7	0.1
3.3	0.1	0.7	0.6	-0.5	2.5	0.2	0.7	0.7	-0.3					

## 14 Term Spread

### 14.1 Data analysis

Figure 14.1 Term Spreads calculated as the difference between the Long Term Interest Rates and the Short Term Interest Rates.



- The difference between the long and short term interest rate is called the *term spread* which in general is a compensation (risk premium) for all kinds of risk incurred by a lender when lending money for a long instead of a short period of time. Possible sources of this risk are (averse) changes in the interest rates during the lending period, liquidity risk and inflation risk. In general agents require and extra compensation to be persuaded to lend money for a long period of time because they prefer liquid investments (liquidity preference theory). Because the previous two chapters already analyzed the properties of both short and long term interest rates, a separate analysis of the difference between long and short term interest rates seems superfluous. In principle, all the interesting properties of the term spread can be obtained by combining the relevant conventional statistics and spectral properties of the long and short term interest rates as described in the previous chapters. Nevertheless a separate analysis of the term spread was performed. The most important reason for this is that the term spread is a very important and often studied variable within macroeconomics. Therefore the objective is to obtain the properties of the term spread (just as for any other variable) with as great a precision as possible. Furthermore, a direct analysis of the term spread is more easily performed, given the readily available methodological framework. Were possible, in the subsequent analysis it will be checked how the results for the term spread relate to the results for the short and long term interest rates separately.
- Figure 14.1 shows both the historical evolution of the term spread in the three countries and the constituting long and short term interest rates for the NL. From the latter we can already learn several things about the dynamics of the term spread. First, on average short term interest rates lie below the long term interest rates, causing a positive term spread (i.e. a normal term structure) most of the times. Second, the long term fluctuations in the short term interest rates seem

more accentuated than in the long term interest rates, causing also (inverted) long term fluctuations in the term spread. Finally, because of their higher volatility, it are the short term interest rates that determine most of the short term fluctuations in the term spread. When interest rates go down, they do so more sharply for the short term interest rates, causing an increase in the term spread. Similarly the term spread decreases when interest rates go up. On occasions (less frequent than a normal upward sloping term structure) this even causes a negative term spread (i.e. an inverted term structure).

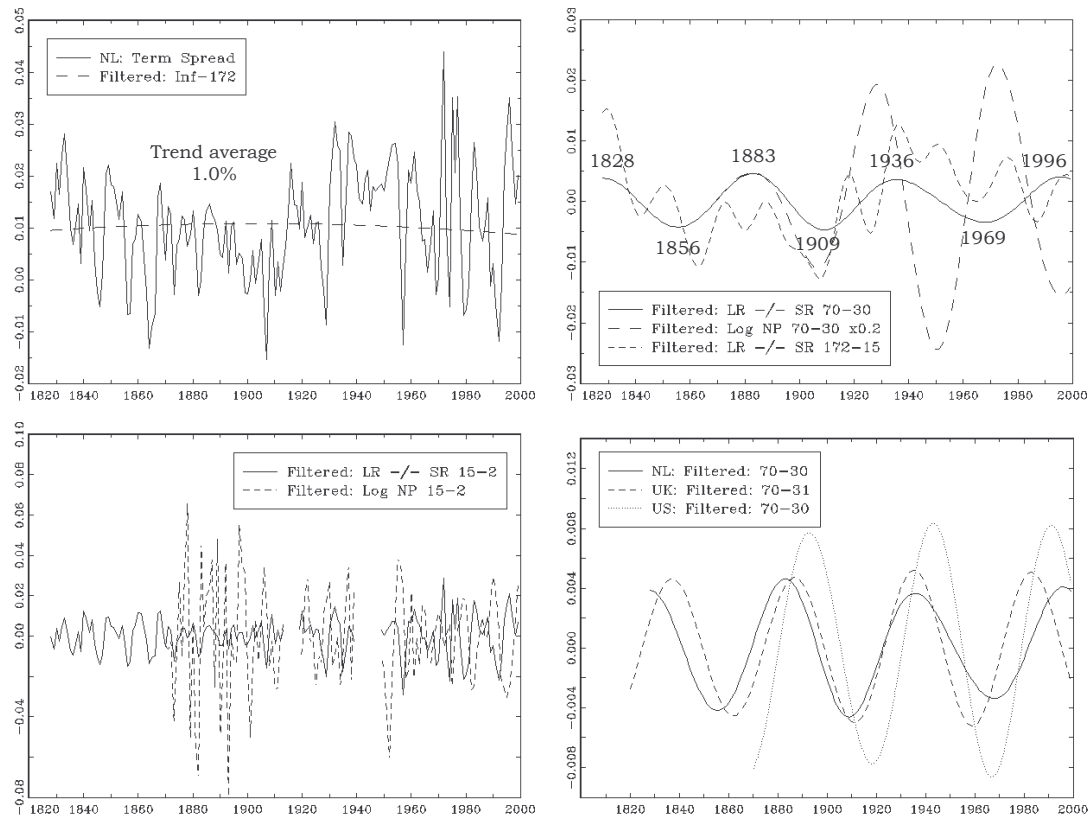
- As already indicated in the previous chapter, the *long term average* value of the term spread is around 1.0%. The only exception is the 19<sup>th</sup> century US data for which very large negative term premiums are reported. From section 12.1 we already know that these are probably caused by the significant (credit) risk premiums present in the 19<sup>th</sup> century short rates.
- Table 14.1 also suggests that the term premium can be approximately 0.5% lower or higher during sustained periods of time. Note that, given the presence of *long term fluctuations* in the constituting long and short term interest rates, long term fluctuations in the term spread are to be expected. The only prerequisite is that the long term fluctuations in long and short term interest rates are of different amplitude. We will return to this in the next section.
- Because the *shorter term fluctuations* in long and short term interest rates have different volatility and are not exactly in phase, it is logical that also the term spread shows significant short term fluctuations. That is, the shape of the term structure is not constant through time but changes with changing economic conditions. At 1.5% the *volatility* of the term spread is approximately between the volatility of (the changes of) the long and short term interest rates. The *correlation* of the term spread with the growth of the national product is in general close to zero or even negative. During the second half of the postwar period at approximately 0.4 the correlations are exceptionally high. A possible cause for this could be the increased predictive power of the term spread for economic growth due to changes in the monetary policy of the FED in the US as found by for example Hardouvelis (1988). For this also see section 10.1. More on this will be said in the next section.

Table 14.1 Statistics of a Term Spread for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

<i>Netherlands</i>				<i>United Kingdom</i>				<i>United States</i>			
<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>	<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>	<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>
1828-1999	1.1%	1.1%	-	1820-1999	0.5%	1.6%	-	1831-1999	-0.7%	2.5%	-
1870-1999	1.1%	1.1%	0.1	1855-1999	0.7%	1.6%	0.5	1870-1999	0.1%	1.7%	0.3
1870-1913	0.5%	0.7%	0.1	1855-1913	0.0%	1.2%	0.4	1870-1913	-1.5%	1.1%	0.2
1919-1939	1.4%	1.1%	0.2	1919-1939	1.3%	1.1%	0.2	1919-1939	0.3%	1.6%	0.2
1950-1999	1.3%	1.3%	0.1	1950-1999	1.0%	2.0%	1.0	1950-1999	1.0%	1.2%	0.5
1950-1975	1.4%	1.3%	0.5	1950-1975	2.0%	1.4%	0.7	1950-1975	0.7%	0.8%	0.3
1976-1999	1.2%	1.3%	0.9	1976-1999	0.1%	2.1%	1.2	1976-1999	1.4%	1.4%	0.7
			0.3				0.3				0.4

## 14.2 Filtering results

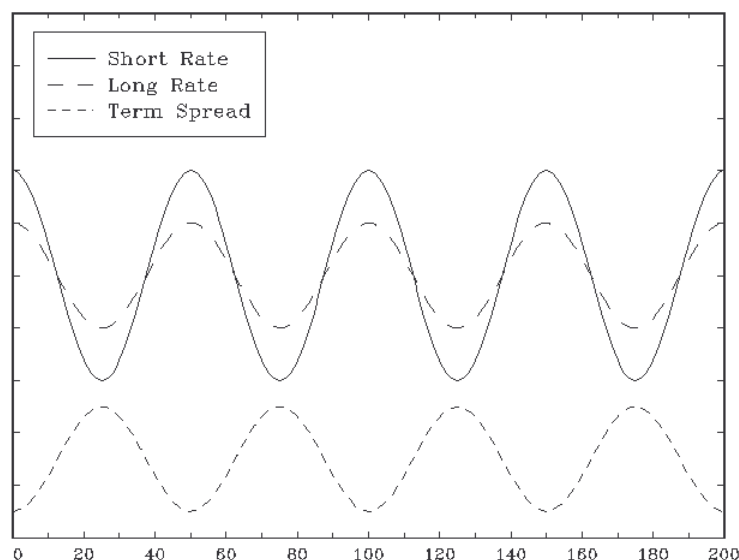
Figure 14.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Term Spread for the NL and long wave component compared to UK and US data (bottom right).



- The *trend component* in the term spread captures the long term averages as reported in Table 14.1 rather accurately.
- In general the *periodic component* shows more types of fluctuations than only the *long wave component*. The latter is very similar across the three countries, both with respect to the dating of peaks and troughs and with respect to the amplitude. The *amplitude* is approximately 0.5% which confirms what was already suspected in the data analysis. A consistent property of the term spread long waves is that they move almost perfectly *contrary* to the long wave in the national product. The average *lag* reported in Table 14.4 is about 20 years which is approximately half of the peak-to-peak and trough-to-trough *period lengths* which are both 53 years. This can be explained as follows. From the previous chapters we know that both the long and short term interest rates contain long wave fluctuations with amplitudes of several percentage points. With respect to the dating of peaks and troughs these long waves were found to move very much in parallel with those in the national product. Now, if the volatility of the long wave in the short term interest rates is larger than in the long term interest rates (compare the volatilities of the periodic and long wave components in Table 12.2 and 13.2 to see that this is actually the case), at a peak in the interest rates the short term interest rates will be higher than the long term interest rates, causing a relatively low term spread. The other way around, at a trough in the interest

rates the short term interest rate will be lower than the long term interest rate, causing a relatively high term spread. These very simple relations are illustrated in Figure 14.2.

Figure 14.2 Stylized representation of long waves in Short Term Interest Rates, Long Term Interest Rates and the Term Spread calculated as their difference.



- The *volatility* of the *stochastic component* in the term spread is on average a little more than 1.0%, which is in between the corresponding volatility of the long and short term interest rates. The *correlation* with the national product is consistently and substantially negative, which is exactly the opposite of what was found for the constituting interest rates. For this see Tables 12.2 and 13.2. Note that for this filtered series the correlation with the national product does not appear to have changed during the postwar period which, casts doubt on the alleged increase in the predictive power of the term spread for economic growth as mentioned in the previous section. In general the negative correlations suggest very different lead / lag relations for the term spread than for the separate interest rates. This will have to be confirmed by the spectral analysis in the next section.
- The results in Table 14.3 show that in general the stochastic component describes almost the entire conventional *volatility* of the historical term spreads. From Figure 14.1 it can already be seen that indeed the shorter term fluctuations determine most of the volatility of the term spread rather than the long term fluctuations. For the UK this ratio is in general somewhat lower than for the NL and the US. From Figure D.17 in Appendix D it can be seen that this is probably due to some relatively large low frequency fluctuations, especially during the postwar period.

Table 14.2 Statistics of the filtered periodic, long wave and stochastic components of a Term Spread for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands				United Kingdom				United States			
		Stddev	Ratio	Corr		Stddev	Ratio	Corr		Stddev	Ratio	Corr
Periodic	1870-1999	0.6%	0.1	-0.3	1855-1999	1.0%	0.1	-0.1	1870-1999	0.9%	0.1	-0.8
Long Wave	1870-1999	0.3%	0.1	-0.1	1855-1999	0.4%	0.1	-0.7	1870-1999	0.6%	0.4	-0.8
Stochastic	1870-1913	0.5%	0.2	0.2	1855-1913	1.0%	0.4	-0.2	1870-1913	1.3%	0.4	-0.3
	1919-1939	0.9%	0.5	-0.2	1919-1939	0.8%	0.2	-0.3	1919-1939	0.9%	0.2	-0.2
	1949-1999	1.2%	0.6	-0.4	1949-1999	1.3%	0.6	-0.3	1950-1999	1.1%	0.5	-0.1
	1949-1975	1.2%	0.6	-0.4	1949-1975	0.8%	0.5	-0.1	1950-1975	0.8%	0.4	-0.1
	1976-1999	1.3%	0.7	-0.6	1976-1999	1.8%	0.7	-0.4	1976-1999	1.3%	0.6	-0.1

Table 14.3 Standard deviation of Term Spread for various sub-periods (*Hist*) versus standard deviation of filtered stochastic components (*Filt*).

	Netherlands			United Kingdom			United States				
	Filt	Hist	Ratio	Filt	Hist	Ratio	Filt	Hist	Ratio		
1828-1913	0.7%	1.2%	0.6	1820-1913	0.9%	1.1%	0.8	1831-1913	2.0%	2.5%	0.8
1919-1939	0.9%	1.1%	0.8	1919-1939	0.8%	1.1%	0.7	1919-1939	0.9%	1.6%	0.6
1949-1999	1.2%	1.3%	0.9	1949-1999	1.3%	2.0%	0.7	1950-1999	1.1%	1.2%	0.9
1949-1975	1.2%	1.3%	0.9	1949-1975	0.8%	1.4%	0.5	1950-1975	0.8%	0.8%	1.0
1976-1999	1.3%	1.3%	1.0	1976-1999	1.8%	2.1%	0.8	1976-1999	1.3%	1.4%	0.9



Table 14.4 Peaks and troughs of the long wave component of a Term Spread. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

	12345					Peak-Peak Trough-Trough Lead			
Netherlands	Peak	1828	55	1883	53	1936	60	1996	56
	Trough		1856	53	1909	60	1969		57
	Lead			0	-2	-8	-19	-24	-11
	Peak	1837	50	1887	48	1935	48	1983	49
United Kingdom	Trough		1863	48	1911	58	1969		53
	Lead			-31	-28	-25	-31	-15	-26
	Peak			1893	41	1934	57	1991	49
United States	Trough		1870	48	1918	49	1967		49
	Lead				-28	-16	-21	-17	-21
	Peak	1833	55	1888	47	1935	55	1990	53
Average	Trough		1863	50	1913	56	1968		53
	Lead			-18	-19	-16	-24	-19	-19

## 14.3 Spectral analysis

### 14.3.1 Static

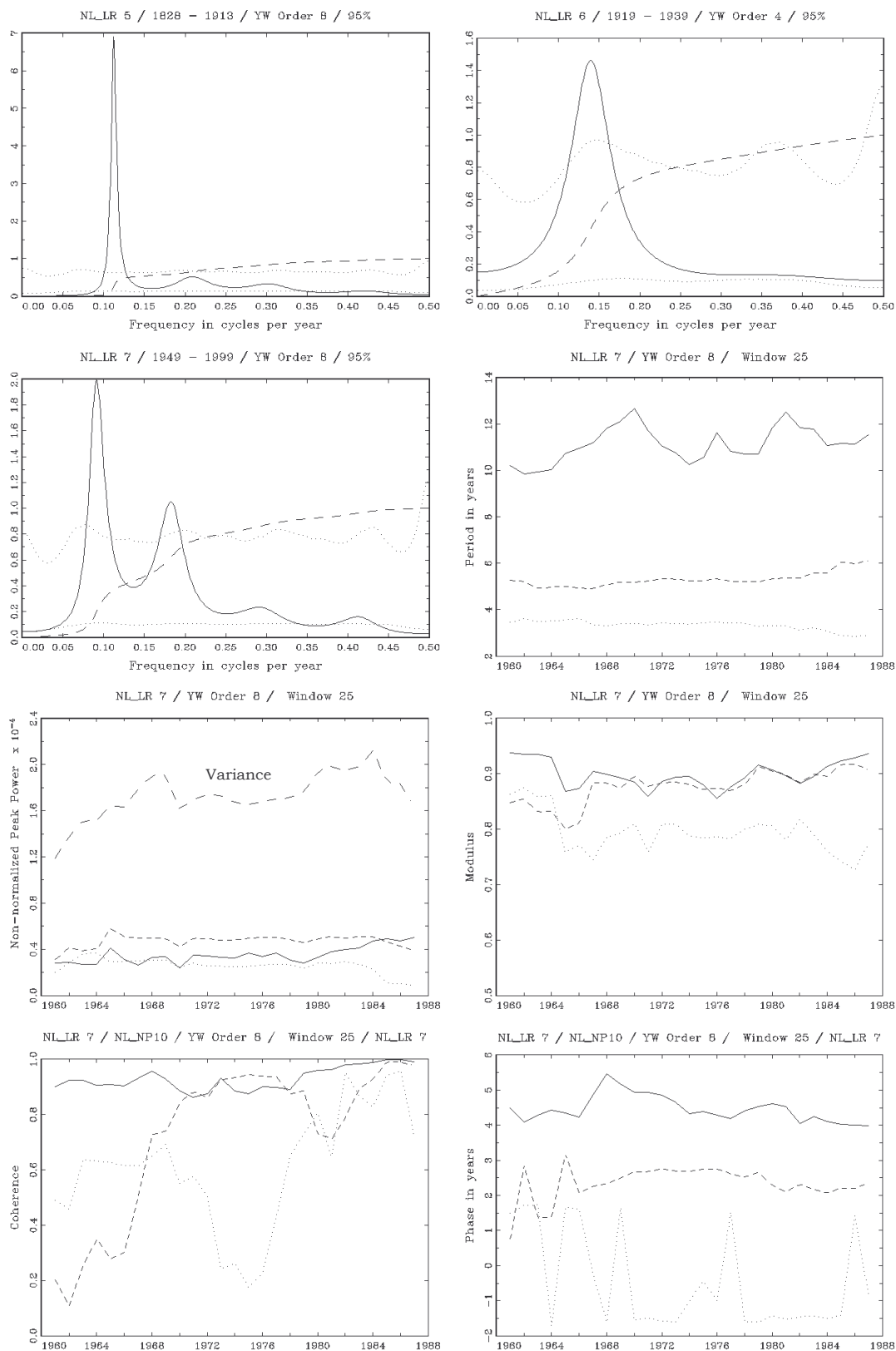
- With respect to the *peak period*, *peak power*, *modulus* and *coherence* the estimated term spread spectral densities are very similar to those of the short and long term interest rates from the previous chapters. This is not very surprising if one realizes that a linear combination of sinusoids of similar frequency is again a sinusoid of the same frequency. What is a bit surprising is that, besides the Juglar peak in all countries and sub-periods, in the postwar NL period also the Kitchin peak is *significant* at the 95% confidence level. Neither in the corresponding spectral densities of the short term interest rate nor of the long term interest rate this was the case.
- Due to the differences in *volatility* and *phase* between the spectral properties of the short and long term interest rates, the lead / lag relations of the term spread are markedly different from those of the interest rates. While the long and short term interest rates were *lagging* the national product by two to three years, the results in Table 14.6 indicate that for the postwar period the term spread actually *leads* the national product by several years, both at the Juglar and the Kitchin frequencies. For the earlier periods the findings are similar<sup>60</sup>.
- These leading or predicting properties of the term spread with respect to the national product are well known in the literature<sup>61</sup>. Examples of the literature on this subject are Estrella and Hardouvelis (1991), Harvey (1991), Berk (1998) and Estrella and Mishkin (1998). Also especially noteworthy is the early work of Kessel (1965) who, besides this topic, also discusses many other aspects of the behavior of the term structure of interest rates. Two important aspects of this lead of the term structure deserve further attention.

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<sup>60</sup> Note that, as described in Chapter 6, we chose to always represent the phase within the interval of a lag and a lead of half a period length. The occasionally large lags reported for the prewar and interwar periods in Table 14.5 can be transformed in substantial leads by adding a full indicated period length. After all, a phase is only unique up to an integer times the relevant period length. In that case the phases reported in Table 14.5 show much more consistency with respect to a leading term spread than they may do on first sight.

<sup>61</sup> The fact that the term spread consistently leads the business cycle fluctuations in the national product is the reason that it is often included in leading indicators.

Figure 14.3 Static and dynamic estimates of the spectral density of the stochastic component of a Term Spread for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.

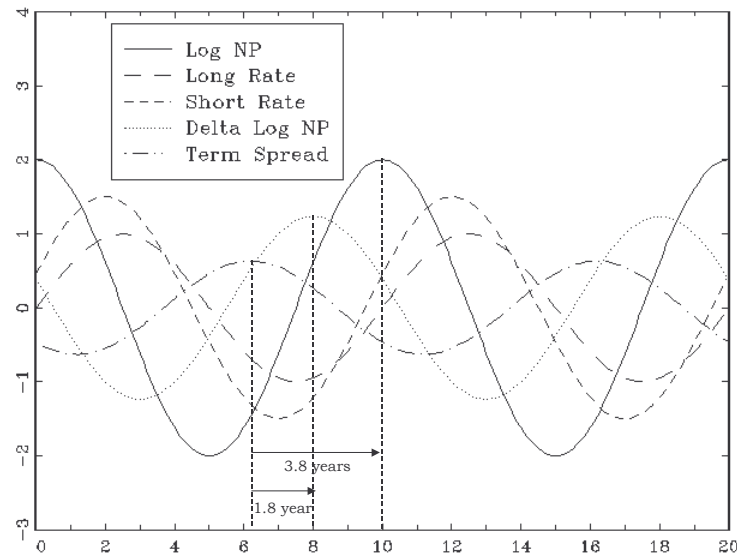


- The first issue is that most of the times much smaller leads are reported in the literature than those found here. The explanation for these apparent differences lies in the simple fact that often the relation between the term spread and the *growth* instead of the *level* of the national product is analyzed. A lead of say three years at the Juglar period length of about ten years over the (logarithmic) level of the national product is transformed into a lead of only one year over the growth of the national product by applying the phase of the first order differencing operator as shown in Figure 5.1 in section 5.2.1.
- The second issue is of course how the lead of the term spread can be consistent with the lag of long and short term interest rates as found in the previous chapters. To illustrate this in a rather stylized fashion, Table 14.5 first summarizes some properties of the postwar spectral densities of the national product and the short and long term interest rates as reported in respectively Chapter 7, 12 and 13. Based on this information, Figure 14.4 then plots sinusoids to represent in a stylized fashion the Juglar components in each of the three variables. A uniform period length of ten years was used together with an amplitude for the national product of two, for the long term interest rate of one and for the short term interest rate of one and a half. The short and long term interest rate sinusoids lag the national product by respectively two and two and a half years. From these three sinusoids we then calculated the corresponding Juglar cycle in the *growth* of the national product and in the difference between the long and short term interest rates (i.e. the term spread). First of all, this figure shows that, because of the larger volatility of the short term interest rate, the lags of the interest rates indeed imply a *lead* of three to four years as found in the spectral analysis. Second, the figure shows, as explained earlier, that such a large lead over the *level* of the national product implies a much smaller lead of say one to two years over the Juglar cycle in the *growth* of the national product. Finally, the figure confirms what economists have long recognized about the causal relation between the shape of the term structure and the stage of the business cycle. Term structures tend to be low and upward sloping in the early stages of an economic expansion (consider the values at period six) and high and downward sloping close to the peak of a cycle (consider the values at period twelve). For this see for example Kessel (1965) and Fama and French (1989). Overall it is therefore fair to say that the lead / lag relations from the spectral analysis of both the long and short term interest rates and the term spread are consistent with each other and with what is already known in the literature about their behavior.

Table 14.5 Standard deviation of stochastic component and period length and phase of Juglar component in National Product, Long Term Interest Rate, Short Term Interest Rate in the indicated countries for the 1949-1999 period.

	<i>Log National Product</i>		<i>Long Term Interest Rate</i>			<i>Short Term Interest Rate</i>		
	<i>Stdev</i>	<i>Period</i>	<i>Stdev</i>	<i>Period</i>	<i>Phase</i>	<i>Stdev</i>	<i>Period</i>	<i>Phase</i>
Netherlands	2.0%	9.9	0.8%	10.1	-2.7	1.7%	10.1	-1.9
United Kingdom	2.1%	11.1	1.0%	7.7	-1.8	1.6%	9.3	-1.6
United States	2.2%	10.9	0.8%	9.3	-3.0	1.6%	8.7	-2.4
Average	2.1%	10.6	0.9%	9.0	-2.5	1.6%	9.4	-2.0

Figure 14.4 Stylized representation of Juglar component in National Product, Long Term Interest Rate, Short Term Interest Rate and in the corresponding growth of National Product and the Term Spread.



### 14.3.2 Dynamic

- In general the *average* spectral measures in Table 14.7 are very consistent with the static estimates for the postwar period in Table 14.6. Two exceptions are the *coherence* for the Kitchin peak in both the NL and the UK, which are on average much higher than the static estimates suggest. The corresponding phases, which are not very different from the static estimates, can therefore be interpreted with greater confidence.
- Only for the NL slight indications of a lengthening of the *period length* at both the Juglar and Kitchin frequencies can be observed.
- In each of the three countries the *variance* has increased, driven by both an increase in the *peak power* of the Juglar and the Kitchin peak. This is probably due to the level effect of interest rate volatility as analyzed in sections 12.2.1 and 13.2.1. Striking is the fact that for the NL the Kitchin peak power has been consistently higher than the Juglar peak power, while this is seldom the case for other variables and countries. Also the *modulus* for this peak is relatively high (0.9 on average for the NL and 0.8 for the UK and US). The combination of the relatively high peak power and the high modulus indicate that the Kitchin type fluctuations play an important role in the business cycle fluctuations of the NL term spread. Note that this is consistent with the fact that for the postwar static estimates, the NL is the only country for which the Kitchin peak is significant.
- The *modulus* of many of the spectral peaks are lower in the middle section of the sample, indicating less clear periodic behavior during those times. With the exception of the NL, the modulus of the Kitchin peak is much lower than that of the Juglar peak.
- The *coherence* of the Juglar peak has increased in all countries towards values close to one. After some very low initial values (0.2) in the NL this is also the case for the Kitchin peak while this coherence has been exceptionally high in the US

throughout the entire sample. Only for the UK the coherence of the Kitchin peak has stayed very low around 0.6.

- For the UK the *phase* of the Juglar peak indicates a lead of approximately three years both in the beginning and at the end of the sample. Although because of the low coherence the phase of the UK Kitchin peak has to be interpreted with caution, it seems that it has increased towards a lead of approximately one year. These UK figures are pretty consistent with the US results, with the exception that for the US the leads have been very constant throughout the entire sample. The most striking are the larger, though constant, phases for the peaks in the NL spectra. For the Juglar and the Kitchin peaks, leads of respectively four and two years are obtained, which are about one year longer than those found for the UK and the US. One hypothesis for this marked difference between the NL and the other two countries is that we know that in general the NL economy (national product) lags about one to two years behind the UK and US economies, which move pretty much in parallel. For this see section 7.3. If the increased internationalization of the world financial markets cause the term spreads to synchronize while this is not the case for the economy (national product) itself, then this could explain the greater lead of the NL term spread over the relevant national product. Note however that for the long and short term interest rates no indications of such differences were found.

Table 14.6 Statistics of static spectral estimates of the filtered stochastic components of a Term Spread for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States							
	Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase		Period	PP	Mod	Coh	Phase
1828-1913*	8.9#	0.3	1.0	0.4	-0.1	1820-1913**	9.2#	0.3	0.9	0.9	4.3	1831-1913***	11.3#	0.3	0.9	0.7	-4.6
	4.8	0.2	0.9	0.7	-1.5		4.0	0.2	0.8	0.2	-0.9		4.4	0.1	0.6	0.5	1.3
	3.3	0.1	0.8	0.7	0.5		2.7	0.2	0.7	0.3	-0.6		2.8	0.2	0.8	0.7	0.4
	2.4	0.1	0.8	0.4	-0.4												
1919-1939	7.2#	0.5	0.8	0.8	-2.7	1919-1939	11.5#	0.3	0.9	1.0	5.2	1919-1939	11.4#	0.4	0.9	0.8	-5.2
	2.7	0.2	0.5	0.4	-0.7		4.1	0.2	0.8	0.8	0.9		4.0	0.2	0.7	0.5	0.1
							2.6	0.2	0.8	0.7	-0.2		2.7	0.1	0.8	0.7	1.0
1949-1999	11.1#	0.2	0.9	0.9	4.3	1949-1999	9.9#	0.3	1.0	1.0	3.2	1950-1999	8.0#	0.3	0.9	0.8	2.6
	5.5#	0.2	0.9	0.4	2.0		4.4	0.2	0.8	0.2	0.9		4.1	0.3	0.8	0.9	1.3
	3.4	0.1	0.8	0.5	1.7		2.8	0.2	0.8	0.4	0.8		2.4	0.2	0.7	0.4	0.9
	2.4	0.1	0.8	0.5	-1.1												

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1828-1913 frequencies / period lengths.

\*\* Coherence and phase come from the 1855-1913 estimates of the co-spectrum for the indicated 1820-1913 frequencies / period lengths.

\*\*\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1831-1913 frequencies / period lengths.

Table 14.7 Average statistics of dynamic spectral estimates of the filtered stochastic components of a Term Spread for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 14.6.

Netherlands				United Kingdom				United States						
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
11.1	0.2	0.9	0.9	4.5	10.7	0.3	0.9	0.9	2.5	8.0	0.3	0.9	0.8	2.1
5.3	0.3	0.9	0.7	2.3	4.2	0.2	0.8	0.6	0.2	4.2	0.3	0.8	1.0	1.2
3.3	0.2	0.8	0.6	-0.5	2.8	0.2	0.7	0.8	0.8	2.4	0.1	0.7	0.7	0.6

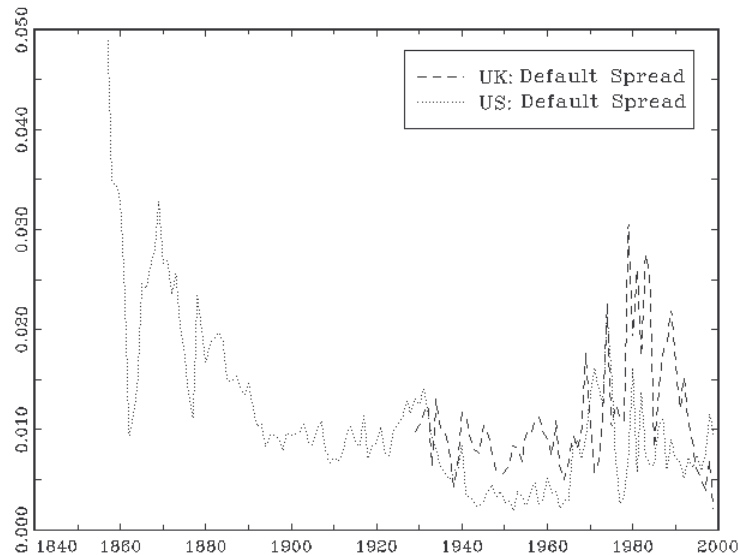




## 15 Default Spread

### 15.1 Data analysis

Figure 15.1 Default Spreads calculated as the difference between the yield on high grade corporate bonds and Long (government) Interest Rate.



- The default spreads analyzed in this chapter are the difference between the yield on high grade corporate bonds and the yield on government bonds as analyzed in Chapter 13 as the default free long term interest rates. In general such spreads are a compensation for the lenders risk that the borrower will not be able to fulfil its obligations in terms of interest and redemption payments. Here, we are especially interested in how default spreads behave in general rather than that the exact rating (i.e. the credit worthiness of the borrower) associated with the default spread is of particular importance. Unfortunately (long) time series of yields on corporate bonds are not readily available for the NL, which is why the analysis in this chapter is limited to the UK and the US. Because of the longer sample, the graphical results for the US are shown in the main text. As usual, the results for the UK are contained in Appendix D.
- The default spreads do not show any clear *trending behavior*. Instead the default spreads have evolved around a *long term average* of approximately 1%. For this also see Table 15.1. One exception is the early 19<sup>th</sup> century US data, which shows some very high levels of the default spread. Around the long term average, also some long periods of relatively high and relatively low default spreads can be observed. The spreads have for example been high during the 1930's while they have been relatively low during the first half of the postwar period, until around 1970. We know that the 1930's were a period of general economic uncertainty and low economic growth, while the first half of the postwar period is characterized as a period of stable economic conditions with high economic growth. The long term level of the default spread therefore seems to have moved contrary to the long term growth of national product. This opposite pattern in the

long term averages is also the prime reason for the strong negative *correlations* between default spreads and economic growth as reported in Table 15.1. The long term swings in default spreads are also observed by for example Fama and French (1989) who state that “...its major swings seem to go beyond the business cycles...”. Finally note that these long term swings move very much in parallel in the two countries.

- Around the long term swings, the default spreads also clearly show some *shorter term fluctuations* at the business cycle frequencies. Such cyclical variation in default spreads was to be expected since one intuitively feels that default risk is to a large extent driven by the state of the economy. Default risk increases during recessions and decreases during periods of economic expansion. For example Chen et al. (1986) argue that the default spread is a measure of business conditions. Wilson (1998)<sup>62</sup> explicitly models the link between the state of the economy and default risk and finds that default *rates* can be almost perfectly explained from some general macroeconomic variables such as the growth of national product, unemployment rates and interest rates. Wilson (1997) gives a more direct available summary of the results. Since default *spreads* are a compensation for the default *rates* it is to be expected that also the default spreads show cyclical variation.
- The *volatility* of the default spread is approximately 0.5% which is much smaller than the 1.5% standard deviation of the term spread. So, default spreads evolve about three times as stable as term spreads. Just as for the short and long term interest rates, there also seems to be a *level effect* for the volatility of the default spreads. For example, (short term) volatility is high during the 1980's in which the general level of the default spreads is high, while the volatility is low during the first half of the postwar period in which the level of the default spreads is rather low. The level effect of default spreads will be further investigated in section 15.2.1.

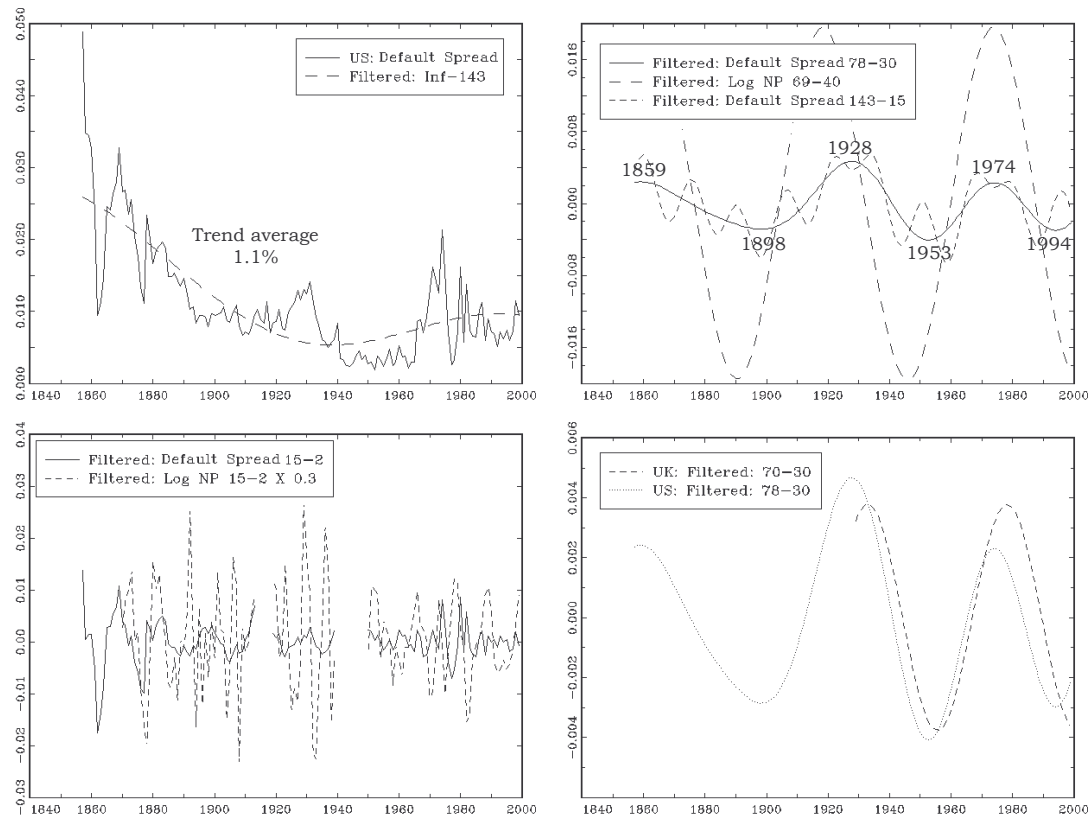
Table 15.1 Statistics of a Default Spread for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index.

<i>United Kingdom</i>					<i>United States</i>				
	<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>		<i>Avg</i>	<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>
1929-1999	1.1%	0.6%	0.2	-0.1	1857-1999	1.1%	0.8%	-	-
					1871-1999	0.9%	0.5%	0.1	-0.1
					1871-1913	1.3%	0.5%	0.1	0.1
1929-1939	0.9%	0.3%	0.1	-0.2	1919-1939	0.9%	0.3%	0.0	-0.5
1950-1999	1.2%	0.7%	0.3	-0.3	1950-1999	0.7%	0.4%	0.2	-0.4
1950-1975	0.9%	0.4%	0.2	-0.4	1950-1975	0.7%	0.5%	0.2	-0.4
1976-1999	1.5%	0.8%	0.4	-0.2	1976-1999	0.8%	0.3%	0.2	-0.5

<sup>62</sup> Thanks go to Pieter Klaassen for pointing out this reference.

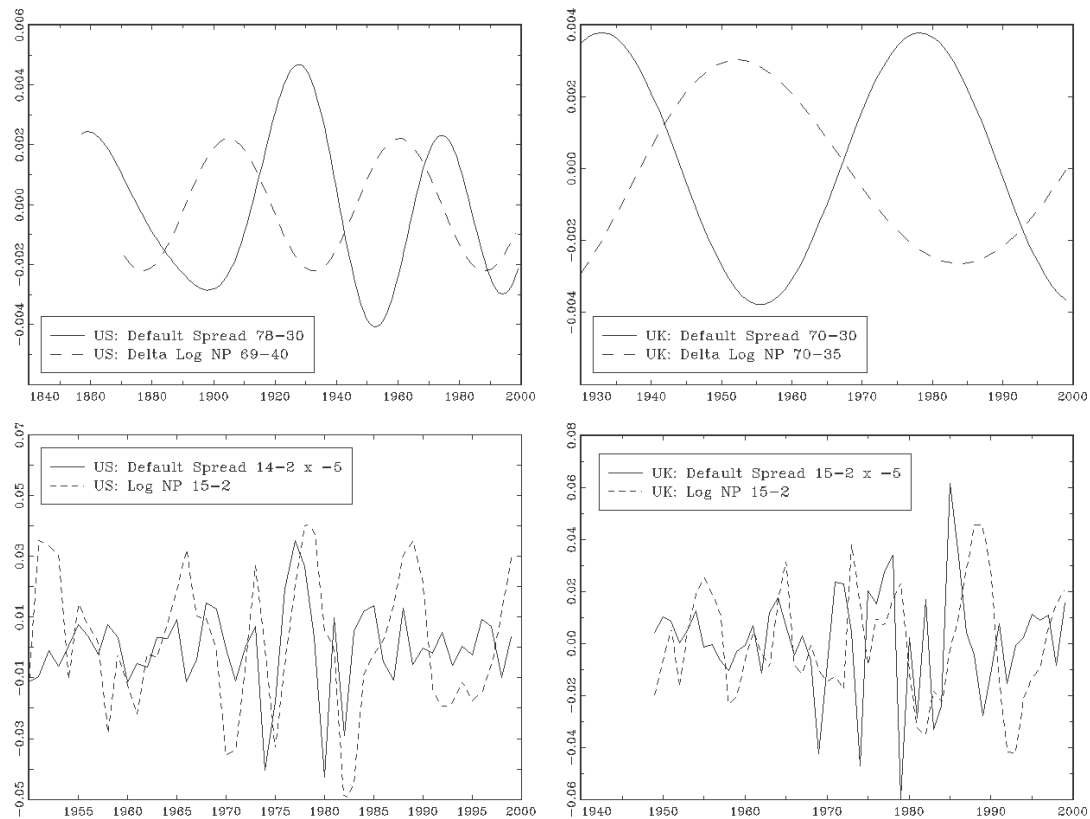
## 15.2 Filtering results

Figure 15.2 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Default Spread for the US and long wave component compared to UK data (bottom right).



- The averages of the *trend components* for both the US and the UK are 1.1% and therefore capture the long term averages from Table 15.1 rather well.
- As already noted in the previous section, in both the US and in the UK the volatility of the *periodic component* is largely determined by the long term fluctuations. However, around this *long wave component* also fluctuations with a period length of between some fifteen to twenty years are visible which are difficult to interpret. Table 15.2 shows that the *amplitude* of the long wave component in the default spread is about five to ten times as small as that of the national product. Due to these long waves, default spreads can be approximately 0.4% above or below their long term average of approximately 1%. The average peak-to-peak and trough-to-trough *period lengths* are respectively 46 and 49 years. With respect to the dating of the peaks and troughs of the long wave, the default spread *lags* the national product by about 10 to 15 years. An exception is the postwar US cycle. See Table 15.4. By looking at the *growth* instead of the level, the long wave in the national product shifts backwards in time by about 10 years. As the top two pictures of Figure 15.3 show, this causes the long wave in the default spread to behave almost perfectly countercyclical to the long wave in the growth of the national product. That is, the default spread is relatively high during periods of low growth and relatively low during periods of high growth, just as noted with the data analysis.

Figure 15.3 *Top two pictures*: Long wave components in Default Spread together with long wave components in *growth* of National Product index. *Bottom two pictures*: (Scaled) *inverted* postwar stochastic component of a Default Spread for the US and the UK together with the stochastic component in a relevant National Product index.



- The *volatility* of the *stochastic component* in the default spreads is approximately 20% of the national product volatility. Again this is about two to three times as small as for the term spread. The *correlations* at the business cycle frequencies are in general less negative than they were for the conventional statistics reported in Table 15.1. This illustrates that the strong negative conventional correlations are indeed to a large extent determined by the countercyclical long term fluctuations in the default spread. At first sight the stochastic component in the default spread does not seem to show a clear link to the business cycle fluctuations in the national product. The low correlations in Table 15.2 indicate acyclical or slightly countercyclical behavior. However, comparing the *negative* of the stochastic component of the default spreads to the national product shows a more clear relation. After all, default spreads are high when the economy performs badly and vice versa. From the bottom two pictures in Figure 15.3 it is clear that the *inverted* default spreads move very similar to the national product, only with a lead of a few years. If we were to transform these pictures even further by considering the *growth* of the national product, this lead would become smaller and the business cycle fluctuations in the inverted default spread would be very similar to those in the growth of the national product. The exact lead / lag relations are further explored in the next section on spectral analysis.

- Table 15.3 shows that the stochastic component describes only about 60% of the total *volatility* of the default spread. This is a direct consequence of the relative large volatility of the long wave component. For this, also compare the standard deviations of the various components in Table 15.2 to the standard deviation of the original series reported in Table 15.1. This shows that the long wave component and stochastic component both describe about half of the total volatility of the default spread<sup>63</sup>.

Table 15.2 Statistics (standard deviation) of the filtered periodic, long wave and stochastic components of a Default Spread for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National product index.

	<i>United Kingdom</i>				<i>United States</i>			
		<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>		<i>Stdev</i>	<i>Ratio</i>	<i>Corr</i>
Periodic	1929-1999	0.4%	0.0	-0.1	1870-1999	0.3%	0.0	0.0
Long Wave	1929-1999	0.3%	0.1	-0.1	1870-1999	0.2%	0.2	0.7
Stochastic					1870-1913	0.3%	0.1	0.3
	1929-1939	0.2%	0.1	0.2	1919-1939	0.2%	0.0	0.0
	1949-1999	0.4%	0.2	0.0	1950-1999	0.3%	0.1	-0.2
	1949-1975	0.3%	0.2	0.0	1950-1975	0.2%	0.1	-0.2
	1976-1999	0.5%	0.2	0.0	1976-1999	0.3%	0.1	-0.3

Table 15.3 Standard deviation of a Default Spread for various sub-periods (Hist) versus standard deviation of filtered stochastic components (Filt).

	<i>United Kingdom</i>			<i>United States</i>		
	<i>Filt</i>	<i>Hist</i>	<i>Ratio</i>	<i>Filt</i>	<i>Hist</i>	<i>Ratio</i>
				1857-1913	0.5%	0.9%
1930-1939	0.2%	0.3%	0.7	1919-1939	0.2%	0.3%
1950-1999	0.4%	0.7%	0.6	1950-1999	0.3%	0.4%
1950-1975	0.3%	0.4%	0.9	1950-1975	0.2%	0.5%
1976-1999	0.5%	0.8%	0.6	1976-1999	0.3%	0.3%

<sup>63</sup> Because of the (theoretically) zero correlation between the filtered components, the *variance* of the trend, periodic and stochastic components should add up to the total variance of the series.

Table 15.4 Peaks and troughs of the long wave component of a Default Spread. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

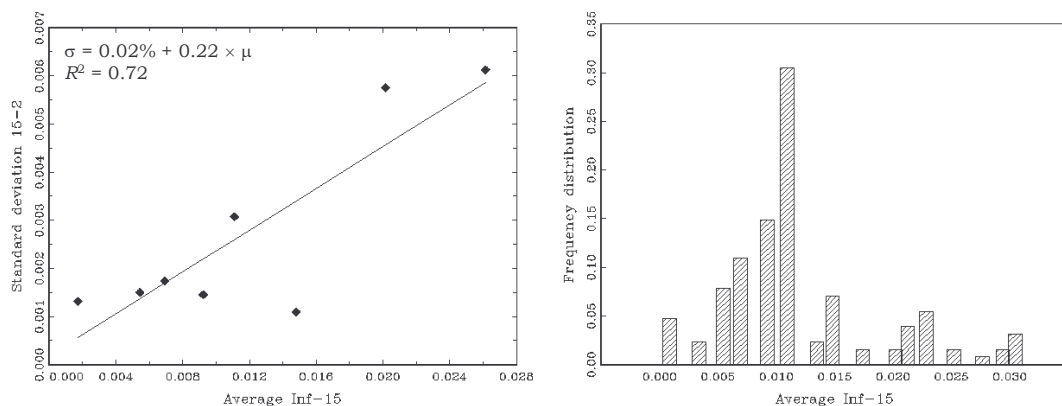
	1	2	3	4	5	Peak-Peak Trough-Trough Lead
United Kingdom			1933	45	1978	45
Peak				1956	43	1999
Trough			-23	-18	-10	0
Lead			1928	46	1974	46
United States			1898	55	1953	41
Peak			-8	-10	-7	0
Trough						5
Lead						-4
Average			1931	46	1976	46
Peak			57	1955	42	1997
Trough			-5	-12	-10	-5
Lead						-8



## 15.2.1 Volatility versus level

- To further investigate the relation between the *level* of the default spread and its *volatility*, a similar analysis was performed as for the short and long term interest rates. For this, the level is defined as the value of the sum of the filtered trend and periodic components while the volatility is the standard deviation of the stochastic component. For more information on the exact procedure see section 12.2.1.

Figure 15.4 For a Default Spread for the US, the frequency distribution of the sum of the filtered trend and periodic component (right) and an estimated relation between this “level” of the spread and its “volatility” represented by the standard deviation of the stochastic component (left).



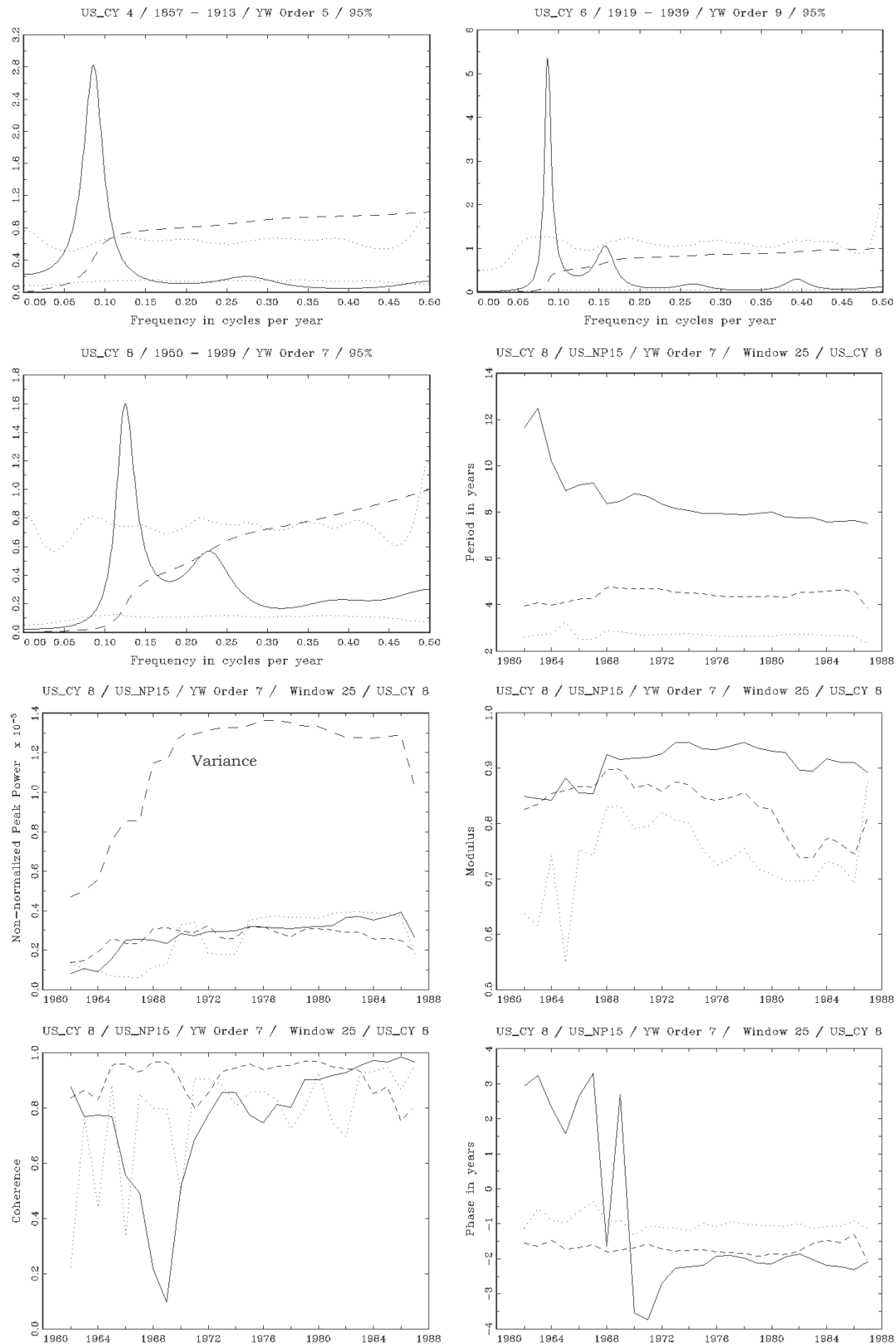
- The *frequency diagram* on the right hand side of Figure 15.4 shows that for the US the level of the defaults spread is skewed to the left. Almost 50% of the observations is close to a value of 1%. Higher values occur with much smaller probability. From Appendix D.9 it can be seen that the same holds for the UK.
- The left hand side picture in Figure 15.4 shows a *positive relation* between the level and the volatility of the default spread. For the US and the UK the volatility approximately increases by respectively 0.2% and 0.4% for every 1% increase in the underlying level. That is, for the UK the volatility of the default spread increases more rapidly than for the US. The size of these level effects for the default spread are comparable to those found for the short term interest rates. The major difference with the results for the interest rates is however that the level has less explanatory power for the volatility. For the short interest rates the  $R^2$  was very close to one while here it is “only” 0.72 and 0.83 for respectively the US and the UK. Apparently the volatility of the default spread is driven by more factors than only the underlying level. At the most common level of the default spread of 1% the standard deviation of the stochastic component is about 0.2% for the US and 0.3% for the UK. This is about half of the volatility of the long term interest rates.

## 15.3 Spectral analysis

### 15.3.1 Static

- Also the default spread clearly shows the by now familiar types of business cycle fluctuations. In general this should come as no surprise because of the intuitive logic that default risk (and therefore also spreads) tend to increase during recessions and decrease during economic expansions. Table 15.6 shows that the Juglar peak is found in all cases, except for the (eleven observations) interwar UK period. The Juglar peak is *significant* in all cases and has a *period length* of approximately ten years. Also the Kitchin peak is present in most of the samples. In the postwar UK period it is significant while during the interwar period it is almost significant in both the UK and the US. Also its relatively high *peak power* suggests that the Kitchin type fluctuations are rather important in the default spread process.
- Especially during the postwar period there is relatively a lot of peak power around the very *high frequency fluctuations* with a period length of between two and three years. Note that these high frequency fluctuations may be an explanation for the low  $R^2$  in the volatility / level regressions in section 15.2.1. In general such very high frequency fluctuations may be associated with “random” type events or shocks in the underlying process.

Figure 15.5 Static and dynamic estimates of the spectral density of the stochastic component of a Default Spread for the United States for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.

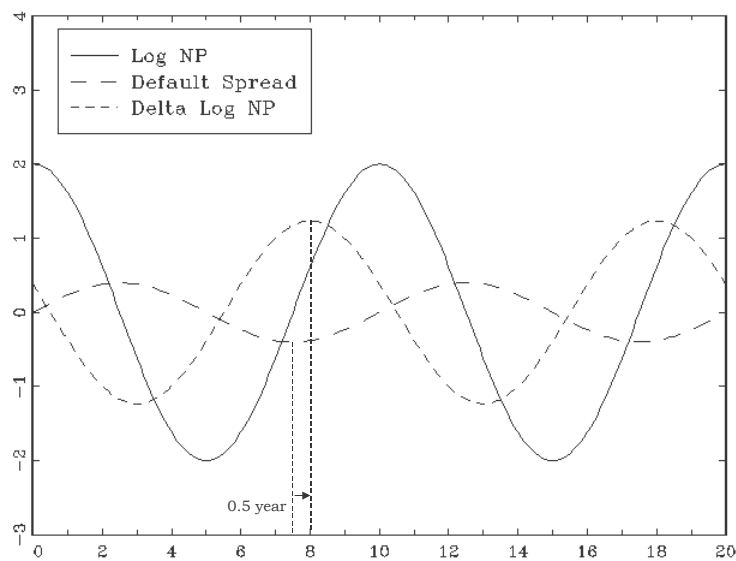


- In general the *coherence* with the national product at the relevant frequencies is rather high at approximately 0.8. The associated *phases* are supportive of a *lag* of around two years over the national product. Because at first sight this result may seem somewhat strange, we constructed a stylized example to see how the business cycle fluctuations in the default spread relate to both the level and the growth of the national product. Based on the summary of results given in Table 15.5, in Figure 15.6 a sinusoid is plotted with a period length of ten years and an amplitude of 2.0 as a representation of the Juglar cycle in the national product. The default spread sinusoid has the same period length but then with an amplitude of only 0.4 and a lag of 2.5 years over (the level of) the national product. The figure also shows the corresponding cycle in the *growth* of the national product. It illustrates that the lag of the default spread over the national product implies that the default spread is lowest just before the growth of the national product reached its highest level. So as expected, default spreads are low during economic expansions and high during economic contractions.

Table 15.5 Standard deviation of stochastic component and period length and phase of Juglar component in National Product and Default Spread in the indicated countries for the 1949-1999 period.

	Log National Product		Default Spread		
	Stdev	Period	Stdev	Period	Phase
Netherlands	2.0%	9.9	-	-	-
United Kingdom	2.1%	11.1	0.4%	10.0	-2.9
United States	2.2%	10.9	0.3%	8.1	-2.1
Average	2.1%	10.6	0.4%	9.1	-2.5

Figure 15.6 Stylized representation of Juglar component in National Product, Default Spread and in the corresponding growth of National Product.



- Finally note that the lag of two to three years of the default spread is very similar to the lag of especially the long term interest rates. However, the *reasons* for this lagging behavior are probably very different. In case of the interest rates the most probable cause is simply a slow adjustment to changing economic circumstances. As we have just seen, in the case of the default spreads the reason is the link with the *growth* of the national product and the inverted relation between default spreads and national product (i.e. spreads are low when growth is high and vice versa).

### 15.3.2 Dynamic

- The *averages* of the spectral measures reported in Table 15.7 are fairly consistent with the postwar static estimates from Table 15.6. The major difference is the *phase* at the Juglar frequencies. For the UK the average lag is approximately half a year longer than the static lag while for the US it is approximately one year shorter. Figures 15.5 and D.21 show that the static phase estimates are consistent with the rolling window estimates during the second half of the sample.
- In general the *period lengths* have been rather constant throughout the postwar period. Only the period length of the Juglar peak in the US seems to have decreased over time.
- The total *variance* of the stochastic component has increased during most of the sample and decreased again at the end. Just as with the interest rates, this pattern is probably due to the level effect as analyzed in section 15.2.1. This pattern is most clearly visible in the *peak power* of the Kitchin peak. In the UK both this Kitchin peak and the very high frequency fluctuations (with a period length between two and three years) contribute more to the total variance than the Juglar peak. For the US the Juglar and Kitchin peak have approximately equal peak power and at the end of the sample also there the high frequency fluctuations have the highest peak power. In general this again stresses the relatively high importance of both the Kitchin type fluctuations and the very high frequency fluctuations in the default spread process.
- The *modulus* of the Juglar peaks is higher at the end than at the beginning of the sample. For the Kitchin peak it was highest in the middle of the sample while in recent decades it seems to have decreased.
- Again the *coherence* at the Juglar frequencies has increased to values close to one at the end of the sample. For the Kitchin frequencies it has been reasonably high throughout the entire period. However, at the end of the sample, similar to the modulus, it has decreased rather substantially, especially in the UK.
- In general the *phase* of the Juglar and Kitchin peak have been rather constantly around the lags found in the static estimates. The only exception is the first half of the sample for the US Juglar peak. The *leads* of two to three years found there are very unusual in the light of the other findings. However, the very low coherence during at least a part of that period indicate that not too much confidence should be attached to this finding.

Table 15.6 Statistics of static spectral estimates of the filtered stochastic components of a Default Spread for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

<i>United Kingdom</i>						<i>United States</i>					
<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>		<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>	
1857-1913*						1857-1913*	11.7 <sup>#</sup>	0.4	0.9	0.8	0.8
							3.6	0.1	0.8	0.1	0.3
						1919-1939	11.6 <sup>#</sup>	0.3	1.0	0.8	0.2
							6.3	0.2	0.9	1.0	-2.5
1929-1939	4.6 <sup>#</sup>	0.2	0.9	0.9	-1.1		3.7	0.1	0.9	0.8	-1.8
	2.5 <sup>#</sup>	0.4	0.9	1.0	0.2		2.5	0.1	0.9	0.9	-0.1
						1950-1999	8.1 <sup>#</sup>	0.2	0.9	0.8	-2.1
							4.3	0.3	0.8	0.8	-1.9
1949-1999	9.9 <sup>#</sup>	0.1	0.9	0.9	-2.9						
	4.8 <sup>#</sup>	0.2	0.9	0.7	0.1						
	3.1	0.1	0.6	0.2	-0.9						
	2.4	0.2	0.9	0.8	-0.7		2.6	0.3	0.7	0.6	-1.1

# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.

\* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1857-1913 frequencies / period lengths.

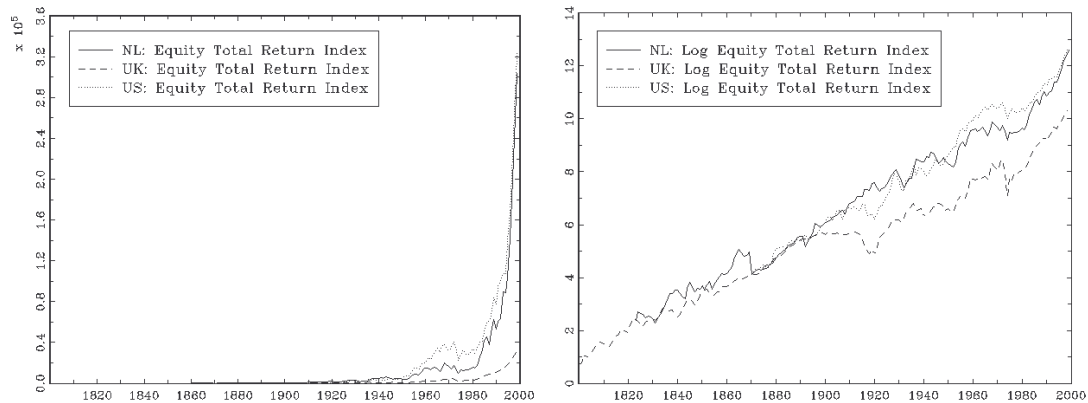
Table 15.7 Average statistics of dynamic spectral estimates of the filtered stochastic components of a Default Spread for the 1949-1999 period. These can be compared to the statistics of the static estimates for the last sub-period from Table 15.6.

<i>United Kingdom</i>						<i>United States</i>					
<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>		<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>	
10.0	0.2	0.9	1.0	-3.6		8.5	0.2	0.9	0.8	-0.9	
5.0	0.3	0.9	0.8	-0.4		4.4	0.2	0.8	0.9	-1.7	
3.3	0.1	0.7	0.6	-0.3							
2.4	0.2	0.9	0.9	-0.7		2.7	0.2	0.7	0.8	-1.0	

## 16 Equities

### 16.2 Data analysis

Figure 16.1 Real Total Return equity indices and their (natural) logarithm. The series pass through a common point in 1870 (start US series).



- The equity indices analyzed here are real total return indices. That is, they describe both capital gains and reinvestments of dividends. The original nominal indices were divided by the value of the relevant consumer price index to obtain real indices<sup>64</sup>. Just as with the other series, the equity indices are local indices denominated in the local currency. That is, currency effects are excluded here.
- As one would expect, equities show a clear *exponential trending* behavior which translates into linear trends for the (natural) logarithmic transforms. For example one Euro invested in Dutch equities in 1823 would have accumulated to more than 83.500 Euro in 1999, excluding inflation. This is an average real return of approximately 6.5% per annum. Including inflation this would have been more than 2.200.000 Euro, an average annual nominal return of 8.5%<sup>65</sup>.
- Table 16.1 shows both the *average geometrical returns* and the *average arithmetic returns* for the three countries for various sub-periods in real terms. Unlike for the other variables, for equities the returns are exactly calculated as  $return_t = (x_t - x_{t-1})/x_{t-1}$  instead of approximated as  $return_t \approx \ln(x_t) - \ln(x_{t-1})$ . Because of the large volatility of the equity returns the latter approximation can become rather inaccurate. From this table we see that the geometrical averages are always lower than the arithmetic averages. This comes from the simple effect that starting from a lower value, higher returns are needed to obtain the same increase as from a higher level<sup>66</sup>. Under the assumption of returns following a Normal distribution

<sup>64</sup> Note that because, as we shall see, the volatility of the real equity returns is much larger than that of the price changes, reported statistics as standard deviations and correlations are not very different for nominal equity indices. The only major difference concerns the long term average returns which are of course several percentage points higher in nominal terms.

<sup>65</sup> As noted by Siegel (1999) the *actual* return realized by an investor would probably have been lower because of transaction costs and lack of diversification. He estimates the difference in real terms at approximately 1 to 2% per annum. Note that indexing and mutual funds did not exist in the early years of equities.

<sup>66</sup> As a simple example of this effect suppose that an initial value of 100 decreases to 80 in the first year and increases again to 100 in the second year. The geometrical average return is obviously 0%. The



with some arithmetic average of  $\mu$  and a standard deviation of  $\sigma$ , the (long term) geometric average return is  $\mu^* = \mu - \sigma^2/2$ . So the value of a geometrical average return is determined by both the expected value and the standard deviation of the annual returns. As the standard deviation increases, the geometrical average decreases<sup>67</sup>. Geometrical averages give better information on the difference between the initial and the terminal wealth. Together with the standard deviations, arithmetic averages however reflect the true probability distribution of what returns might actually occur during one year. They are therefore better suited for modeling and decision making purposes.

- For the NL and the US the results in Table 16.1 indicate long term *average annual geometric real returns* of 6 to 7%. The arithmetic averages are approximately 1.5% higher. For the UK the long term real returns are “only” 5% and thereby approximately 2% lower than for the NL and the US. From Figure 16.3, this seems mostly because of the aberrant 1900-1920 period during which UK equities approximately stabilized in real terms while substantial positive real returns were obtained in the other two countries<sup>68</sup>. The finding of a long term geometric real return on equities of approximately 6 to 7% is consistent with results found by others. Siegel (1999) finds long term US geometrical returns to be remarkably stable around 7% per annum for the 1802-1998 period, despite of the large price changes during this period. Arnott and Bernstein (2002) report an average real return of 7.2% for the US as well for the 1870-2001 period. Based on 1871-1995 US data, Bernstein (1997) finds an average real return of 5.7% when adjusting for changes in valuation<sup>69</sup>. He acknowledges that this, what he calls real *basic return*, excludes a substantial portion of the historical return due to an upward revaluation of equities (i.e. selling at higher price-earnings ratios). Dimson et al. (2000) investigated equity returns in twelve countries for the 1900-2000 period. They found an overall geometrical average return of 5.6%. Their figures for the NL, UK and US are respectively 6.0%, 5.9% and 6.9%. Finally Eichholtz et al. (2002) report an arithmetic annual real equity return of 5.3% for the NL during the 1900-2001 period.
- Both Siegel (1999) and Arnott and Bernstein (2002) note that an average US real return of approximately 7% is consistent with the average *price-earnings ratio* observed of approximately fourteen since 1871 in the US. The reciprocal of the price-earnings ratio is the *earnings yield* (not the same as the dividend yield) which is then on average 7.1% and closely approximates the long term real return on equities. Apparently the real rate of return on capital has been the same as the return investors require on equities. Phillips (1999) gives a formal development why this might be so. The reason why the earnings yield approximates the real

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arithmetic average however is 2.5% which is the average of the subsequent annual returns of -20% and +25%.

<sup>67</sup> This effect is sometimes called the *volatility drain*.

<sup>68</sup> These marked differences might be caused by two effects. The first is that the UK was much more adversely effected by World War I during the 1914-1918 period than the NL and the US. The second is that following this great war the UK was confronted with a civil war during the 1922-1924 period which led to the independence of Southern Ireland. This issue was not further investigated here. However, the fact that for example during the postwar period the UK results are very similar to those for the other two countries indicates that “nowadays” real UK equity returns are not structurally lower than in the other two countries.

<sup>69</sup> He did this by searching for widely separated years in which either dividend yields or price-earnings ratios were identical and then calculating the (geometrical) average annual return between two such observations. In this way he in fact compared the value of the same “goods” at different points in time.

and not the nominal return to shareholders is that stock returns are based on the production of real assets whose value should, in the long run, change with the overall price level. We say more about this alleged *inflation hedge* of equities in section 16.4.3.

- Overall it is fair to say that the average equity returns found here are pretty much consistent with previous research in this field. However, the differences observed may have three causes. The first is the large *volatility* of equity returns due to which long samples of data are needed to obtain statistical significant results. Based on the familiar *t*-test at the 5% level, Jorion and Goetzman (1999) show for example that with an average return of 6% and a standard deviation of 20% (which is not unrealistic given the results in Table 16.3), a sample of 44 years is needed to even tell whether the observed return is significantly different from zero or not. In case of an average return of 3% this even becomes 178 years<sup>70</sup>. The second reason for differences in the results might be the extent to which has been relied on the most easily available data. Dimson et al. (2000) estimate that on average this *easy data bias* may lead to more than 2% higher estimated returns. The third and last possible reason for differences in the results is a possible *survivorship bias* and the fact that indices are often constructed with hindsight. Jorion and Goetzman (1999) show for example that by considering only US data one in fact conditions upon the best performing market in the 20<sup>th</sup> century, which has not been plagued as much by political upheaval, war and financial crises as most other countries in the world. For the 1921-1999 period they report an average capital return (i.e. excluding the dividend yield) of 4.3% for the US, while this median return for all the 39 countries analyzed is only 0.8%<sup>71</sup>. Dimson et al. (2000) describe a similar effect by noting that equity indices are often constructed with *hindsight* in the sense that only stocks of those equities are included in the index that are known to have existed (survived) throughout the entire or most part of the sample. Thereby losses on investments in companies that did not survive are excluded from the reported returns, obviously causing an upward bias.
- From Table 16.3 and Figure 16.1 it can be seen that equities have known long periods of both higher and lower total returns than their long term average. For example during the first half of the postwar period returns were lower while during the second half of the postwar period they were (much) higher than their long term averages. In general such periods are known as *secular bear markets* and *secular bull markets*. See for example Van Duijn (1999). The last secular bull market is supposed to have started around 1980.
- From Figure 6.1 it is immediately clear that besides such long term swings, equities are also to a large extent driven by very volatile *shorter term fluctuations*. The long term *standard deviations* of the real returns reported in Table 16.3 are

<sup>70</sup> As a simple example of this volatility effect consider the case where a historical average equity return has been estimated on 30 years of data. Now suppose an extra annual return of +30% is added to the sample. Because of this single observation the estimated average return already increases by one full percentage point (i.e.  $30\%/30 \text{ years} = 1\%$ ).

<sup>71</sup> Siegel (1999) rightfully notes that by itself this finding does not say anything about the value of the important concept of the *equity risk premium*, which is defined as the *difference* between the expected return on equities and treasury bills or government bonds. We say more on the equity risk premium in section 16.2.1.

remarkably consistent at around 17% in each of the three countries<sup>72</sup>. The high volatility of more than 30% for UK equities during the first half of the postwar period is mostly due to the exceptional returns in 1974 (minus 60%) and 1975 (plus 100%). Without just these two years the standard deviation reduces to approximately 22% for the first half of the postwar period and 18% for the entire postwar period. Upon visual inspection the volatility of both the postwar NL and US returns are to some extent determined by a long term swing in the returns during that period. As we shall see in the next section removing these long term fluctuations brings the postwar standard deviation for the NL closer to a value of around 17%. For the US this reduces the volatility even further. Especially for the postwar period it seems that US equities have experienced a lower volatility than the other two countries. A possible cause for this might be a higher degree of diversification in the US due to the sheer size of the market. The *correlations* of the real equity returns with the growth of national product are close to zero or slightly negative, suggesting an acyclical relation between these variables.

- In order to further investigate the properties of the equity return *distribution function*, also higher moments in terms of the *skewness* and the *kurtosis* are estimated besides the average and standard deviation. These are reported in Table 16.2 together with the result of the Jarque-Bera test for Normally distributed returns. These results show that *annual* real equity returns are mostly symmetrical distributed (i.e. have a skewness of approximately zero) and do not show fatter or thinner tails than a Normal distribution (i.e. have a kurtosis of approximately three). The Jarque-Bera test clearly does not reject the null-hypothesis of Normality for most sub-periods and countries. The most important exceptions are several (overlapping) periods for the UK but these are all very likely due to the extreme 1974 and 1975 returns which cause fatter tails.

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<sup>72</sup> Just as average returns, also standard deviations of equity returns are hard to estimate on small samples of data because of the large volatility.

Table 16.1 Statistics of return on real Total Return equity indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to (the growth of) the National Product index. *Avg* indicates the arithmetic annual average and *Geo* indicates the geometric annual average.

Netherlands						United Kingdom						United States					
	Avg	Stdev	Geo	Ratio	Corr		Avg	Stdev	Geo	Ratio	Corr		Avg	Stdev	Geo	Ratio	Corr
1824-1999	8.0%	17.0%	6.5%	-	-	1801-1999	6.2%	16.1%	5.0%	-	-	1871-1999	8.4%	17.4%	6.8%	3.2	0.1
1871-1999	8.2%	17.1%	6.8%	2.3	0.0	1856-1999	6.4%	17.2%	5.0%	5.0	-0.1	1871-1999	6.8%	13.7%	5.5%	3.0	0.1
1871-1913	7.7%	10.7%	7.0%	2.2	-0.2	1856-1913	4.3%	5.8%	3.9%	1.7	0.0	1871-1913	11.6%	24.1%	8.6%	3.6	0.2
1919-1939	7.1%	19.6%	3.9%	3.6	0.5	1919-1939	9.6%	16.4%	7.5%	3.6	0.1	1919-1939	10.1%	16.1%	8.6%	6.8	-0.1
1950-1999	10.7%	20.5%	9.2%	9.1	-0.1	1950-1999	10.9%	23.9%	8.1%	12.2	-0.2	1950-1999	8.7%	18.5%	6.3%	6.8	-0.1
1950-1975	6.4%	21.8%	4.6%	9.2	0.1	1950-1975	9.6%	30.5%	4.7%	14.9	-0.2	1950-1975	11.5%	12.9%	10.0%	6.8	-0.2
1976-1999	15.3%	17.9%	14.1%	12.0	0.0	1976-1999	12.4%	13.3%	12.1%	7.6	-0.1	1976-1999					

Table 16.2 Further statistics of return on real Total Return equity indices for various sub-period to test whether the returns satisfy a Normal distribution. For a Normal distribution the *Skewness* and *Kurtosis* are respectively zero and three. *JB* is the Jarque-Bera test statistic which has a Normal distribution as the null-hypothesis. *Prob* is the probability of this test statistic exceeding the value in the *JB* column under the null-hypothesis (Chi-Square distribution with two degrees of freedom). If this probability is greater than the desired confidence level, the null-hypothesis of the returns satisfying a Normal distribution cannot be rejected.

Netherlands					United Kingdom					United States				
	<i>Skew</i>	<i>Kurt</i>	<i>JB</i>	<i>Prob</i>		<i>Skew</i>	<i>Kurt</i>	<i>JB</i>	<i>Prob</i>		<i>Skew</i>	<i>Kurt</i>	<i>JB</i>	<i>Prob</i>
1824-1999	0.3	3.2	4.0	14%	1801-1999	0.9	9.5	375.2	0%	1871-1999	-0.1	2.9	0.3	87%
1871-1999	0.4	3.5	4.9	9%	1856-1999	1.0	9.9	311.6	0%	1871-1913	0.1	3.5	0.6	75%
1871-1913	-0.7	6.3	23.0	0%	1856-1913	0.8	3.3	6.3	4%	1919-1939	-0.3	1.9	1.2	54%
1919-1939	0.4	2.9	0.5	77%	1919-1939	0.2	3.5	0.3	84%	1950-1999	-0.2	3.0	0.3	85%
1950-1999	0.4	2.5	1.5	46%	1950-1999	0.5	6.5	27.5	0%	1950-1975	-0.1	2.9	0.0	98%
1950-1975	0.7	2.9	2.1	35%	1950-1975	0.7	4.7	5.1	8%	1976-1999	-0.3	2.1	1.1	57%
1976-1999	0.1	2.3	0.5	76%	1976-1999	-0.7	2.4	2.1	35%					

### 16.2.1 Capital growth and dividend yields

- We know that total returns on equities consist of two components: the *capital growth* and the *dividend yield*. Although they are not analyzed elaborately, it is interesting to get at least some idea of how these two constituents influenced the total equity return. To do this, separate series of equity price indices and dividend yields are used, which are described in Appendix F. With the exception of the UK these separate series were also used to construct the total rate of return indices analyzed in this chapter. Tables 16.3 and 16.4 contain some statistics for respectively the *real* capital growth and the dividend yields for various sub-periods in each of the three countries. From these tables we can observe several things.
- The first observation is that the *dividend yields* are very consistently around an average value of 4 to 5% with a small standard deviation of say 1.5%. That is, 95% of the dividend yields have been between approximately say  $4.5\% - 2 \times 1.5\% = 1.5\%$  and  $4.5\% + 2 \times 1.5\% = 7.5\%$ . The relatively high standard deviations for the prewar and interwar periods in the NL are probably due to a poor quality of the data. Such average values of dividend yields are consistent with conventional knowledge. For example Arnott and Bernstein (2002) report an average dividend yield in the US for the 1802-2001 period of 4.9%.
- The second observation is that most of the *volatility* of the total returns comes from the capital returns. The standard deviations of the capital returns are virtually identical to those of the total returns reported in Table 16.1. Together with the only small volatility of the dividend yields this implies that total rate of return indices and *dividend indices* follow a pattern very similar to the equity *price indices*. This is illustrated in Figure 16.2 for each of the three countries analyzed. Note that this observation renders a separate analysis of the dynamics of the equity price indices and dividend indices unnecessary. By studying (in principle) only the total return indices we are therefore able to obtain all the relevant information about the behavior of equities.
- The third observation is that in the very long run, as a rough indication, the *nominal* (arithmetic) total rate of return consists for 50% of the nominal capital growth and for 50% of the dividend yield. To see this note that the average inflation rates from Table 10.1 for the longest possible samples for the NL, UK and US are respectively 1.5%, 2.7% and 2.3%. Together with an average real total return of approximately 7% these result in an average nominal return of approximately 9% which is about double the average dividend yield of some 4 to 5%. This shows at least that the dividend yields play a very important part in constituting the long term total return on equities which is contrary to the widespread belief that capital gains are the most important source of equity return. Perhaps this belief is founded on the very high average rates of capital growth observed during the postwar period and especially during the second half of this period. The geometrical average capital appreciation for the US is 1.7%, which is consistent with for example Arnott and Bernstein (2002), who report a 20-fold increase from 1802 price levels which means an annual geometric return of 1.5%. The finding that real equity capital returns are much lower than dividend yields and sometimes even zero while the total return index increased rapidly is also found by Dimson et al. (2001). Note that the marked differences between the

terminal wealth of the total return indices and the equity price indices in Figure 16.2 illustrate this importance of the reinvestment of dividends with respect to long term equity behavior.

Figure 16.2 Logarithms of real Total Return, Price and Dividend indices.

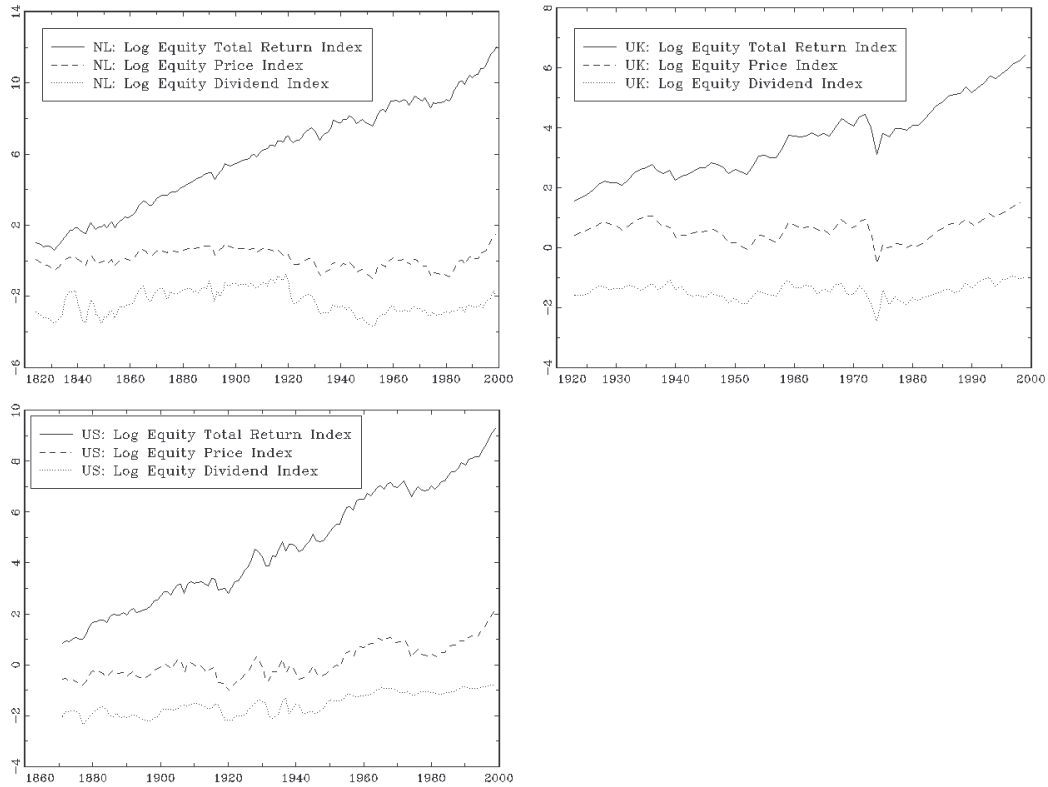




Table 16.3 Statistics of real equity Capital Growth for various sub-periods. *Avg* indicates the arithmetic annual average and *Geo* indicates the geometric annual average.

Netherlands			United Kingdom			United States					
	Avg	Stdev	Geo		Avg	Stdev	Geo		Avg	Stdev	Geo
1817-1999	2.6%	16.7%	1.2%	1801-1999	2.2%	17.3%	0.8%	1821-1999	3.3%	17.8%	1.7%
1871-1999	2.2%	17.3%	0.8%	1871-1999	2.3%	17.5%	0.7%	1871-1999	3.9%	18.1%	2.2%
1817-1913	2.4%	13.3%	1.5%	1801-1913	1.7%	14.2%	0.8%	1821-1913	2.0%	15.7%	0.7%
1919-1939	-0.7%	19.5%	-3.2%	1919-1939	4.6%	15.6%	2.8%	1919-1939	6.3%	25.3%	3.1%
1950-1999	6.5%	20.7%	4.8%	1950-1999	5.7%	22.8%	3.0%	1950-1999	6.5%	16.5%	5.0%
1950-1975	2.4%	21.9%	0.6%	1950-1975	4.1%	29.0%	-0.3%	1950-1975	5.0%	18.6%	2.7%
1976-1999	10.9%	18.4%	9.8%	1976-1999	7.3%	12.8%	7.3%	1976-1999	8.1%	13.6%	6.7%

Table 16.4 (arithmetic) Average and standard deviation of equity Dividend Yield for various sub-periods.

<i>Netherlands</i>			<i>United Kingdom</i>			<i>United States</i>		
<i>Avg</i>	<i>Stddev</i>		<i>Avg</i>	<i>Stddev</i>		<i>Avg</i>	<i>Stddev</i>	
1824-1999	5.7%	2.5%	1923-1999	4.8%	1.0%	1871-1999	4.5%	1.4%
1824-1913	5.6%	2.0%				1871-1913	4.7%	1.0%
1919-1939	7.8%	2.9%	1923-1939	4.5%	0.7%	1919-1939	5.2%	1.5%
1950-1999	4.4%	1.3%	1950-1999	4.9%	1.0%	1950-1999	3.7%	1.2%
1950-1975	4.2%	0.8%	1950-1975	5.2%	1.0%	1950-1975	3.9%	1.1%
1976-1999	4.6%	1.7%	1976-1999	4.6%	1.0%	1976-1999	3.6%	1.3%



- In their article Arnott and Bernstein (2002) also report a strong similarity between the real growth in share prices and the real *per capita* growth of national product. The intuition here is that share holders participate in the *productivity* growth of the enterprises they are investing in. That they cannot benefit from the *total* growth of the national product is because a substantial part of economic growth comes from new enterprises in which cannot be invested yet or which do not even exist yet. The authors do not explain how the substantial dividend yields fit into this explanation<sup>73</sup>. After all, they are a part of the total return earned by share holders. Nevertheless we checked the link between real national product growth and equity price changes as found by Arnott and Bernstein (2002) for the US data analyzed here and extended the analysis the NL and UK data. The results are shown in Figure 16.3 and Table 16.5. These confirm that in the US the real long term per capita growth of national product indeed matches nicely with the long term geometrical real return on an equity price index. However the results also show that this finding cannot be generalized to the NL and UK situations. In these countries, in real terms, equity prices only grew half as much as per capita national product. Especially during the 20<sup>th</sup> century equities stayed far behind. What is striking however from the second and third line in Table 16.5 is that the long run *arithmetic* average change in stock prices is very close to the long term *total* real growth of national product in each of the three countries. Apparently the long term expected annual increase in real stock prices is approximately the same as the expected total real economic growth<sup>74</sup>. From Table 16.3 and Figure 16.2 we know however that equity capital gains can be much higher and lower than the expected economic growth for several decades in a row. The fact that, as observed for the NL and the UK, the *geometric* average equity capital gain is lower than the national product growth than seems to be a direct result of the much higher volatility of the equity prices (i.e. the *volatility drain* as described in section 16.1). This complex issue of the link between economic growth and equity returns definitely requires further research.

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<sup>73</sup> An explanation with considerable intuitive appeal is that the real capital growth is a direct result of the participation in the growth of the economy, while the dividend yields (profits) are the compensation for the business risks taken by investing in some business enterprise (i.e. the *equity risk premium* to be discussed in section 16.2.1). Also note that, as indicated in section 13.2.2, the real interest rate is the compensation for the opportunity costs incurred by not being able to participate in the economic growth. This participation in economic growth might very well be seen as investing in equities. This assertion is supported by the remarkable resemblance between the arithmetic equity capital growth and the ex post real interest rates in Table 16.5. Of course the real interest rate does not contain some (equity) risk premium. Please note the resemblance between these possible empirical relations and the familiar Dividend Discount models for equity prices in which the total return on equities is equal to the real interest rate + dividend yield + inflation.

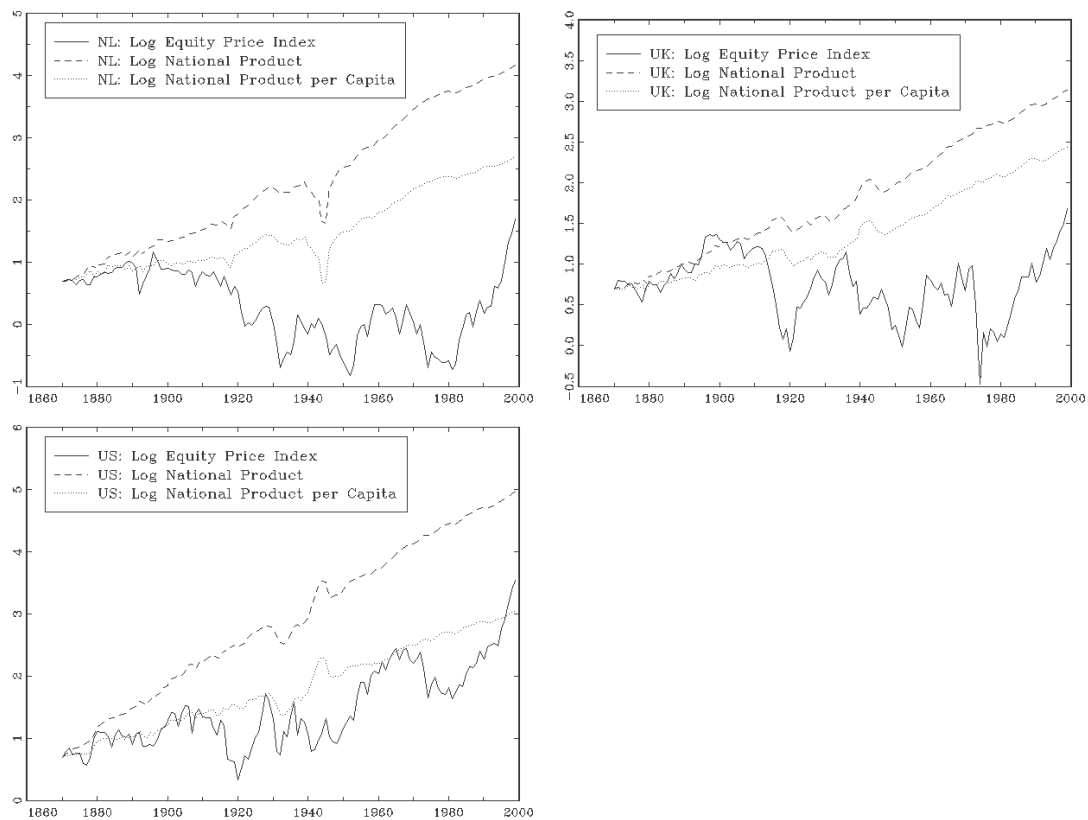
<sup>74</sup> That real long term equity capital growth is comparable to the real growth of the *total* economy instead of only the *per capita* growth might be explained by the fact that in general not all corporate earnings are paid out as dividend to the shareholders. These retained earnings might be used for the acquisition of young enterprises enabling the shareholders to benefit from the growth of new business enterprises as well.

Table 16.5 Long term average real equity Capital Growth and growth of real total and per capita National Product and ex post real long Interest Rates for 1870-1999.

	<i>Netherlands</i>	<i>United Kingdom</i>	<i>United States</i>
Equity Prices (geometric)	0.8%	0.7%	2.2%
Equity Prices (arithmetic)	2.2%	2.3%	3.9%
Total National Product*	2.7%	1.9%	3.3%
Per capita National Product*	1.6%	1.4%	1.8%
Real long Interest Rate	2.3%	2.3%	2.4%

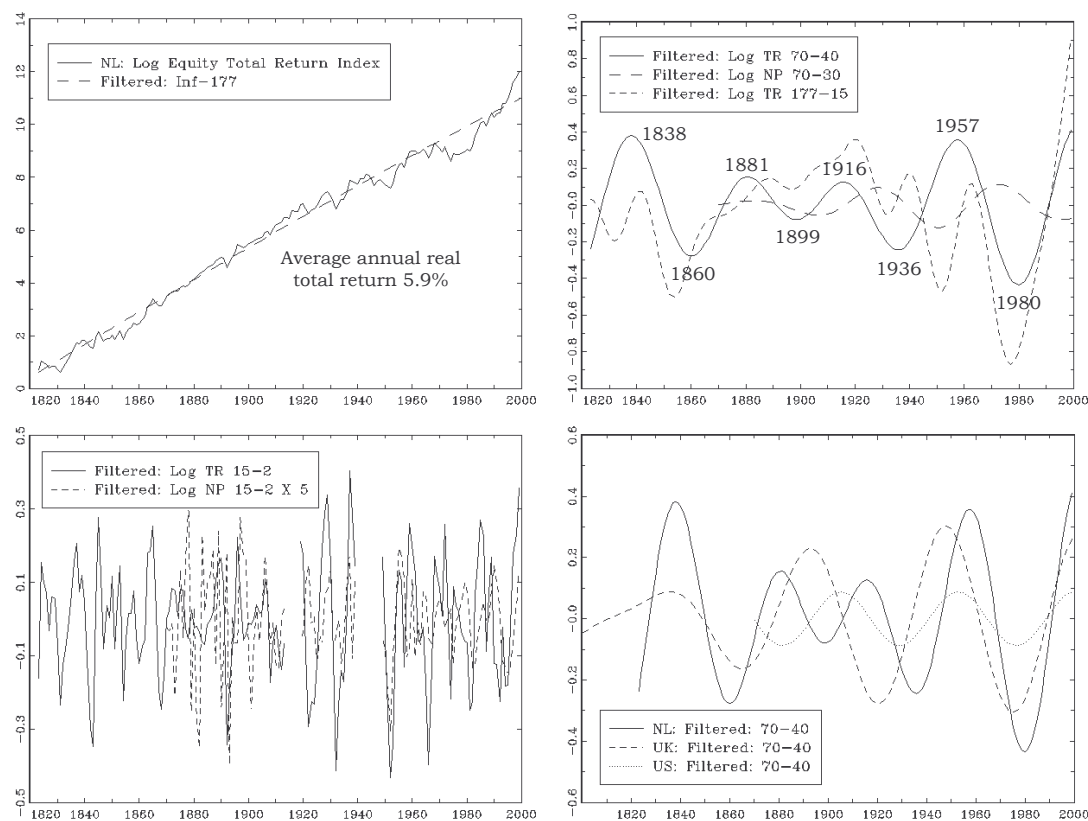
\* For these series the arithmetic and geometric averages are virtually identical because of the low standard deviations.

Figure 16.2 Logarithms of real equity Price indices for the NL, UK and US together with corresponding total National Product and per capita National Product indices. All series start from the same value to facilitate the comparison of growth rates.



### 16.3 Filtering results

Figure 16.4 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a real Total Return equity index for the NL and its long wave component compared to UK and US data (bottom right).



- In each of the three countries the *trend component* closely resembles a logarithmic linear trend with an average growth that approximates the *geometric* average returns from Table 16.1. The trend average growth is in general between a half and one full percentage point lower than the geometric average return. This is mostly due to the very high levels of capital gains at the very end of the sample, starting around 1980.
- In general the *long wave component* does a rather poor job at capturing all the fluctuations in the *periodic component*. It is clear however that the *volatility* of the long wave component in equities is approximately five times as large as that in the national product. Also the pictures and the negative long wave correlations in Table 16.6 indicate that the long wave in equities is clearly out of phase with that in the national product. Table 16.8 shows that the *lead* is rather consistently some 15 to 20 years. This can for example be seen from the final trough in all equity indices around 1980, while in the national product indices this trough is only dated somewhere during the 1990's. With an average peak-to-peak and trough-to-trough *period length* of respectively 38 and 39 years this implies a lead of between a quarter and half a cycle. Van Duijn (1999) states that secular bull markets in general already begin during the recovery phase of the long wave in the real economy. The intuition behind this long wave lead of equities is that stocks are discounting mechanisms of the future. During recovery phases new

technological innovations lay the foundations for a new long period of prosperity. Equities anticipate on these developments in the real economy. For example Hobijn and Jovanovic (2000) claim that the IT revolution was already anticipated as early as 1973 and was the driving force behind the most recent secular bull market. Furthermore, they argue that stock prices are likely to decline and then rise before any major technological shift. The finding of long term fluctuations in equities is consistent with for example Siegel (1998) and Bernstein (1997) who both find evidence of long term mean reversion of real stock returns, while this is clearly not the case for (nominal) bonds, which in fact show mean *aversion* because their real return is heavily influenced by inflation and inflation cumulates over time. As early as Rhea (1932), economists already observed alternating patterns of bull and bear equity markets. Finally note that if the observed pattern of long wave fluctuations in equity prices repeats itself in the future, the long waves in Figure 16.3 show that the “current” secular bull market may very well reach its end somewhere in the first decade of the 20<sup>th</sup> century.

- Due to the large volatility it is more difficult to see the familiar dating of business cycle peaks and troughs in the *stochastic components* of the equity indices than it is to see them in variables from the real sector of the economy such as industrial production and employment rates. Nevertheless the in general positive *correlations* in Table 16.6 of approximately 0.3 with the business cycle fluctuations in the national product show that “underneath” the volatility there indeed is some connection. The negative correlations for the first half of the postwar period in the UK are probably the result of the extreme equity returns in 1974 and 1975. Note that these positive correlations for the stochastic components are very different from the zero or negative correlations from the conventional data analysis reported in Table 16.1. One possible explanation for this might be the observed negative long wave correlations because of the lead of the long wave in equities over the national product. To some extent these long term movements are still present in the conventional growth and return rates, causing a lower correlation while they are completely eliminated from the stochastic components. Besides the apparently high correlation between equities and the national product business cycle, a visual inspection of the stochastic components also suggests a slight *lead* of equities over the national product. This will have to be confirmed by the spectral analysis in the next section.
- The finding of a link between equities and business cycle fluctuations in the real sector of the economy should come as no surprise. After all, business cycle fluctuations are especially observed in variables related to the work organized in business enterprises. Stocks reflect the ownership of a part of these enterprises and are therefore at a very fundamental level related to business cycle fluctuations. Another way of looking at this connection is that future dividend payouts constitute the value of stocks while the dividends are determined by the profits made in the enterprises. These profits in turn are directly related to economic growth and hence, to the business cycle. Fama (1990) and Schwert (1990) formally confirm that stock prices are positively correlated to (future) economic activity by means of statistical regressions. Fama (1990) also finds that variables that reflect expectations on production and dividends explain approximately 43% of the total variance of equity returns. Siegel (1999) demonstrates the link between equities and the business cycle in another way.

For the US he very consistently finds that by switching between stocks and bonds a few months before the NBER dated business cycle peaks and troughs, an investor can outperform a buy and hold investment strategy by as much as 4 to 5% on an annual basis. Note that if investors would be able to perfectly predict the business cycle and, related to that, the dividend payments they will receive and interest rates would not change, stock prices would not change with the business cycle. The fact that they do so, shows that predicting remains a difficult exercise and that interest rates also change with the business cycle.

- The conclusions with respect to the *volatility* of the stochastic component are not that clear. With the exception of the turbulent interwar period, the standard deviation for the US is consistently around 10%. The sometimes much higher standard deviations for the conventional US returns as found in the data analysis were apparently caused by the long term fluctuations in equity prices rather than by their business cycle fluctuations. For the UK the standard deviations are around 10% most of the times as well. Exceptions are the first half of the postwar period with a large volatility, probably caused by the extreme 1974 and 1975 returns<sup>75</sup>, and the second half of the prewar period with an exceptionally low standard deviation of only 4%. This last aberrant behavior can already be observed for the original index in Figure D.21 of Appendix D. This probably has something to do with a poor quality of the data underlying the constructed index. The low standard deviations are also observed for the prewar NL data. The postwar NL data however show consistently higher standard deviations of around 18%. In short, both the US and the UK stochastic equity component seem to have a standard deviation of around 10% while it is significantly higher at 18% for the postwar NL series. Supposing a 10% standard deviation for equities and a typical 2% standard deviation for the national product it follows that business cycle fluctuations in equities are approximately five times as volatile as those in the national product. Such a ratio is also mentioned by Van Duijn (1999).
- From Table 16.7 it is clear that the business cycle fluctuations in equities describe virtually the total *volatility* of the conventional total returns. So by focussing on the stochastic components of equity indices almost no variance is lost. Note that the historical standard deviations reported in this table are based on the delta logarithmic approximation of the equity returns and are therefore (slightly) different than the standard deviations of the actual returns reported in Table 16.1.

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<sup>75</sup> Without the 1974 and 1975 values, the postwar standard deviation of the UK stochastic component reduces from 18.7% to 15.8% while for the first half of this period it reduces from 24.1% to 20.5%.

Table 16.6 Statistics of the filtered periodic, long wave and stochastic components of real Total Return equity indices for various sub-periods. The (standard deviation) ratio and the correlation numbers hold relatively to the (filtered) National Product index.

	Netherlands			United Kingdom			United States							
	Stdev	Ratio	Corr	Stdev	Ratio	Corr	Stdev	Ratio	Corr					
Periodic	1870-1999	33.7%	2.7	-0.3	1855-1999	42.7%	5.4	0.6	1870-1999	27.7%	2.9	0.1		
Long Wave	1870-1999	20.1%	3.2	-0.5	1855-1999	18.6%	7.9	-0.6	1870-1999	6.1%	4.5	-0.4		
Stochastic	1800-1913			8.4%	-	-								
	1800-1854			11.3%	-	-								
	1855-1913			3.9%	1.6	-0.1	1870-1913	10.2%	3.0	0.4	0.4			
	1919-1939			12.4	0.3	0.3	1919-1939	23.0%	5.1	0.2	0.2			
	1949-1999			17.9%	9.1	0.4	1949-1999	18.7%	8.8	-0.1	1950-1999	10.5%	4.7	0.3
	1949-1975			18.9%	8.8	0.4	1949-1975	24.1%	14.5	-0.3	1950-1975	11.6%	5.6	0.2
	1976-1999			16.7%	9.6	0.3	1976-1999	9.0%	3.6	0.2	1976-1999	9.1%	3.8	0.4

Table 16.7 Standard deviation of growth (delta logarithm) of real Total Return equity indices for various sub-periods (Hist) versus standard deviation of first differences of filtered stochastic components (Filt).

Netherlands				United Kingdom				United States			
	Filt	Hist	Ratio		Filt	Hist	Ratio		Filt	Hist	Ratio
1823-1913	13.0%	13.5%	1.0	1800-1913	8.9%	9.5%	0.9	1870-1913	12.7%	13.1%	1.0
1919-1939	17.9%	18.0%	1.0	1919-1939	12.2%	13.9%	0.9	1919-1939	23.7%	23.6%	1.0
1949-1999	17.4%	18.5%	0.9	1949-1999	22.5%	23.2%	1.0	1950-1999	13.9%	15.4%	0.9
1949-1975	19.5%	19.8%	1.0	1949-1975	29.0%	29.6%	1.0	1950-1975	16.3%	17.9%	0.9
1976-1999	14.7%	15.7%	0.9	1976-1999	11.8%	12.5%	0.9	1976-1999	10.7%	11.9%	0.9

Table 16.8 Peaks and troughs of the long wave component of real Total Return equity indices. The *Italic* numbers are the peak-to-peak and trough-to-trough periods to estimate the period length. The reported leads are the leads over the relevant peak or trough in the corresponding National Product index (to the extent that the “same” peaks and troughs are present).

	<i>1</i>					<i>5</i>					<i>Peak-Peak Trough-Trough Lead</i>
Netherlands	1838	43	1881	35	1916	41	1957				40
		1860	39	1899	37	1936	44	1980			40
			2	8	12	14	15	16			11
United Kingdom			1836	57	1893	55	1948				56
											55
			20	18	17	18	20	24			20
United States											48
											48
											16
Average	1838	21	1859	46	1905	48	1953				38
		1860	22	1882	47	1928	49	1977			39
			11	12	14	16	19	21			15



### 16.3.1 Equity Risk Premium

- Before turning to the spectral analysis of the filtered stochastic components in the equity indices, this sub-section pays specific attention to a related concept that many finance professionals and financial economists regard as the single most important number in finance: the *equity risk premium*. Formally, the equity risk premium is the *expected* extra return obtained by investing in equities rather than in some “risk-free” asset, typically treasury bills but sometimes also government bonds. The equity risk premium is the long term compensation for the (short term) business risk, as incurred by investing in some business enterprise. Sometimes people speak of the equity risk premium in terms of the difference between equity and say bond returns in one single year. Strictly speaking this is not correct. In each separate year the difference between equity and bond returns can be very different from the expected long term behavior because of the high volatility of equities observed in the previous sections. Here the focus will therefore be on the *long term* aspects of the equity risk premium. Furthermore, because we do not have any bond indices directly available, we define the equity risk premium as the expected difference between the nominal total return on equities and the nominal long term interest rate. Note that, if in the long run interest rates do not change, the long term return on some bond portfolio should correspond to the long term average interest rate.
- Table 16.9 shows the average *arithmetic* equity risk premium calculated for various sub-periods for the three countries analyzed<sup>76</sup>. Note that the equity risk premium is a “ratio” and is therefore independent of currency fluctuations and inflation, which is why the risk premiums across the three countries are directly comparable. The *arithmetic* average risk premium is preferred over the *geometric* average because the first gives a better representation of the expected one year extra return of equities over bonds. Together with the annual volatility the arithmetic average determines the geometric average. Because of the very high volatility of the equity returns, and hence of the one year “risk premiums”, statistical sound estimates of the equity risk premium require as long samples as possible. Even with several centuries of data the estimates still contain considerable uncertainty. For this, note the example given by Jorion and Goetzman (1999) as already mentioned in section 16.1. Assuming an average risk premium of 3% with a 20% annual standard deviation implies that already at least 178 observations are needed to even tell whether the estimated average risk premium statistically differs from 0% at a 95% confidence level. The results show that for the longest samples the arithmetic risk premium is between 3 and 6% per annum. In geometric terms, over some bond index, this will be 1 to 2% lower.
- Mehra and Prescott (1985) introduced the *equity premium puzzle* by showing that the historical difference between the average return on equities and the risk free rate is too great to be explained by the consumption version of the Capital Asset Pricing Model (CAPM), unless one would assume unrealistic high levels of risk aversion for economic agents. Since that time, many have tried to solve this

<sup>76</sup> Because of the possible errors due to the large volatility, just as in section 16.1 these estimates use the *actual returns* instead of the *approximate returns* obtained by the first order differences of the natural logarithms of the equity index.

puzzle by reexamining historical data for the equity risk premium<sup>77</sup>. Table 16.10 summarizes some empirical findings of the equity risk premium from this line of research. When comparing the long term results in Table 16.9 with those in Table 16.10 it is fair to say that the long term historical estimated risk premiums for the NL and the US are very consistent with those found by others. The UK estimate of only 3.3% seems rather low. However this seems to be mostly caused by the 19<sup>th</sup> century data. The UK risk premium for 1801-1869 is only 1.4%, while it is 4.4% for the 1870-1999 period. For the 1900-1999 period it is 5.2%, which is much more in line with the other countries. The overall conclusion seems to be an ultra long historical *arithmetic* equity risk premium over bonds of 5 to 6%. Assuming a 17% annual standard deviation (see Table 16.1) this implies an approximate 4% *geometric* risk premium<sup>78</sup>.

Table 16.9 Average (ex post) equity Risk Premium for various sub-periods as the (arithmetic) average of the nominal Total Return minus the Long Interest Rate.

<i>Netherlands</i>		<i>United Kingdom</i>		<i>United States</i>	
<i>Avg</i>		<i>Avg</i>		<i>Avg</i>	
1824-1999	5.2%	1801-1999	3.3%	1871-1999	6.2%
1900-1999	6.8%	1900-1999	5.2%	1900-1999	7.2%
1824-1913	3.7%	1801-1913	1.0%	1871-1913	3.6%
1919-1939	2.3%	1919-1939	1.4%	1919-1939	7.4%
1950-1999	8.3%	1950-1999	9.4%	1950-1999	7.9%
1950-1975	5.8%	1950-1975	8.9%	1950-1975	7.7%
1976-1999	11.1%	1976-1999	10.0%	1976-1999	8.2%

Table 16.10 Empirical findings of the equity risk premium reported in the literature.

<i>Source</i>	<i>Country</i>	<i>Sample</i>	<i>Versus</i>	<i>Arithmetic</i>	<i>Geometric</i>
Bernstein (1997)	US	1871-1995	Bonds	-	5.3% / 3.3%*
Siegel (1999)	US	1802-1998	Bonds	4.7%	3.5%
Dimson et al. (2000)	12 countries	1900-2000	Bonds	7.4%**	5.0%
Ibbotson (2001)	US	1926-2000	Bonds	-	5.7%
Eichholtz et al. (2002)	NL	1900-2001	Bonds	5.3%	-

\* The first / second number is with / without the upward revaluation of equities.

\*\* Estimated from their results by subtracting the reported difference between the geometric risk premium over bills (5.7%) and bonds (5.0%) from the reported arithmetic risk premium over bills (8.1%).

<sup>77</sup> This is an example of the rather awkward methodology, often encountered in macroeconomics, as discussed in Chapter 2. A thorough analysis of the data is in this case inspired by a theoretical model while this should be exactly the other way around. Models should be build and tested to try an explain observed empirical behavior as best as possible.

<sup>78</sup> Note that these values of the equity risk premium are very similar to the dividend yields reported in section 16.1.1. This confirms the idea suggested in section 16.1.1, that it are the dividend yields that form the compensation for the risk taken by investing in business enterprises and that the real interest rate and the real equity capital growth resemble the long real rate of economic growth.

- Given the fact that the filtering results revealed long term fluctuations in both long interest rates and equity indices, we might expect that also the equity risk premium is driven by *long term fluctuations* around the ultra long term averages. Indeed, the historical statistics in Table 16.9 show some long periods with both a relatively high and relatively low equity risk premium. Especially the second half of the postwar period shows very high risk premiums of some 8 to 11%. The risk premiums for the entire postwar period are very consistently around 8% which is similar to Siegel (1999) who reports a 7.3% arithmetic risk premium for the US during the 1946-1998 period. In order to further investigate such possible long term fluctuations in the equity risk premium we filtered the (ex post) difference between the nominal equity returns<sup>79</sup> and the long interest rates for the long term fluctuations in the familiar [1/70,1/30] frequency range<sup>80</sup>. The lower frequency segments basically contain the long term average of the risk premium, while the higher frequency ranges are not very informative because the risk premium by definition is related to the long term, low frequency, fluctuations in the difference between equity returns and long interest rates. The filtering results in Figure 16.5 and Table 16.12 show a remarkable resemblance between the long wave fluctuations in the equity risk premium in the three countries with respect to both the dating of the peaks and troughs and the amplitude. As was to be expected from the observed long wave lead of equities over the national product, also the equity premium long wave consistently shows a substantial *lead* over (the level of) the national product. This lead is some 15 to 20 years which by the phase of the first order differencing operator (see section 5.2.1) implies, that the equity risk premium leads the *growth* of the national product by some 5 to 10 years. At first sight it may seem a little strange that both the real equity index and the equity risk premium lead the level of the national product by approximately 15 years. For this compare Tables 16.8 and 16.12. After all, the risk premium is related to the *returns* which by definition lead the *indices* by approximately a quarter cycle. We should however not forget that the risk premium is defined as the *difference* between the *nominal* equity returns and the long interest rates. That is, besides the long wave in the real equity indices, also the long wave in inflation rates and in the long (real and nominal) interest rates influence the long wave in the equity risk premium. The *amplitude* of the long waves is approximately 5%, which is pretty consistent with the differences between the averages for the various sub-periods in Table 16.9. Note that the samples used there almost never fall exactly around the peaks or troughs of the long wave fluctuations. Therefore it is no surprise that the average risk premiums reported in the table show less extreme fluctuations than the filtering results. With an (arithmetic) average long term risk premium of 5 to 6% this implies that there are times when the risk premium is around 10% while there are also times when it is approximately 0%, depending on the phase of the long wave. The average peak-to-

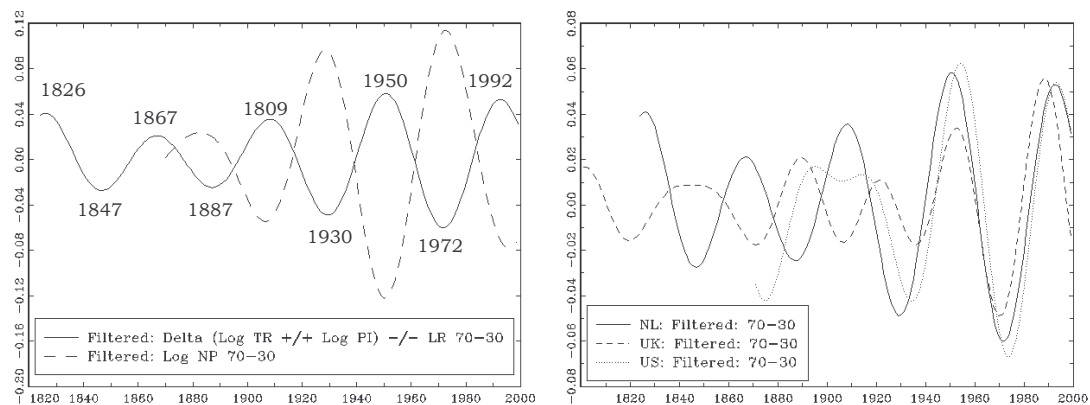
<sup>79</sup> Here the first order differences of the logarithmic index were used as an approximation of the (nominal) returns. We checked that the use of the actual returns leads to practically identical results for the filtered long wave components. The reason is that the large volatility of the ex post risk premiums is high frequency volatility, which is therefore excluded from the filtered long wave components anyway.

<sup>80</sup> Just as with the analysis of the real interest rates in section 13.2.2, an alternative to this *direct* filtering of the risk premiums would have been the *indirect* approach of constructing the risk premium long waves from transformations of the long waves in respectively the equity indices, consumer price indices and long interest rates. For the sake of robustness, the direct approach was here while at the same time we checked that the indirect approach leads to virtually identical results.

peak and trough-to-trough *period lengths* are respectively 42 and 46 years. To be complete, Table 16.11 gives the usual statistics for both these long wave components as well as the *periodic components* in the ex post equity risk premium.



Figure 16.5 Long wave component in the ex post equity Risk Premium versus long wave in the National Product index for the NL (left) and a comparison to the long wave component in the equity Risk Premium for the UK and the US (right).



- It is interesting to say the least that the end of the filtered long waves in the equity risk premium in each of the three countries indicates that the decline of the high risk premium of the secular bull market that started around 1980 has already set in around the end of the 1990's. Extrapolation of the risk premium long waves in the NL and US suggest that starting somewhere towards 2010 the risk premiums will become below average. Inspired by the historically very high valuation levels of equities (price-earning ratios of say thirty while the long term average is around fifteen) recently many researchers suggest a rather low *forward looking* equity risk premium. Siegel (1999) for example states that historical risk premiums are very unlikely to apply to the near future because of the (at that time) very high valuation levels. Arnott et al. (2001) expect real equity returns of between 3 and 4% and thereby an equity risk premium of approximately 0%. Also Claus and Thomas (2001) forecast substantially lower risk premiums, while Fama and French (2000) show that the exceptionally high risk premiums observed during the postwar period were probably not expected beforehand (and should therefore also not be expected for the future?). Finally, based on the empirical length of the long wave and business cycles, Van Duijn (1999) predicts that the "current" bull market will end around 2008.
- In the light of all these low (academic) expectations it may seem rather surprising that in a survey of over 200 financial economists, Welch (2000) finds that most of them estimate the arithmetic equity premium for the next thirty years at about 7%. A pessimistic outcome would be an equity premium of 2 to 3%, while the consensus regarding an optimistic outcome is a 12 to 13% equity risk premium. Dimson et al. (2000) shows that both the expected risk premium and the uncertainty around it are exactly the same as arises from the statistical analysis of historical (US) returns. That is, professionals heavily base their forward looking estimates on what (the same) historical data tell them. Given that still very little is known about the behavior of the equity premium and no way has been found to robustly predict the equity premium out of sample, this is no surprise. Also see Goyal and Welch (1999). Note that the extremes of 2 to 3% and 12 to 13% are rather consistent with the amplitude of approximately 5% of the equity premium long waves in Figure 16.5.

- Despite all the pessimistic views on the future equity risk premium, Dimson et al. (2000) conclude that also in the 21<sup>st</sup> century investments will remain risky because business itself is risky and that because of this inherent risk, equity investors should continue to expect a positive risk premium. However, this risk premium will very likely be lower than the estimates based on the (recent) historical periods. The long wave results found here suggest that after the secular bear market to come, there will also be a secular bull market again with an above average risk premium.



## 16.4 Spectral analysis

### 16.4.1 Static

- The results in Table 16.13 and Figure 16.6 clearly show that also the short term fluctuations in equities are characterized by both Juglar and Kitchin type business cycle fluctuations. Given the relatively high correlations with the national product business cycle components as found in Table 16.6 this is not very surprising. Also see section 16.2 for more background on the link between equities and the business cycle. An important difference with the variables analyzed in the previous chapters is however, that the *Kitchin* type fluctuations are relatively more important here. This assertion is based on the following three observations. First, contrary to what we saw for most of the other variables, the Juglar peak is not *significant* in all sub-periods. This holds for the postwar US period and the second half of the prewar period in the UK<sup>81</sup>. Second, the Kitchin peak itself is more often significant at the 95% confidence level than usual. For example during the postwar period it is significant in each of the three countries analyzed while for the other variables this was only the case for the industrial production indices in Chapter 8. Third and last, the *peak power* of the Kitchin peak is often the same or even higher than that of the Juglar peak. The relative dominance of the Kitchin type fluctuations is confirmed in the literature by for instance Siegel (1991) and Van Duijn (1999) who both report equity business cycle fluctuations with a length of four to five years.
- The estimated spectral densities do not show exceptionally many high frequency fluctuations as one might have expected because of the sensitivity of equities for historical incidents, sentiment, crashes, etc. Note that, because of the amplification of *high frequency fluctuations* by the first order differencing operator, these high frequency fluctuations may be more clearly present in the *returns* instead of the (logarithmic) levels. This however also holds for the other variables analyzed.
- With regard to the relation between the equity business cycle fluctuations and those in the national product the results most of the times show a *lead* of approximately one year, both at the Juglar and at the Kitchin frequencies. The lead at the Juglar frequencies for the postwar period in the NL and the UK are examples of exceptions on this finding. In these cases however the corresponding coherence is extremely low (only 0.2), which renders the interpretation of the phase not very reliable. In most other cases the *coherence* is at the usual high levels of around 0.9. A more detailed inspection of the complete coherence and phase spectra for the postwar NL and UK estimates show a typical pattern. The coherence is very high for almost all frequencies but shows an extreme dip for frequencies just below 1/10 (i.e. the Juglar frequency). Around this frequency the phase shows extreme fluctuations between lead and lags of several years. Striking is however that for all higher frequencies in the range [1/10, 1/2] the phase consequently indicates a lead of between zero and one year. For example, the

<sup>81</sup> We estimated separate spectral densities for the first and second half of the 1800-1913 prewar period in the UK. The reason is that the 1800-1854 and 1855-1913 periods show markedly different behavior. This can be seen from the original series and the filtered stochastic components shown in Figure D.21 in Appendix D.

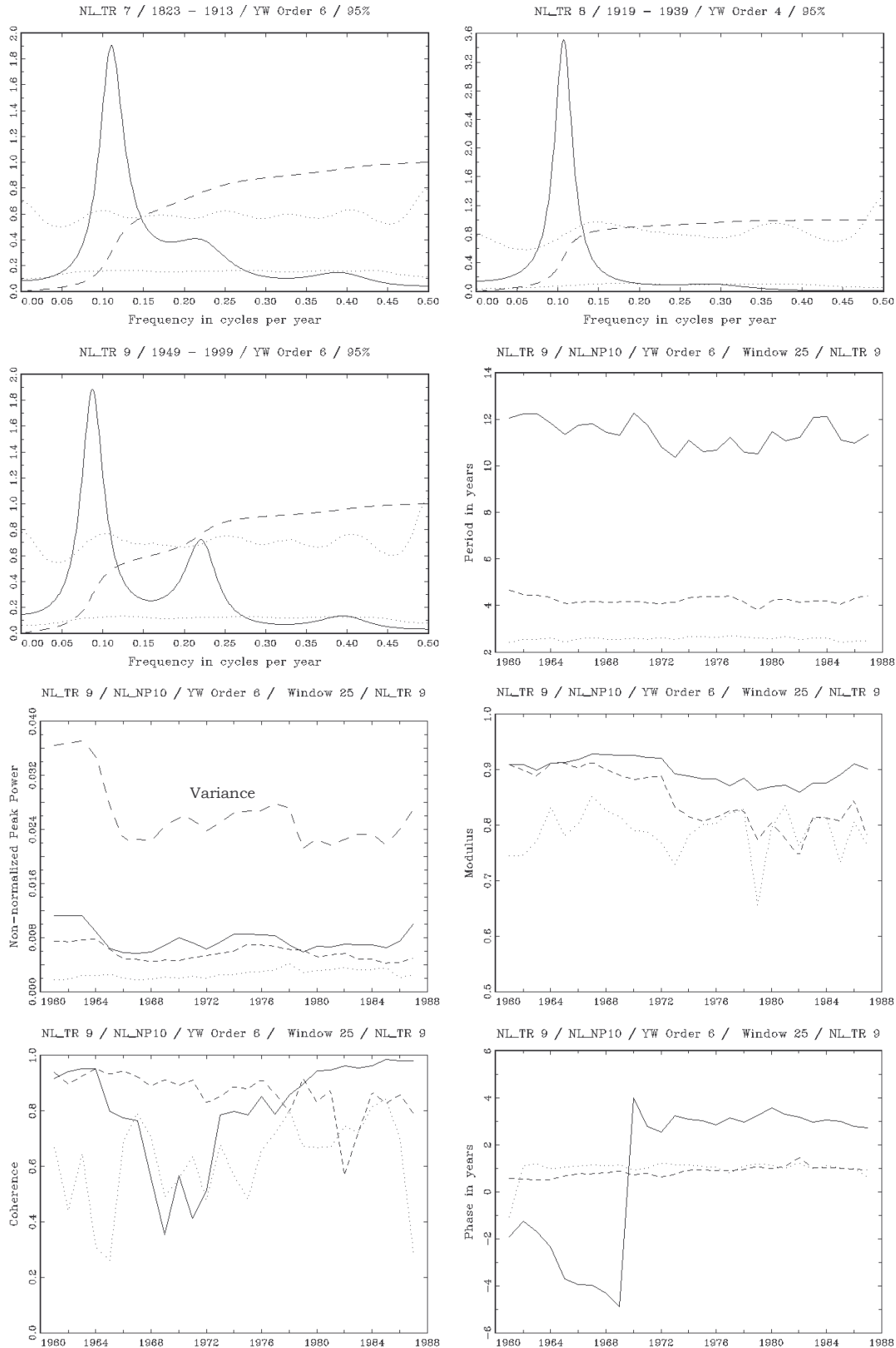
values of the phase spectrum at the frequency  $1/9$  (i.e. period length of nine years) for the NL and the UK are respectively 0.3 and 1.1.

- The observed *lead* of equities over the real sector of the economy in terms of for instance the national product or industrial production is consistent with what is known in the literature on this subject. Siegel (1991) for example reports that 93% of the forty-one US recessions that occurred during the 1802-1990 period have been preceded by significant declines in equity prices of 8% or more. He finds a lead of between zero and fourteen months and an average lead of six months. Van Duijn (1999) gives as a rule of thumb for both the US and the NL that equities lead the cycle in the real economy by approximately one year. Based on data for ten European countries, Aspren (1989) finds a strong positive relation between equity returns and the growth of industrial production several quarters later. Based on one full century of data, Barro (1990) finds a similar strong predictive power of equity returns for investment growth approximately one year later. He also finds that the strength of this relation seems to have decreased during the postwar period, though it is still highly significant. Also Fama (1981) and Bong-Soo Lee (1992) confirm the positive leading relation between equities and industrial production for the postwar period. Before the great stock market crash in 1987 Fisher and Merton (1984) and Barro (1989) reached the conclusion that the stock market is the best predictor of the business cycle. The 1987 crash and the fact that this was not followed by an economic recession, growth even accelerated, is the most often cited example of the fact that equities are also known for sometimes giving false signals about the future state of the economy. Samuelson (1966) expresses the tendency of stock markets to give false alarms of an oncoming recession in his famous line “*The stock market has predicted nine out of the last five recessions*”. Siegel (1991) modified this remark to “*Declines in the stock market have predicted fifteen of the last nine recessions, missed one downturn, and called two others late!*”. This behavior of equities merely reflects the fact that, besides macroeconomic fundamentals, equities are also very sensitive to more fickle influences, such as politics and public sentiment. Such influences may at times disturb the true underlying connection between equities and the real economy. Finally, note that the predictive qualities of equities have traditionally led to them being included in leading *business cycle indicators*. See section 2.2.4. However, because of their false alarm property and the alleged reduction of their predictive power during the postwar period, they have occasionally been excluded again. An example is Stock and Watson (1989)<sup>82</sup>.
- Apart from the sometimes low coherence and the corresponding aberrant value of the phase, the spectral properties in Table 16.13 do not show any significant changes of the business cycle equity behavior between the prewar, interwar and postwar period. If we would have to name something anyway, it would probably be the relatively low Juglar period length for the first half of the prewar UK period (7.5 year).

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<sup>82</sup> They are also no longer present in the leading indicator of the Dutch economy as constructed by the Dutch central bank (DNB). The reason was that equities did not provide enough extra explanatory power over the other variables included, such as the term spread.

Figure 16.6 Static and dynamic estimates of the spectral density of the stochastic component of a real Total Return equity index for the NL for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures.



## 16.4.2 Dynamic

- With respect to the *average spectral measures* of the rolling window estimates for the postwar period, the peak period, peak power and modulus give very similar results as the static estimates. The same holds for the coherence and phase of the Juglar and Kitchin peak in the US and the Kitchin peak in the NL and the UK. However, the average *coherence* of the Juglar peak in the latter two countries is much higher than it was in the static estimates. Instead of only 0.2 it is now approximately 0.8. The corresponding average phase of one year for the NL seems much more consistent with the other results. For the UK however, the average lead remains exceptionally high at around three years.
- In general the *period length* of the Juglar and Kitchin peaks seem fairly constant during the postwar period. Only for the UK the length of the Juglar peak seems to have decreased from more than ten years to approximately eight years. The most recent UK estimate is however again ten years. The very large fluctuations in 1974 and 1975 may have an important part in this because the indicated year, in which the period length is ten years again, is 1987. This is the middle of the first rolling window sub-sample of twenty-five years, from which the disturbing 1974 and 1975 years are excluded.
- In all three countries the *variance* has decreased throughout the postwar period. For the NL and the US the reduction is approximately 30% while it is as much as 80% for the UK, which is again probably due to the extreme fluctuations during the 1970's. In the end, the UK and US standard deviations are both approximately 10% while it is 15% for the NL. Note that this can also be seen in Table 16.6. In the NL and the UK the *peak power* of the Juglar fluctuations is consequently bigger than that of the Kitchin fluctuations. In the US however the Kitchin peak most of the time contributes more to the total variance than the Juglar peak. This can also be seen from the average peak powers reported in Table 16.14. Nevertheless, also in the US at the end of the sample the Juglar peak power is again greater than the Kitchin peak power.
- In the NL the *modulus* of the Juglar peak has been relatively constant at a level of around 0.9. For the Kitchin peak it has decreased from 0.9 to 0.8. In the UK the modulus of both peaks has decreased from 0.9 to respectively 0.8 and even 0.6. At the very end of the sample however, they have again strongly increased. Just as with the period lengths this may be because of the 1974 and 1975 fluctuations falling out of the rolling window sub-sample. In the US the Kitchin modulus has been constantly at a high level. Although the Juglar modulus is also high at the end of the sample it has been rather low at around 0.6 in the beginning.
- The *coherence* at the Kitchin frequencies has been constantly high in all the three countries although it has decreased from very high values close to one in the beginning towards lower values of around 0.8 at the end. The coherence at the Juglar frequencies in the US has been high throughout the entire sample. For the NL and the UK it shows an extreme decrease during the middle of the sample (as low as 0.4 and 0.2), which is probably the cause of the very low values of the coherence found in the static estimates.
- The *phase* at the Kitchin frequencies indicates an exceptionally constant lead of approximately one year. For the Juglar frequencies the results are somewhat confusing. The NL Juglar phase shows a large switch from a lag of three years towards a lead of three years. Because of the high coherence during the second

part of the sample the lead of three years also seems pretty reliable. Note that the average lead of one year reported in Table 16.14 is therefore not very informative. Given the rather volatile behavior of the coherence and phase spectrum around the Juglar frequencies as noted in the previous sub-section, this may be a consequence of the rather long period length of eleven to twelve years for the Juglar peak in the (univariate) equity spectrum. At the frequency  $1/9$  (i.e. a period length of nine years), the lead is indeed very constantly lower at around two years. In the US the Juglar phase shows a lead of between one year and approximately zero, while it is between a lead of half and one full year in the second half of the UK sample.

Table 16.13 Statistics of static spectral estimates of the filtered stochastic components of real Total Return equity indices for various sub-periods. The coherence and phase hold relatively to the National Product index. The phase is to be interpreted as the expected lead over the (filtered) National Product index.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
1823-1913*	9.2#	0.4	0.9	0.8	-0.2	1800-1854	7.5#	0.4	0.9	-	-	-	-	-
	4.4	0.2	0.8	0.4	2.1		4.8	0.1	0.7	-	-	-	-	-
	2.5	0.1	0.7	0.8	0.9		2.8	0.1	0.6	-	-	-	-	-
						1855-1913	11.9	0.2	0.9	0.4	-5.5			
1919-1939	9.4#	0.5	0.9	0.9	1.1		5.3#	0.3	0.9	0.3	1.1			
	3.3	0.1	0.7	0.8	-0.9		3.4	0.1	0.7	0.4	-1.3			
							2.3	0.1	0.9	0.7	0.7			
1949-1999	11.5#	0.3	0.9	0.2	4.7	1919-1939	9.1#	0.3	0.9	1.0	1.9			
	4.5#	0.2	0.9	0.9	0.7		5.4	0.3	0.8	0.7	0.9			
	2.5	0.1	0.8	0.4	1.2		3.1	0.2	0.8	0.9	0.1			
						1949-1999	10.0#	0.2	0.9	0.2	3.7			
							4.4#	0.2	0.9	0.9	1.3			
							2.6	0.2	0.6	0.7	0.8			
# The relevant spectral peak is significantly different from the contribution these frequencies have in a Yule-Walker estimated white noise process.														
* Coherence and phase come from the 1870-1913 estimates of the co-spectrum for the indicated 1823-1913 frequencies / period lengths.														

Table 16.14 Average statistics of dynamic spectral estimates of the filtered stochastic components of real Total Return equity indices for the 1949-1999 period. These can be compared to the statistics of the static spectral estimates for the last sub-period from Table 16.13.

Netherlands					United Kingdom					United States				
Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase	Period	PP	Mod	Coh	Phase
11.4	0.3	0.9	0.8	1.0	9.2	0.3	0.9	0.7	3.0	9.4	0.2	0.8	0.9	0.2
4.2	0.2	0.9	0.9	0.9	4.2	0.2	0.8	0.8	1.3	3.9	0.3	0.9	0.9	1.0
2.6	0.1	0.8	0.6	1.0	2.5	0.2	0.7	0.8	0.6	2.5	0.1	0.7	0.8	0.9



## 16.5 Some important aspects reconsidered

Besides the equity risk premium as already discussed in section 16.2.1, several other important aspects of equities have traditionally received much attention among economists as well. In this extra section we reconsider three of them in the light of the information obtained in the previous sections. Each of the next sub-sections starts with a short summary of the literature after which these findings are confronted with those found here.

### 16.5.1 Mean reversion and autocorrelations

The first aspect of the behavior of equities that has traditionally received much attention in the literature is whether or not equities show some kind of *mean reversion* or that they are completely unpredictable and behave like a random walk. The first would imply that equity returns show significant autocorrelation patterns while the latter would imply all autocorrelations being equal to zero because of the independence of subsequent returns. Empirically there are indications that equity returns show positive and negative autocorrelations, depending on the frequency of observation. For example Poterba and Summers (1988) and Fama and French (1988) show that on longer horizons equity prices show significant mean reversion and hence some negative autocorrelations. Lo and MacKinlay (1988) find the opposite, positive autocorrelations, in weakly and monthly data. Much effort has been put into the development of better statistical testing procedures such as for example variance ratio tests to test whether multiple year equity returns are mean reverting or not. As these tests developed, the evidence of rejecting the random walk hypothesis got smaller. Examples of literature rejecting mean reversion in long term stock returns in favor of the random walk model are Kim et al. (1988), Coggin (1998), Lo (1991), McQueen (1992), Richardson (1993) and Richardson and Stock (1989). Poterba and Summers (1988) rightfully argue that the distinction between low frequency mean reversion and complete unpredictability can only be made based on sufficiently long samples of data. Based on two and three centuries of data for respectively the US and the UK, Goetzmann (1993) finds evidence of a persistent mean-reverting component in stock market prices, especially in the detrended (!) price data. More references to research on equity autocorrelations are given by Lo and Wang (1995). Finally Balvers et al. (1990) and Cecchetti et al. (1990) show that mean reversion in stock prices is consistent with efficient markets if the underlying fundamentals (profits, industrial production etc.) also are mean reverting. In efficient markets all (new) mean reverting information of the fundamentals will be incorporated in the stock prices, causing them to show mean reverting behavior. As noted before, this assumes that investors are not able to perfectly predict all future business cycle fluctuations and the corresponding profits. That is, at least some part of the fluctuations in the fundamentals should come as a surprise. Otherwise investors would simply look through all future fluctuations and simply discount them into a fixed stock price (given that the interest rates used for the discounting do not change).



From a frequency domain perspective, autocorrelations depend on both the period length of the fluctuations considered and the frequency of observation. A simple example are business cycle type fluctuations with a period length of several years. Annual data of such fluctuations may very well show negative, autocorrelations, while monthly observations of the same phenomenon may very well show positive autocorrelations (of the same order but with a very different time span). Note that the finding of positive autocorrelations in weekly and monthly data reported by Lo and MacKinlay (1988) and the negative multi-year autocorrelations found by others are probably an illustration of this phenomenon. Another example concerns the distinction between long wave type fluctuations on the one hand and business cycle type fluctuations on the other. Considered in terms of annual data, the long wave will show positive (low order) autocorrelations due to its slow moving behavior, while the business cycle fluctuations will probably show negative (low order) autocorrelations.

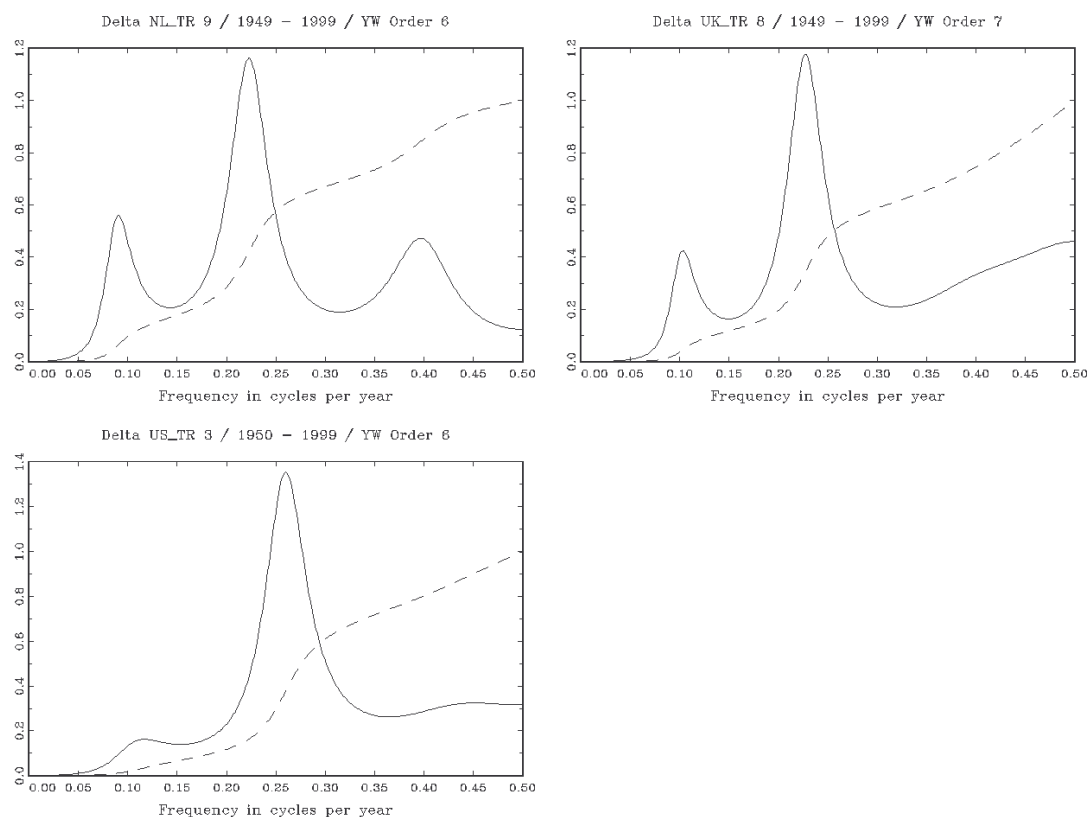
Here we investigated the autocorrelations for the equity returns as caused by their business cycle type fluctuations. For this we took the filtered stochastic components of the postwar (logarithmic) equity indices in the three countries and applied the first order differencing operator. Because, as we know, theoretically the order in which the pass-band and differencing filter are applied makes no difference for the final series, this gives us the business cycle fluctuations in the equity *returns* instead of in the original equity indices. Because autocorrelations and spectral densities are different representation of the same thing, let's first look at the spectral densities of these return components. Figure 16.7 shows these spectral densities, as obtained by multiplying the estimated static spectral densities of the logarithmic equity indices by the Power Transfer Function (PTF) of the first order differencing operator as given in section 5.2.1<sup>83</sup>. Because this PTF suppresses low frequency fluctuations and amplifies high frequency fluctuations, in the spectral densities of the returns the Juglar peak is much lower while the Kitchin peak and also the very high frequency fluctuations now describe a larger portion of the total return variance. This typical shape of a spectral density also holds for other return or growth statistics as for example the growth of the national product<sup>84</sup>. Because of the relative dominance of the Kitchin type fluctuations, the four to five year fluctuations play an even more important role in the equity returns.

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<sup>83</sup> Another, theoretical identical, way of obtaining these spectral densities would be to first calculate the equity returns, filter them with a  $[1/15, 0.5]$  band-pass filter and then estimate a spectral density for the resulting series. We applied this approach as well and found that this yielded almost identical spectral densities, just as theory predicts. If one does not apply the band-pass filter and directly estimates a spectral density on the equity returns this results in almost identical spectral densities as well. The only difference, again just as theory predicts, is that these contain some variance mass at the very low frequencies because these are not completely suppressed by the first order differencing operator.

<sup>84</sup> For a complete random walk model, the returns or growth rates follow a white noise process with all autocorrelations equal to zero. Section 4.5.1 shows that such a process has a completely flat spectral density with an equal importance for all possible frequencies. Note that spectral densities such as those in Figure 16.7 clearly do not fit this description. It is therefore highly unlikely that the underlying process is indeed a random walk.

Figure 16.7 Normalized spectral densities of real Total equity Returns for the postwar period calculated as the static spectral density estimates of the stochastic component of the real Total Return equity indices multiplied by the Power Transfer Function of the first order differencing transformation.



Given this knowledge of the spectral properties of the equity returns it is now instructive to consider the corresponding autocorrelations of these business cycle components in the equity returns and compare these to the autocorrelations of the conventional, not filtered, returns. Table 16.15 shows the empirical autocorrelations up to order  $k=5$  for both the stochastic components and the original (real) returns. From this we see that the stochastic components of the returns consistently show negative autocorrelations up to the third order while at order four and five they are again positive. The final column shows the autocorrelations of a perfect four and a half years sinusoid. The second and third order autocorrelations are very negative because these compare peaks and subsequent troughs of the sinusoid, while the fourth and fifth order autocorrelations are very positive, because these compare subsequent peaks of the sinusoid. This illustrates that the dominant Kitchin type fluctuations with a period length of between four and five years are the primary cause of the observed autocorrelation pattern in the business cycle fluctuations of the returns<sup>85</sup>. The fact that the first order autocorrelation of the sinusoid is positive, while for the stochastic components it is consistently negative, is probably due to the highest frequency fluctuations observed in the spectral densities in Figure 16.7. If we now look at the autocorrelations of the original, not filtered, equity returns than we

<sup>85</sup> Note that basically it makes no difference whether we look at the *level* or the *returns* (first order differences) of a perfect sinusoid. After all, the first order differencing filter only reduces the amplitude (gain of the filter) of the sinusoid and shifts it backwards in time (phase of the filter). Both these operations have no effect on the autocorrelations.

also consistently see negative autocorrelations at the second, and for the UK also third, order. It is now clear that these negative empirical autocorrelations are mostly due to the Kitchin type business cycle fluctuations observed in the equity indices. The only difference between the stochastic components and the historical returns are the low frequency fluctuations in the range  $[0,1/15]$ . These very low frequency fluctuations are very likely the cause of the positive first order autocorrelations observed in the original returns. One could say that the autocorrelations of the returns are the “sum” of the business cycle autocorrelations on the one hand and the autocorrelations of the (partly suppressed) long term fluctuations on the other. The main conclusion is that the negative autocorrelations in equity returns, as found in the literature and in the data analyzed here, are consistent with the Kitchin type business cycle fluctuations found in equity prices and returns. An important implication is that, due to these fluctuations, negative equity returns in one year show a tendency to be followed by positive returns in subsequent years and vice versa.

Table 16.15  $k$ -th order autocorrelations of real Total equity *Returns* based on the filtered stochastic components by applying the first order differencing operator on the filtered stochastic (logarithmic) components (Filt) compared with the auto correlations of the original returns (Hist).

$k$	<i>Netherlands</i>		<i>United Kingdom</i>		<i>United States</i>		<i>Theoretical 4.5 year sinusoid</i>
	<i>Filt</i>	<i>Hist</i>	<i>Filt</i>	<i>Hist</i>	<i>Filt</i>	<i>Hist</i>	
1	-0.1	0.1	-0.2	0.2	-0.3	0.0	0.2
2	-0.3	-0.1	-0.2	-0.2	-0.1	-0.2	-0.9
3	-0.1	0.0	-0.2	-0.2	-0.2	0.2	-0.5
4	0.1	0.2	0.2	0.2	0.0	0.4	0.8
5	0.2	0.3	0.1	0.1	0.0	-0.1	0.8

### 16.5.2 Equities and the term spread

A second aspect of the behavior of equities that is well known in the literature is the positive relation between equity returns and the interest rate term spread. Examples that apply to US data are Keim and Stambaugh (1986) and Boudoukh et al. (1997). For the 1802-1990 period the latter find a correlation of approximately 0.2 to 0.3. Aspremi (1989) reports positive correlations for ten European countries. Reviews of the literature on the relation between (expected) equity returns and the term spread are given by Fama (1991) and also by Hawawini and Keim (1995).

This positive relation between equity returns and the term spread can easily be explained from the spectral analysis results for the term spread in section 14.3 and for equities in the previous sub-section of this chapter. As an example, the left hand side picture in Figure 16.8 shows the stochastic components of the term spread and the (logarithmic) equity *index* in the NL for the postwar period. The right hand side picture does the same but now for the stochastic component of the *returns*. Especially for the latter, the positive relation between the term spread and equities in terms of the business cycle fluctuations is very clearly visible.

Figure 16.8 Postwar stochastic components for the NL real (logarithmic) Total Return equity *index* and the Term Spread (left hand side) and the stochastic components in the equity *returns* as obtained by applying the first order differencing operator and the Term Spread (right hand side).

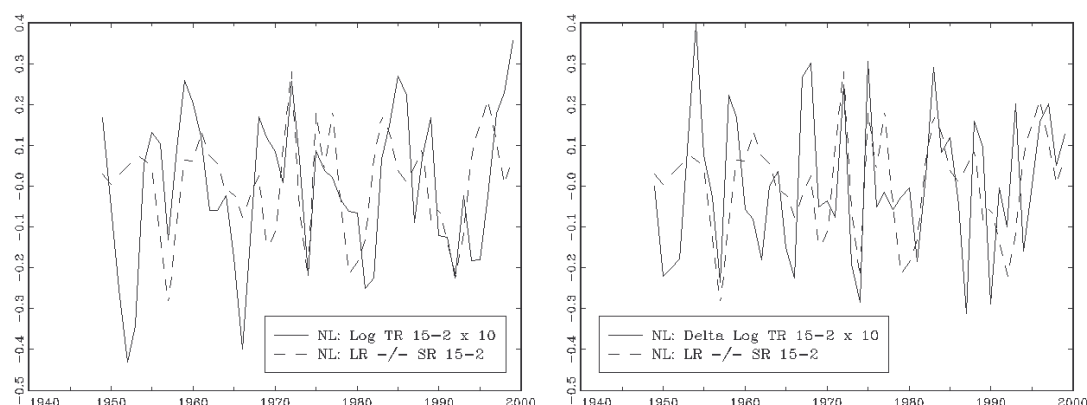


Table 16.16 reports the correlations between these types of series for each of the three countries analyzed. In the first line it also gives the conventional correlation between equity returns and the term spread. For the NL and the US all correlations are consistently around 0.3. The UK results seem very different. A further investigation reveals that the negative conventional correlations in the UK are to a large extent caused by the very high levels of the term spread of around 6% during the 1970's. For this see for instance Figure D.17 in Appendix D. Despite the fact that these long term spread movements are excluded from the stochastic component by the filtering process, also for the stochastic component the correlations are approximately zero, and thereby much lower than for the other two countries. The fact that for the second half of the postwar period the correlations are again positive at around 0.2 indicates that probably the exceptional UK equity returns in 1974 and 1975 are to blame once more. At second sight the UK results are therefore not very different from those for the other two countries. Now, from the spectral analysis we know that both the term spread and equities show business cycle type fluctuations at both the Juglar and the Kitchin frequencies. We also know that at these frequencies the spread leads the national product by about two to three years, while equities were shown to lead by approximately one year. This means that the term spread leads the equity *index* by approximately one to two years, which is clearly visible in the left picture of Figure 16.8. The phase of the first order differencing operator shifts the equity *returns* one to two years back in time, causing them to move almost exactly in line with the term spread. Especially for the second half of the postwar period this is very clear from the right hand side picture of Figure 16.8. In short, at the business cycle frequencies the term spread leads the equity *prices*, while it moves approximately contemporaneously with the equity *returns*. These lead / lag relations between equities and consumer prices imply the positive correlation between these variables.

Table 16.16 Historical correlations between real Total equity Returns (TR) and the Term Spread (TS) for the postwar period (first line), the correlations between the stochastic component of the Term Spread and the (logarithmic) equity *index* (second line) and between the stochastic component of the Term Spread and the stochastic component of the equity *returns* as obtained by the first order differencing operator (third line).

	<i>Netherlands</i>	<i>United Kingdom</i>	<i>United States</i>
Historical real TR and TS	0.4	-0.1	0.3
TR [1/15,1/2] and TS [1/15,1/2]	0.3	0.0	0.3
Delta TR [1/15,1/2] and TS [1/15,1/2]	0.4	0.0	0.3

### 16.5.3 Inflation hedge

The third and final often studied aspect of equities concerns their inflation hedging capacities. The intuition is that, contrary to (nominal) bonds, equities are a claim on real assets and that their value should therefore move in line with (expected) changes in the general price level. Many research effort has been put into the testing of this Fisher hypothesis for equities<sup>86</sup>. Fama and Schwert (1977) were one of the first to show that contrary to what this hypothesis predicts, (US) stock prices are *negatively* related to current and lagged inflation. Both Fama (1990), Lintner (1973), Nelson (1976), Bodie (1976) and Lee (1992) find evidence for rejection of the Fisher hypothesis and conclude that equities are a poor hedge against inflation. Also Fama (1981) and Lee (1992) find a negative correlation between equity returns and inflation. They add that this is not a causal correlation but a consequence of the underlying stochastic process (business cycle?). Solnik (1983) summarizes the rather counter intuitive finding of a negative correlation by stating that “*there is a consistent lack of positive association between stock returns and inflation covering several countries*”. Most of these reported correlations are annual contemporaneous correlations and therefore hold for the *short term* relation between equities and inflation. Based on two centuries of US and UK data Boudoukh and Richardson (1993) report a *positive* correlation between five year equity returns and inflation and thereby find evidence for the existence of a positive *long term* (Fisher) relation between equities and inflation. Also Evans and Lewis (1995) show that there is a positive relation when longer horizons are considered. Schotman and Schweitzer (2000) show that stocks can be a hedge against inflation, even if stock returns are negatively correlated with unexpected inflation shocks, and only moderately positively related to expected inflation. Depending on the investment horizon, the optimal hedge ratio can be either positive or negative.

Based on the evidence in the literature mentioned here, we made a split between the short and the long term relation between equities and inflation. Given the frequency domain filtering approach, it is very natural to define the business cycle fluctuations (i.e. the stochastic components) as the short term fluctuations and the long wave and, more general, the periodic components as the long term fluctuations. Let's first look at the *short term*. The first line in Table 16.17 gives the conventional correlation between *nominal* equity returns and inflation for the postwar period in the various countries. The consistently negative correlations confirm the findings in the



literature. The second and third line give respectively the correlation between the *stochastic components* in the (logarithmic) nominal *level* of equities and in the (logarithmic) nominal consumer price *level* and the same for the nominal equity *returns* and the consumer price *changes*. Note that to enable a fair comparison, here equities hold in *nominal* terms and not in *real* terms as was the case in the previous parts of this chapter<sup>87</sup>. Also these correlations are negative most of the times. The only exception are (“filtered”) UK return and inflation data with a correlation of approximately zero. This is probably due to the extreme 1974 and 1975 returns. For the 1976-1999 period this correlation indeed becomes negative again (-0.2). In general the level and return correlations will not be very different, provided that this transformation is applied on both series. To see this, note that the first order differencing operator shifts two sinusoids of equal frequency backwards in time by the same amount. This does not alter the correlations. The (identical) reduction in amplitude does not change the correlation either. Now these negative correlations are easy to understand from the spectral analysis of the stochastic components. We know that consumer prices *lag* the business cycle in the national product by approximately two to four years. See section 10.2. From the results in this present chapter we know however that real equity indices *lead* the business cycle by approximately one year. It is not hard to imagine that these very different phases imply a negative short term correlation between equities and consumer prices. Of course, by definition, the stochastic price components are also a part of stochastic components in the *nominal* stock prices. However, the large volatility in the underlying *real* stock prices dominates the behavior of the nominal stock prices. So, in short, the negative short term relation between equities and consumer prices is a direct consequence of the their lead / lag relations at the business cycle level.

Table 16.17 Historical correlations between *nominal* Total equity Returns (TR) and the Price Inflation (PI) for the postwar period (first line), the correlations between the stochastic component of the (logarithmic) price *index* and the (logarithmic) nominal equity *index* (second line) and between the stochastic component of the price *inflation* and the stochastic component of the nominal equity *returns*, both obtained by the first order differencing operator (third). The nominal equity components are constructed as the sum of the real components and the corresponding component of the consumer prices, used to construct the real indices in the first place.

	<i>Netherlands</i>	<i>United Kingdom</i>	<i>United States</i>
Historical nominal TR and PI	-0.4	-0.1	-0.5
TR [1/15,1/2] and PI [1/15,1/2]	-0.1	-0.2	-0.3
Delta TR [1/15,1/2] and Delta PI [1/15,1/2]	-0.1	0.0	-0.4

To investigate the *long term* relation between equities and prices, a similar exercise was performed for both the *periodic* and the *long wave components* in both variables by calculating the relevant correlations. The correlations are calculated for levels and growth numbers and for the original real equity indices and derived nominal indices.

<sup>86</sup> This hypothesis is very comparable to the decomposition of nominal interest rates into a real interest rate and a compensation for the expected inflation rate, as described in section 13.2.2.

<sup>87</sup> As indicated in the description of Table 16.17, the nominal indices are constructed as the sum of the filtered components of the real equity indices and the consumer prices. A theoretically identical approach would be to actually filter the original nominal indices. This approach gave practically identical results.

The nominal indices are again constructed as the sum of the components in the real equity indices and the same components in the consumer price indices<sup>88</sup>. The resulting correlations are given in Table 16.18. In case of the *real* equity indices the correlations are mostly negative, which is logical because the consumer prices were used to turn the original nominal equity indices into real equity indices. However, in case of the more relevant *nominal* equity indices the correlations are consistently positive. The only exception is the long wave component for the NL. This could be because of the aberrant dating of the long wave in the NL price index as found in section 10.2. For the NL, the correlations for the broader periodic components are clearly positive. The positive correlations confirm the indications in the literature for the existence of a positive *long term* correlation between equities and prices.

The conclusion must be that, in the short run, equities are a poor hedge against inflation because of the substantial phase differences between equities and prices at the business cycle frequencies. In the long run equities do provide a good hedge against inflation. Of course we should not forget that the consistently positive long term average real equity returns as reported in Table 16.1 by themselves already reflect the long term inflation hedging capacities of equities.

Table 16.18 Correlations for longest possible samples between long wave and periodic components in Total Return equity indices and Consumer Price indices. The first two rows use the filtered components of the (logarithmic) price indices and the original (logarithmic) equity indices in *real* terms. The bottom two rows use the filtered components of the price indices and the equity indices in *nominal* terms. The latter are constructed as the sum of the real components and the corresponding components of the consumer prices, used to construct the real indices in the first place. The *level* columns hold for the original filtered (logarithmic) series, while the *growth* columns hold for the first order differences of the filtered series. These give the correlations between the relevant components in *total rate of returns* and the *inflation rates*.

	<i>Netherlands</i> 1823-1999		<i>United Kingdom</i> 1800-1999		<i>United States</i> 1870-1999	
	Level	Growth	Level	Growth	Level	Growth
Long wave component / Real	-0.5	-0.5	-0.3	-0.3	-0.5	-0.4
Periodic component / Real	0.1	-0.2	0.2	-0.3	-0.5	-0.5
Long wave component / Nominal	-0.1	-0.2	0.4	0.4	0.9	0.8
Periodic component / Nominal	0.5	0.2	0.9	0.5	0.3	0.1

<sup>88</sup> Again the theoretical identical direct approach of filtering the original nominal equity index gave very similar results. Note that the filtering process was not repeated in the case of different sample sizes for equities and consumer prices.



## 17 Stylized Facts

The foregoing chapters of Part III contain the results with respect to the first research objective as posed in Chapter 1. In short, the objective was to gather broad empirical information about the behavior of macroeconomic variables. The methodology followed for collecting this information is described in Chapter 6. The Chapters 7 until 16 contain the specific and detailed results for all the macroeconomic variables analyzed. In this final chapter of Part III these specific results are summarized in terms of a number of stylized facts. In general we can call some observed phenomenon a stylized fact if it is found to be robust with respect to the historical time period and / or the country of origin for which it is observed. Where needed and possible, we also give a short discussion with some (groups of) stylized facts. Based on the formulation of the first research objective in section 1.3, this chapter is divided into four sub-sections. The first is about the long run behavior of macroeconomic variables, the second about their business cycle behavior and the third about possible changes in their (business cycle) behavior over time. The fourth section presents an overall variance decomposition. It is important to remember that all findings are based on data for the Netherlands, the United Kingdom and the United States and therefore “only” hold for what might be called the most developed countries in the world today and in the past. Finally note that in this concluding chapter the results are formulated at a rather general level. In this way, besides summarizing, this chapter also serves the purpose of bringing out the most important empirical properties of macroeconomic behavior as clearly as possible. For more specific information on some topic, more elaborate discussions of the results and references and comparisons to related literature, the reader is referred to one or more of the previous chapters which describe the underlying results for each of the variables in greater detail.

## 17.1 The long run

With respect to the long term behavior of macroeconomic variables we made a methodological distinction between possible *trends* on the one hand and *long term fluctuations* around these trends on the other. Next, separate sub-sections are devoted to the stylized facts for each of these two components of long run behavior.

### 17.1.1 Trends

As described in Chapter 6, the *trend component* of a time series is defined in the frequency domain as all fluctuations with a period length exceeding the sample size  $T$ . That is, trends comprise all components in the frequency range  $[0, 1/T]$ . Table 17.1 summarizes the results with respect to the trend components. The results in this table support the following Stylized Facts (SF).

- SF 1:* In the long run, *national product volume indices* show *exponential trending* behavior with an average annual rate of growth of approximately 3%. Due to population growth, the per capita growth is approximately 1% lower.
- SF 2:* In the long run, *industrial production volume indices* show *exponential trending* behavior with an average annual rate of growth of approximately 0.5% higher than national product volume indices. One reason for these higher growth rates might be the transformation into manufacturing economies that has taken place during the past centuries.
- SF 3:* The long run average *employment rate* is 95%.
- SF 4:* With respect to their long term behavior, *consumer price indices* show a *structural break* around 1900. Before that time they show no trending behavior with an average annual inflation rate of approximately 0%. After that time they show *exponential trending* behavior with an average inflation rate of approximately 3.5%. For what its worth, the overall long term average inflation rate is approximately 1.5%.
- SF 5:* In the long run, *real industry wages* show *exponential trending* behavior with an average real wage inflation of approximately 1% per annum.
- SF 6:* *Nominal short term (default free) interest rates* show no trending behavior and their long term average is approximately 4%.
- SF 7:* *Nominal long term (default free) interest rates* show no trending behavior and their long term average is approximately 5%.
- SF 8:* The empirical *frequency distribution* of both long and short term interest rates is *skewed* to the left.

- SF 9:* The *term spread* as the difference between the nominal (default free) long term and short term interest rates shows no trending behavior and its long term average is approximately 1%.
- SF 10:* The (ex post) *real long term interest rate* shows no trending behavior and its long term average is approximately 3%.
- SF 11:* In the long run, “the” *default spread* shows no trending behavior and its long term average is approximately 1%.
- SF 12:* In the long run, *real equity total return indices* show *exponential trending* behavior with a geometrical annual average total rate of return of approximately 6% (capital growth + dividend yield). The arithmetic average annual total rate of return is approximately 7.5%.
- SF 13:* The empirical *frequency distribution* function of *annual* real equity returns shows *no* significant deviations from a Normal distribution. Their distribution is therefore not skewed and also does not have fatter tails than a Normal distribution.
- SF 14:* On the one hand, *real equity capital returns* contribute relatively little to the long term real total return on equities. The approximate *geometrical* average annual capital growth is only 1%. The *arithmetic* average annual capital growth is approximately 3%. On the other hand, real equity capital returns determine most of the short term volatility of the total return indices.
- SF 15:* In the long run, *equity dividend yields* are very *stable* around a long term annual average of 5%. Thereby they contribute relatively much to the long term total return on equities but very little to their short term volatility.
- SF 16:* The (ex post) *equity risk premium* over long term bonds shows no trending behavior and its *arithmetic* long term annual average is approximately 5%. In *geometric* terms this is approximately 4%.

Besides information on the typical long term levels or growth rates of the various macroeconomic variables, these stylized facts also suggest some interesting long term (equilibrium) relations. First of all, the real long term interest rate of some 3% (*SF 10*) is approximately the same as the long term real growth rate of the national product indices (*SF 1*). As more elaborately discussed in section 13.2.2 this might be because, as long term growth theory predicts, real interest rates are the compensation for the opportunity costs incurred by a lender of money by not being able to participate in economic growth. Second and related to this, it is interesting to note that the long term (arithmetic) average equity capital growth is also around 3% (*SF 14*). Seen from a long run perspective, equities apparently approximately grow at the same rate as the economy itself which also has a great intuitive appeal. The 3% seems to represent some “true” time value of money. In principle, in the (very) long run it therefore

makes no difference whether an investor invests (“lends”) his money in terms of buying bonds or equities. One important difference is however the third and final interesting long term relation. In the long run the (arithmetic) equity risk premium of some 5% (*SF 16*) is approximately the same as the average dividend yield. Seen in this way, it are the dividend yields (profit pay-outs) that form the actual compensation for the risk taken by investing in business enterprises instead of in “safe” long term bonds. For more information on these topics see sections 16.1.1 and 16.2.1.

Table 17.1 On the *left hand side* the long term historical average levels or growth rates calculated on the longest possible (individual) samples are plotted. Unless indicated otherwise, growth rates are approximated as the first order differences of the natural logarithm of the series. Also unless indicated otherwise, the averages are arithmetic averages. On the *right hand side* the average levels or growth rates according to the filtered trend components are plotted. *Legend:* NP: National Product volume index, IP: Industrial Production volume index, EM: Employment rate, PI: Consumer Price index, WI: Real industry Wage Index, SR: Short term interest Rate, LR: Long term interest Rate, TS: Term Spread, DS: Default Spread, TR: Real equity Total return Index, RR: (ex post) Real long term interest Rate, CG: Real equity Capital Growth, DY: Equity Dividend Yield and RP: (ex post) equity Risk Premium. In gray are important parts of the table that are referred to in the stylized facts.

	Type	Historic			Avg	Filtered trend component				
		NL	UK	US		Type	NL	UK	US	Avg
NP	Growth	2.7%	2.0%	3.3%	2.7%	Log Linear*	2.8%	1.9%	3.2%	2.6%
IP	Growth	3.6%	2.1%	4.0%	3.2%	Log Linear	4.0%	2.1%	4.0%	3.4%
EM	Level	94.9%	95.1%	93.1%	94.4%	Average	95.1%	95.3%	93.3%	94.6%
PI**	Growth	-0.4% / 3.2%	0.2% / 4.1%	0.2% / 3.1%	0.0% / 3.5%	Log Linear	-0.2% / 3.4%	0.4% / 4.3%	0.5% / 3.1%	0.2% / 3.6%
WI	Growth	1.2%	1.2%	1.0%	1.1%	Log Linear	1.6%	0.3%	1.1%	1.0%
SR	Level	3.6%	4.2%	5.5%***	4.4%	Average	3.4%	4.0%	5.4%	4.3%
LR	Level	4.8%	4.6%	5.0%	4.8%	Average	4.7%	4.6%	4.9%	4.7%
TS	Level	1.1%	0.5%	-0.7%***	0.3%	Average	1.0%	0.4%	-0.4%	0.3%
DS	Level	-	0.9%	1.1%	1.0%	Average	-	1.1%	1.1%	1.1%
TR****	Growth	8.0% / 6.5%	6.2% / 5.0%	8.4% / 6.8%	7.5% / 6.1%	Log Linear	5.9%	4.1%	6.1%	5.4%
RR	Level	3.3%	3.2%	3.0%	3.2%	-	-	-	-	-
CG****	Growth	2.6% / 1.2%	2.2% / 0.8%	3.3% / 1.7%	2.7% / 1.2%	-	-	-	-	-
DY	Level	5.7%	4.8%	4.5%	5.0%	-	-	-	-	-
RP	Level	5.2%	3.3%	6.2%	4.9%	-	-	-	-	-

\* This means that the natural logarithm of the series has a linear trend and therefore that the original variable has an exponential trend with the indicated average growth rate.

\*\* Contains a structural break around 1900. Therefore separate results for the pre-1900 and post-1900 periods are reported.

\*\*\* Unusual high or low value probably because of the large credit risk premiums present in the 19<sup>th</sup> century US short term interest rates. See section 12.1.

\*\*\*\* First is the arithmetic average of the actual returns, not approximated as the first order differences of the natural logarithm of the series. Second is the geometrical average.

### 17.1.2 Long waves and periodic components

Besides the trend components, in Chapter 6 two further components of the long term behavior of macroeconomic variables were defined. While the trend components by definition consist of all non-periodic components of a time series, all long term fluctuations with a period length longer than 15 years are contained in the *periodic components*. In the frequency domain these fall in the frequency range  $[1/T, 1/15]$ . As a special case of these periodic components we also analyzed *long wave components* falling in the frequency range  $[1/70, 1/30]$ . The results for these two types of components are summarized in Tables 17.2 and 17.3. The results in these tables support the following stylized facts.

- SF 17:* Around the long term trends or averages, the development of both real and financial macroeconomic variables is without exception for a substantial part characterized by worldwide *long wave* or *Kondratieff* type fluctuations with an average period length of approximately 50 years. The approximate dating of the subsequent peaks and troughs is given in the bottom rows of Table 17.3.
- SF 18:* These long wave components on average describe approximately 60% of the *volatility* (standard deviation) of the broader periodic components (36% of the variance).
- SF 19:* Because of the amplitude of the long wave, the *national product indices* can be approximately 5% above or below their underlying exponential trend. In terms of the growth rates this implies an amplitude of approximately 0.3%<sup>89</sup>. There are indications that the NL *lagged* the other two countries by 10 to 20 years for the earlier cycles. Also, the UK seems to have had a *leading* role in the earlier cycles.
- SF 20:* The amplitude of the long wave in the *industrial production indices* is approximately 2.5 times as large as in the national product indices. In terms of the growth rates this implies an amplitude of approximately 0.7%. The industrial production long wave moves in *parallel* with the national product.
- SF 21:* The amplitude of the long wave in the *employment rates* is approximately 4%. The employment rate long wave *leads* the national product index by approximately a quarter of a cycle. This implies that it moves approximately in *parallel* with the growth of the national product.

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<sup>89</sup> For this assume a long wave period length of 50 years and substitute the corresponding frequency of  $1/50=0.02$  cycles per year into the Gain of the first order differencing filter as given by (5.2.4). This gives  $\text{Gain}(0.02)\approx 0.06$  as the factor with which to multiply the average amplitude of the logarithmic index as given in Table 17.2 to obtain the approximate amplitude of the long wave in the growth rate of the index. The same factor is applied to obtain the amplitude of the long wave components in the growth rate of industrial production (*SF 20*), consumer price inflation (*SF 22*), real wage inflation (*SF 23*) and real equity returns (*SF 29*).

- SF 22: The amplitude of the long wave in the *consumer price indices* is approximately 3.5 times as large as in the national product indices. In terms of the inflation rates this implies an amplitude of approximately 1%. The consumer price long waves in the UK and the US *lag* the national product by approximately a quarter of a cycle. This implies that the long wave in inflation rates moves approximately in *parallel* with the level of the national product.
- SF 23: The amplitude of the long wave in the *real industry wages* is approximately 3 times as large as in the national product indices. In terms of the wage inflation this implies an amplitude of approximately 0.8%. Based on the results found here it is hard to draw a general conclusion about the lead / lag relation between the long wave in real industry wages and national product indices.
- SF 24: The amplitude of the long wave in the *short term interest rates* is approximately 1.7%. The short term interest rate long wave moves in *parallel* with the national product.
- SF 25: The amplitude of the long wave in the *long term interest rates* is approximately 1.5% and therefore approximately 90% of that in the short term interest rates. The long term interest rate long wave moves in *parallel* with the national product.
- SF 26: The amplitude of the long wave in the *term spread* is approximately 0.6%. The term spread long wave leads / lags the national product by approximately half a cycle and therefore moves *contrary* to the national product.
- SF 27: The amplitude of the long wave in the (ex post) *real long term interest rates* is approximately 2%. The real interest rate long wave *lags* the national product by approximately a quarter of a cycle and therefore moves contrary to the growth of the national product. Another way of looking at it is that the long wave in real interest rate year to year changes moves approximately in *parallel* with the national product indices.
- SF 28: The amplitude of the long wave in the *default spread* is approximately 0.4%. The default spread long wave *lags* the national product by approximately a quarter of a cycle and therefore moves *contrary* to the growth of the national product.



- SF 29:* The amplitude of the long wave in the *real equity total return indices* is approximately 4.5 times as large as for the national product indices. In terms of the real equity returns this implies an amplitude of approximately 1.5%. The equity indices long wave *leads* the national product index by approximately a quarter of a cycle. The same relation holds between the long wave in the equity returns and in the growth of the national product.
- SF 30:* *Nominal equity total return indices* and *returns* show a consistent positive long run correlation with respectively consumer price indices and inflation rates (i.e. a *long term inflation hedge*). Here the long run relates to all fluctuations in the variables with a period length exceeding 15 years. Also see *SF 63* for the *short term* relation between equities and consumer prices
- SF 31:* The amplitude of the long wave in the (ex post) *equity risk premium* is approximately 4%. The equity risk premium long waves *leads* the national product index by approximately a quarter of a cycle. This implies that the equity risk premium long wave *leads* the growth of the national product by some 5 to 10 years.

Several things can be said about these stylized facts related to the long term fluctuations in macroeconomic variables. The first is that the phase corrected average dating of peaks and troughs as mentioned in *SF 17* very closely resembles the approximate dating of the last four Kondratieff cycles that emerges from the literature as given in Table 2.1. It is important to note that the finding of very similar long wave fluctuations in all the real and financial variables analyzed “merely” proves that such fluctuations have occurred in the past. Given the small number of “observations” it does *not* prove that these fluctuations indeed come from a regular self generating mechanism as the Kondratieff cycle that can be extrapolated into the future (i.e. an endogenous cycle). Another explanation could be that the historical fluctuations are the consequence of very distinct historical time periods during which different types of macroeconomic policies were conducted and very specific major historical events have occurred (i.e. an exogenous cycle). In this view the observed long run fluctuations hold no information for the evolution of economies in the oncoming decades. More information and references on the long wave in economic life are given in section 2.2.3.

Second, note that the information on the *amplitudes* of the long waves together with the long term averages or growth rates as described in the previous section can easily be combined to give information on the extremes of the range in which long term averages (comprising several decades) of the various macroeconomic variables have evolved in the past, and possibly also will evolve in the future. For this it is important to remember that the long wave fluctuations account for approximately 60% of the volatility of the broader periodic components (*SF 18*). As a first interesting example consider the (ex post) *real long term interest rate*. *SF 10* states that its long term average is approximately 3%. *SF 27* states that the amplitude of its long wave fluctuations is approximately 2%. Combining these two, implies that long run averages (comprising several decades) of real interest rates can

be between numbers as far apart as  $3\%-2\%=1\%$  and  $3\%+2\%=5\%$ . As a second interesting example consider the (ex post) *equity risk premium* over long term bonds. Its long term (arithmetic) average is 5% (SF 16) while the amplitude of its long wave fluctuations is approximately 4%. Long term averages (again comprising several decades) of the equity risk premium can therefore be as far apart as  $5\%-4\%=1\%$  and  $5\%+4\%=9\%$ . Such wide ranges in long term averages in both real interest rates and equity premiums can have an enormous impact on for example long term asset allocation decisions.

Third and last, we want to point at the fact that for most of the variables the long wave lead or lag when compared to the national product seems intuitively logical. Think for example of equities leading the national product because of their forward looking behavior (SF 29) and default spreads being low when economic growth is at its highest level when default risk is low and vice versa (SF 28). Perhaps the most surprising are the lead / lag relations of the long waves in real interest rates and consumer price indices. SF 22 and SF 27 respectively indicate that both *changes* in consumer prices (i.e. inflation) and *changes* in real interest rates moved in parallel with the *level* of the national product. Especially for the real interest rates it is hard to imagine why they should be lagging the national product by approximately a quarter of a cycle as they have done in the past. The same holds for the fact that the level of consumer prices and the level of long term real interest rates apparently have moved in parallel with each other.

Table 17.2 *Top left*: Standard deviation of filtered long wave components for the samples for which the national product indices are available (NL: 1870-1999, UK: 1855-1999 and US: 1870-1999) or shorter if required by the relevant series. *Top right*: the approximated amplitude of the long wave components. These are approximated as  $\sqrt{2}$  times the standard deviation. For this see Appendix B.17. *Bottom left*: The part of the standard deviation of the periodic component that is described by the long wave component. *Bottom right*: The correlation of the long wave component with the long wave component in the relevant national product index. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

	Standard deviation				Amplitude				Ratio
	NL	UK	US	Avg	NL	UK	US	Avg	
NP	6.3%	2.3%	1.4%	3.3%	8.9%	3.3%	2.0%	4.7%	1.0
IP	14.7%	2.6%	6.8%	8.0%	20.8%	3.7%	9.6%	11.4%	2.4
EM	2.9%	2.0%	3.0%	2.6%	4.1%	2.8%	4.2%	3.7%	0.8
PI	8.5%	11.5%	13.6%	11.2%	12.0%	16.3%	19.2%	15.8%	3.4
WI	15.8%	8.2%	3.7%	9.2%	22.3%	11.6%	5.2%	13.1%	2.8
SR	0.8%	1.5%	1.3%	1.2%	1.1%	2.1%	1.8%	1.7%	0.4
LR	0.9%	1.3%	1.0%	1.1%	1.3%	1.8%	1.4%	1.5%	0.3
TS	0.3%	0.4%	0.6%	0.4%	0.4%	0.6%	0.8%	0.6%	0.1
DS	-	0.3%	0.2%	0.3%	-	0.4%	0.3%	0.4%	0.1
TR	20.1%	18.6%	6.1%	14.9%	28.4%	26.3%	8.6%	21.1%	4.5
RR	1.5%	1.5%	1.3%	1.4%	2.1%	2.1%	1.8%	2.0%	0.4
RP	3.4%	2.3%	3.4%	3.0%	4.8%	3.3%	4.8%	4.3%	0.9
	Ratio periodic component				Correlation national product				
	NL	UK	US	Avg	NL	UK	US	Avg	
NP	0.5	0.3	0.1	0.3	-	0.1*	0.6*	-	
IP	1.0	0.4	0.5	0.7	1.0	1.0	0.4	0.8	
EM	0.8	0.7	0.7	0.8	-0.2	0.7	0.1	0.2	
PI	0.5	0.1	0.7	0.3	0.6	-0.1	0.3	0.3	
WI	1.0	0.6	0.2	0.6	0.9	-0.5	0.1	0.2	
SR	0.5	0.5	0.7	0.6	0.8	0.6	0.9	0.8	
LR	0.8	0.6	0.5	0.6	0.8	0.4	0.7	0.6	
TS	0.5	0.4	0.7	0.5	-0.1	-0.7	-0.8	-0.5	
DS	-	0.8	0.7	0.7	-	-0.1	0.7	0.3	
TR	0.6	0.4	0.2	0.4	-0.5	-0.6	-0.4	-0.5	
RR	0.6	0.4	0.4	0.4	0.7	-0.4	0.2	0.2	
RP	0.7	0.7	0.8	0.7	-1.0	-0.4	-0.5	-0.6	
Avg	0.7	0.5	0.5	0.6					

\* Correlation with long wave in NL national product.

Table 17.3 Average dating of long wave peaks and troughs per variable, calculated from the separate NL, UK and US results. Also the corresponding peak-to-peak and trough-to-trough period lengths (*Italic*) and the lead compared to the (average) national product long wave are given. The bottom line shows the average dating of peaks and troughs calculated from the (average) long waves in each of the separate variables after correction for the average phase difference reported in the final column. Hence, this can be seen as a dating of “the” long wave in economic life. For an explanation of the abbreviations see the heading of Table 17.1.

	<i>Peak-Peak Trough-Trough Lead</i>								
	1	2	3	4	5				
NP		1870	49	1919	53	1971			51
			1893	51	1945	53	1998		52
IP		1872	44	1916	52	1968			48
			1888	55	1943	51	1994		53
		-3	5	3	2	3	4		2
EM		1858	53	1911	48	1958			50
			1887	48	1935	48	1982		48
		12	6	8	10	13	16		11
PI	1812	53	1865	60	1925	61	1987		58
	1787	50	1837	57	1894	64	1958	41	1999
			5	-1	-7	-13	-15	-1	-5
WI	1846	26	1872	49	1921	54	1974		43
		1866	28	1894	52	1946	50	1996	43
			-3	-1	-2	-1	-3	2	-1
SR		1857	66	1923	55	1978			60
		1838	54	1892	58	1951	48	1999	54
			13	1	-4	-6	-6	-1	-1
LR	1808	57	1865	59	1924	55	1979		57
	1789	41	1830	65	1895	58	1953	46	1999
			5	-2	-5	-8	-8	-1	-3
TS	1833	55	1888	47	1935	55	1990		53
		1863	50	1913	56	1968			53
			-18	-19	-16	-24	-19		-19
DS				1931	46	1976			46
				1898	57	1955	42	1997	49
				-5	-12	-10	-5		-8
TR	1838	21	1859	46	1905	48	1953		38
		1860	22	1882	47	1928	49	1977	39
			11	12	14	16	19	21	15
RR	1826	52	1878	51	1929	51	1980		51
	1802	49	1851	53	1904	51	1955	44	1999
			-9	-11	-11	-10	-9	-1	-8
RP	1826	30	1856	51	1907	46	1952	42	1994
		1834	45	1879	55	1934	39	1973	46
			14	15	12	11	19	25	16
Average	1826	42	1868	52	1920	52	1972		49
	1790	57	1847	47	1893	54	1947	47	1994

## 17.2 Business cycles

From the formulation of the first research objective in section 1.3 it is clear that besides the long run behavior of macroeconomic variables we are also interested in the shorter term business cycle fluctuations. In the filtering process these are supposed to be contained in the *stochastic components* defined as all fluctuations with a period length shorter than fifteen years. That is, all fluctuations within the frequency range  $[1/15, 0.5]$ . In the next sub-sections the focus is on stylized facts with respect to the dynamics of these stochastic components in terms of

1. their conventional volatility and correlations,
2. the Juglar and Kitchin type business cycle fluctuations indicated by their spectral densities and
3. the (spectral) properties of the first order differences (growth rates or returns) of the variables.

### 17.2.1 Volatility and correlations

In principle we analyzed the dynamics of the stochastic components in the frequency domain because this provides information on the dynamics which is the most easy to interpret. However, because the estimated normalized spectral densities “merely” show how the total variance of the stochastic components is distributed over the entire frequency range, Table 17.5 gives a separate summary of the conventional *absolute standard deviations* of the stochastic components and, perhaps even more important, of their *relative standard deviations*. Table 17.6 shows how much of the conventional standard deviations of growth rates and returns is preserved in the filtered stochastic (business cycle) components. Although the interdependence between the business cycle fluctuations in the various variables is best described in the frequency domain by the coherence and the phase spectrum, Tables 17.4 and 17.7 through 17.10 also summarize the conventional contemporaneous correlations between the stochastic components. The results in these tables support the following stylized facts.

- SF 32:* The business cycle fluctuations contained in the filtered stochastic components of the relevant variables, describe more than 90% of the *volatility* (standard deviation) of the standard deviations of the original, not filtered, *growth or return rates*.
- SF 33:* With the exception of the interwar period, the *volatility* of the business cycle fluctuations in the *national product indices* is approximately the same in the NL, the UK and the US at some 2 to 3%. In general the fluctuations are *positively correlated* while the correlation between the UK and the US is on average twice as high as between the NL and both other countries.

- SF 34:* The business cycle fluctuations in the *industrial production indices* are approximately 1.8 times as volatile as those in the national product indices. They show a consistently *high correlation* with the national product fluctuations (overall average correlation of 0.7). Industrial production is therefore *procyclical*.
- SF 35:* The business cycle fluctuations in the *employment rates* are approximately 0.7 times as volatile as those in the national product indices. They are *positively correlated* with the national product fluctuations (overall average correlation of 0.4). Employment is therefore *procyclical*.
- SF 36:* The business cycle fluctuations in the *consumer price indices* are approximately 1.4 times as volatile as those in the national product indices. Their *correlation* with national product fluctuations has undergone marked *changes* through time. During the prewar period it is mildly positive (average correlation of 0.2), during the interwar period it is more strongly positive (average correlation of 0.4) while during the postwar period it is consistently negative (average correlation of -0.3). This suggests *procyclical* behavior of consumer prices for the prewar and interwar periods and *countercyclical* behavior for the postwar period.
- SF 37:* During the postwar period *consumer price inflation* has shown higher *autocorrelations* than during the prewar and interwar periods. Prices have therefore been more *persistent* during the postwar period. Such changes have *not* occurred in the business cycle fluctuations (stochastic components) of consumer prices. The fundamental underlying dynamics of the business cycles and long waves therefore have not changed. Instead, the contribution to the total variance of these phenomena has changed and thereby caused an increase in the autocorrelations. For this see Table 17.5 and *SF 65*.
- SF 38:* The business cycle fluctuations in the *industry wage indices* are approximately 1.1 times as volatile as those in the national product indices. During the postwar period they are *positively correlated* with the national product fluctuations (average postwar correlation of 0.3). Industry wages are therefore *procyclical* during that period.
- SF 39:* The business cycle fluctuations in the *short term interest rates* are approximately 0.5 as volatile as those in the national product indices. At the most common level of short term interest rates of some 3 to 4%, their standard deviation is approximately 1%. They are *positively correlated* with the national product fluctuations (overall average correlation of 0.2). Short term interest rates are therefore *procyclical*.



- SF 40:* The volatility of the business cycle fluctuations in *short term interest rates* are higher / lower when the underlying level of interest rates is higher / lower. Due to this *level effect* their standard deviation increases by approximately 0.3% for every 1% increase in the underlying level of interest rates.
- SF 41:* The business cycle fluctuations in the *long term interest rates* are approximately 0.2 times as volatile as those in the national product indices. Therefore in the business cycle frequency range long term interest rates are about *twice as stable* as short term interest rates. At the most common level of long term interest rates of say 5%, their standard deviation is approximately 0.5%. They are sometimes slightly positive and also sometimes slightly negative correlated with the national product fluctuations (overall average correlation of 0.1) and long term interest rates are therefore *acyclical* or mildly *procyclical*.
- SF 42:* The volatility of the business cycle fluctuations in *long term interest rates* are higher / lower when the underlying level of interest rates is higher / lower. Due to this *level effect* their standard deviation increases by approximately 0.15% for every 1% increase in the underlying level of interest rates. Therefore the strength of this level effect is about twice as small as for the short term interest rates.
- SF 43:* The business cycle fluctuations in the *term spread* are approximately 0.4 times as volatile as those in the national product indices. They are *negatively correlated* with the national product fluctuations (overall average correlation of -0.2). The term spread is therefore *countercyclical*.
- SF 44:* The business cycle fluctuations in “the” *default spread* are approximately 0.1 times as volatile as those in the national product indices. Therefore in the business cycle frequency range default spreads are about four times as stable as term spreads. At the most common level of default spreads of say 1%, their standard deviation is approximately 0.25%. Their low correlations and sometimes also negative correlations with the national product fluctuations indicate that default spreads are *acyclical* or slightly *countercyclical*. Inverted default spreads move very much in line with the growth of the national product (average postwar correlation of 0.4).
- SF 45:* The volatility of the business cycle fluctuations in *default spreads* are higher / lower when the underlying level of default spreads is higher / lower. Due to this *level effect* their standard deviation increases by approximately 0.3% for every 1% increase in the underlying level. Therefore the strength of this level effect is comparable to that for the short term interest rates.



*SF 46:* The business cycle fluctuations in the *real equity total return indices* are approximately 5 times as volatile as those in the national product indices. They show a consistently *positive correlation* with the national product fluctuations (overall average correlation of 0.2) and real equity indices are therefore *procyclical*.

Although it is not reported as a formal stylized fact, section 20.4.2 provides evidence that a *level effect* similar to that described in *SF 40* (nominal short term interest rates), *SF 42* (nominal long term interest rates) and *SF 45* (default spread) is also present in postwar consumer price inflation rates. Because such a level effect is not found for the other variables that all hold in real (volume) terms and hence are all corrected for changes in the price level, this raises the suspicion that the price inflation process is in fact the fundamental cause of the level effects found here. If this is true, for example the volatile inflation rates during the 1970's caused the volatile nominal interest rates during that period, instead of the underlying (real) interest rate process itself.

Table 17.4 Contemporaneous correlations between the stochastic (business cycle) components in the NL, UK and US national product indices for the various sub-periods. All correlations are calculated on the (individual) longest possible samples. In gray are important parts of the table that are referred to in the stylized facts.

	<i>Prewar</i>			<i>Interwar</i>			<i>Postwar</i>			<i>Average</i>		
	<i>NL</i>	<i>UK</i>	<i>US</i>	<i>NL</i>	<i>UK</i>	<i>US</i>	<i>NL</i>	<i>UK</i>	<i>US</i>	<i>NL</i>	<i>UK</i>	<i>US</i>
NL	1.0			1.0			1.0			1.0		
UK	0.2	1.0		0.0	1.0		0.5	1.0		0.2	1.0	
US	-0.1	0.2	1.0	0.5	0.3	1.0	0.2	0.6	1.0	0.2	0.4	1.0

Table 17.5 Absolute and relative *standard deviations* of the filtered stochastic (business cycle) components of the various variables for the three countries and three sub-periods analyzed. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

	Prewar					Interwar					Postwar					Avg Ratio
	NL	UK	US	Avg	Ratio	NL	UK	US	Avg	Ratio	NL	UK	US	Avg	Ratio	
NP	3.3%	2.6%	3.4%	3.1%	1.0	1.8%	3.9%	4.5%	3.4%	1.0	2.0%	2.1%	2.2%	2.1%	1.0	
IP	-	4.3%	7.5%	5.9%	1.9	3.3%	7.3%	10.3%	7.0%	2.0	2.1%	2.8%	4.2%	3.0%	1.4	
EM	-	2.1%	2.3%	2.2%	0.7	2.2%	2.7%	5.0%	3.3%	1.0	1.5%	1.2%	0.9%	1.2%	0.6	
PI	4.9%	4.6%	3.2%	4.2%	1.4	4.4%	8.4%	5.7%	6.2%	1.8	1.7%	2.9%	2.0%	2.2%	1.0	
WI	-	3.1%	3.1%	3.1%	1.0	1.9%	6.8%	4.3%	4.4%	1.3	1.9%	2.6%	2.3%	2.2%	1.1	
SR	0.6%	1.0%	1.5%	1.0%	0.3	0.9%	1.1%	0.8%	0.9%	0.3	1.7%	1.6%	1.6%	1.6%	0.8	
LR	0.2%	0.1%	0.2%	0.2%	0.1	0.2%	0.4%	0.3%	0.3%	0.1	0.8%	1.0%	0.8%	0.9%	0.4	
TS	0.5%	1.0%	1.3%	0.9%	0.3	0.9%	0.8%	0.9%	0.9%	0.2	1.2%	1.3%	1.1%	1.2%	0.6	
DS	-	-	0.3%	0.3%	0.1	-	0.2%	0.2%	0.2%	0.1	-	0.4%	0.3%	0.4%	0.2	
TR	9.7%	3.9%	10.2%	7.9%	2.5	22.8%	12.8%	23.0%	19.6%	5.7	17.9%	18.7%	10.5%	15.7%	7.4	

Table 17.6 Ratio of *standard deviations of first order differences* of filtered stochastic components and standard deviations of original *growth* or *return* (delta logarithm) statistics of unfiltered series. In gray are important parts of the table that are referred to in the stylized facts.

	Prewar			Interwar			Postwar						
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	Avg
NP	0.97	0.99	0.97	0.98	0.67	0.98	0.75	0.80	0.83	0.95	0.93	0.90	0.89
IP	-	0.99	0.99	0.99	0.75	1.00	0.92	0.89	0.72	0.96	0.97	0.88	0.92
PI	0.98	0.98	0.89	0.95	0.98	0.95	0.98	0.97	0.63	0.61	0.60	0.62	0.84
WI	-	0.97	0.99	0.98	0.98	0.99	0.94	0.98	0.78	0.95	0.89	0.86	0.94
SR	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.98	0.98	0.98	0.99
LR	0.97	1.00	0.94	0.97	1.00	1.00	1.00	1.00	0.96	0.94	0.95	0.95	0.97
TR	0.96	0.93	0.97	0.96	0.99	0.88	1.00	0.96	0.94	0.97	0.90	0.94	0.95
Avg				0.97				0.94				0.88	0.93

Table 17.7 Contemporaneous *correlations* between the stochastic (business cycle) components for the various series and sub-periods analyzed for the NL. All correlations are calculated on the (individual) longest possible samples. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

<i>Prewar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	-	-								
EM	-	-	-							
PI	0.1	-	-	1.0						
WI	-	-	-	-	-					
SR	-0.2	-	-	0.1	-	1.0				
LR	0.0	-	-	0.0	-	0.2	1.0			
TS	0.2	-	-	0.0	-	-0.8	0.3	1.0		
DS	-	-	-	-	-	-	-	-	-	
TR	0.0	-	-	-0.4	-	0.2	-0.4	-0.4	-	1.0
<i>Interwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.2	1.0								
EM	0.2	0.3	1.0							
PI	0.2	0.4	0.6	1.0						
WI	-0.5	0.1	0.0	-0.2	1.0					
SR	0.3	0.1	0.2	0.2	-0.4	1.0				
LR	0.2	0.1	0.2	0.5	0.2	0.1	1.0			
TS	-0.2	-0.1	-0.2	-0.1	0.4	-1.0	0.2	1.0		
DS	-	-	-	-	-	-	-	-	-	
TR	0.2	0.2	0.6	0.6	-0.5	0.3	-0.2	-0.4	-	1.0
<i>Postwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.7	1.0								
EM	0.5	0.6	1.0							
PI	-0.4	-0.3	-0.4	1.0						
WI	0.3	0.1	0.4	-0.1	1.0					
SR	0.2	0.3	0.4	-0.1	0.2	1.0				
LR	-0.1	0.1	0.2	0.0	0.0	0.8	1.0			
TS	-0.4	-0.3	-0.5	0.1	-0.2	-0.9	-0.4	1.0		
DS	-	-	-	-	-	-	-	-	-	
TR	0.4	0.2	0.0	-0.2	0.1	-0.4	-0.5	0.3	-	1.0

Table 17.8 Contemporaneous *correlations* between the stochastic (business cycle) components for the various series and sub-periods analyzed for the UK. All correlations are calculated on the (individual) longest possible samples. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

<i>Prewar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.6	1.0								
EM	0.5	0.9	1.0							
PI	0.3	0.4	0.2	1.0						
WI	0.1	0.0	0.1	-0.8	1.0					
SR	0.2	0.5	0.6	0.3	-0.2	1.0				
LR	0.4	0.2	0.2	0.1	-0.3	0.4	1.0			
TS	-0.2	-0.5	-0.6	-0.3	0.2	-1.0	-0.3	1.0		
DS	-	-	-	-	-	-	-	-	-	
TR	-0.1	-0.2	-0.1	-0.5	0.8	-0.4	-0.6	0.4	-	1.0
<i>Interwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.8	1.0								
EM	0.9	0.9	1.0							
PI	0.7	0.7	0.8	1.0						
WI	-0.5	-0.7	-0.6	-0.6	1.0					
SR	0.4	0.3	0.5	0.8	-0.1	1.0				
LR	0.4	0.2	0.4	0.6	-0.1	0.8	1.0			
TS	-0.3	-0.3	-0.4	-0.6	0.2	-0.9	-0.6	1.0		
DS	0.2	0.1	0.1	-0.1	0.2	0.2	-0.2	-0.4	1.0	
TR	0.3	0.4	0.3	-0.1	-0.3	-0.4	-0.6	0.2	0.1	1.0
<i>Postwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.8	1.0								
EM	0.7	0.6	1.0							
PI	-0.1	-0.3	0.2	1.0						
WI	0.2	0.4	0.0	-0.8	1.0					
SR	0.3	0.1	0.7	0.5	-0.2	1.0				
LR	0.1	0.0	0.2	0.2	0.0	0.5	1.0			
TS	-0.3	-0.2	-0.7	-0.4	0.2	-0.8	0.2	1.0		
DS	0.0	0.0	0.3	0.1	-0.1	0.4	0.3	-0.2	1.0	
TR	-0.1	0.0	-0.2	-0.3	0.2	-0.4	-0.7	0.0	-0.3	1.0

Table 17.9 Contemporaneous *correlations* between the stochastic (business cycle) components for the various series and sub-periods analyzed for the US. All correlations are calculated on the (individual) longest possible samples. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

<i>Prewar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.8	1.0								
EM	-0.7	-0.6	1.0							
PI	0.4	0.0	-0.6	1.0						
WI	-0.2	0.1	0.4	-0.6	1.0					
SR	0.2	0.1	-0.2	0.1	-0.1	1.0				
LR	-0.1	-0.1	-0.1	0.1	0.0	0.3	1.0			
TS	-0.3	-0.2	0.2	-0.1	0.0	-1.0	-0.3	1.0		
DS	0.3	0.2	-0.2	0.1	-0.3	0.4	-0.1	-0.5	1.0	
TR	0.4	0.4	-0.3	0.3	-0.3	-0.1	-0.3	0.0	0.1	1.0
<i>Interwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.8	1.0								
EM	0.4	0.4	1.0							
PI	0.4	0.4	0.9	1.0						
WI	0.7	0.6	0.4	0.5	1.0					
SR	0.3	0.1	0.2	0.4	0.6	1.0				
LR	0.1	0.0	-0.5	-0.4	0.2	0.2	1.0			
TS	-0.2	-0.1	-0.4	-0.5	-0.4	-0.8	0.3	1.0		
DS	0.0	-0.3	0.0	0.2	0.2	0.5	0.2	-0.4	1.0	
TR	0.2	0.3	0.7	0.7	0.1	0.2	-0.6	-0.4	-0.1	1.0
<i>Postwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.9	1.0								
EM	0.9	0.8	1.0							
PI	-0.4	-0.3	-0.1	1.0						
WI	0.4	0.5	0.3	-0.8	1.0					
SR	0.2	0.2	0.3	0.7	-0.6	1.0				
LR	-0.1	0.0	0.1	0.7	-0.5	0.6	1.0			
TS	-0.1	-0.1	-0.3	-0.4	0.3	-0.7	-0.1	1.0		
DS	-0.2	-0.3	-0.1	0.5	-0.6	0.3	0.0	-0.4	1.0	
TR	0.3	0.0	0.0	-0.4	0.3	-0.2	-0.2	0.3	-0.2	1.0

Table 17.10 Average contemporaneous *correlations* between the stochastic (business cycle) components for the various series and sub-periods analyzed for the NL, UK and US as reported in Tables 17.7, 17.8 and 17.9. The bottom correlation matrix reports the averages of the (average) correlations for the three sub-periods. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

<i>Prewar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.7	1.0								
EM	-0.1	0.2	1.0							
PI	0.2	0.2	-0.2	1.0						
WI	-0.1	0.0	0.3	-0.7	1.0					
SR	0.1	0.3	0.2	0.2	-0.1	1.0				
LR	0.1	0.1	0.0	0.1	-0.2	0.3	1.0			
TS	-0.1	-0.4	-0.2	-0.1	0.1	-0.9	-0.1	1.0		
DS	0.3	0.2	-0.2	0.1	-0.3	0.4	-0.1	-0.5	1.0	
TR	0.1	0.1	-0.2	-0.2	0.2	-0.1	-0.4	0.0	0.1	1.0
<i>Interwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.6	1.0								
EM	0.5	0.6	1.0							
PI	0.4	0.5	0.8	1.0						
WI	-0.1	0.0	-0.1	-0.1	1.0					
SR	0.3	0.2	0.3	0.5	0.0	1.0				
LR	0.2	0.1	0.0	0.3	0.1	0.4	1.0			
TS	-0.2	-0.2	-0.3	-0.4	0.1	-0.9	0.0	1.0		
DS	0.1	-0.1	0.0	0.0	0.2	0.4	0.0	-0.4	1.0	
TR	0.2	0.3	0.5	0.4	-0.2	0.0	-0.4	-0.2	0.0	1.0
<i>Postwar</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.8	1.0								
EM	0.7	0.7	1.0							
PI	-0.3	-0.3	-0.1	1.0						
WI	0.3	0.3	0.2	-0.6	1.0					
SR	0.2	0.2	0.5	0.3	-0.2	1.0				
LR	0.0	0.0	0.1	0.3	-0.1	0.6	1.0			
TS	-0.3	-0.2	-0.5	-0.2	0.1	-0.8	-0.1	1.0		
DS	-0.1	-0.2	0.1	0.3	-0.4	0.4	0.1	-0.3	1.0	
TR	0.2	0.1	-0.1	-0.3	0.2	-0.3	-0.5	0.2	-0.3	1.0
<i>Average</i>	<i>NP</i>	<i>IP</i>	<i>EM</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TS</i>	<i>DS</i>	<i>TR</i>
NP	1.0									
IP	0.7	1.0								
EM	0.4	0.5	1.0							
PI	0.1	0.1	0.2	1.0						
WI	0.0	0.1	0.1	-0.5	1.0					
SR	0.2	0.2	0.3	0.3	-0.1	1.0				
LR	0.1	0.1	0.1	0.2	-0.1	0.4	1.0			
TS	-0.2	-0.3	-0.3	-0.3	0.1	-0.9	-0.1	1.0		
DS	0.1	0.0	0.0	0.1	-0.2	0.4	0.0	-0.4	1.0	
TR	0.2	0.2	0.1	-0.1	0.1	-0.1	-0.4	0.0	-0.1	1.0

### 17.2.2 Juglar and Kitchin

As explained in section 6.4, the dynamics of the business cycle fluctuations contained in the stochastic components of the variables were investigated by means of (parametric) Maximum Entropy spectral analysis. Virtually all the estimated spectral densities are to some extent characterized by spectral peaks around what can be called the typical Juglar and Kitchin frequencies. See section 2.2.3 for a short description of these two types of business cycle fluctuations. The Tables 17.11 through 17.15 summarize the specific findings with respect to the Juglar and Kitchin spectral peaks. The results in these tables support the following stylized facts.

*SF 47:*        *Juglar* type business cycle fluctuations have been present in all macroeconomic variables for as long as data has been collected for these variables. They are present in 95% of the sub-sample  $\times$  variable combinations for which a spectral density has been estimated. In 99% of these cases their contribution to the total business cycle variance has been significantly different from what it would have been in the case of a white noise process at the 95% confidence level.

*SF 48:*        In principle the same holds for the *Kitchin* type business cycle fluctuations. They are present in 93% of the sub-sample  $\times$  variable combinations. A marked difference is that only in 31% of these cases their contribution to the total business cycle variance has been significantly different from what it would have been in the case of a white noise process at the 95% confidence level<sup>90</sup>.

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<sup>90</sup> In the light of the results in the oncoming section 17.2.3, it is important to note that had we estimated spectral densities for the stochastic components in the *growth* and *return* statistics instead of for the (logarithmic) *levels*, the picture with respect to the significance of the Juglar and Kitchin spectral peaks might very well have been the other way around. In the growth and return statistics the Kitchin peak is in general much higher while the Juglar peak is lower than for the (logarithmic) levels. Therefore, in that case the Kitchin peak might very well be significant more often than the Juglar peak. This shows the potential dangers of clinging on too much to the results of statistical significance testing. The outcomes of such tests, amongst others, depend on what *type* of series (growth rates or levels) are tested for being different than a white noise process while the underlying phenomenon might very well be one and the same (Juglar and Kitchin business cycle fluctuations).



Table 17.11 The number of samples (variables  $\times$  sub-periods) for which the estimated spectral density contains a peak at the Juglar and Kitchin frequencies and the number of samples in which these peaks are significant at a 95% confidence level. In gray are important parts of the table that are referred to in the stylized facts.

	<i>Prewar</i>	<i>Interwar</i>	<i>Postwar</i>	<i>Total</i>
Samples	25	29	29	83
Juglar present	25	25	29	79
Percentage of samples	100%	86%	100%	95%
Juglar significant	24	25	29	78
Percentage of present	96%	100%	100%	99%
Kitchin present	22	27	28	77
Percentage of samples	88%	93%	97%	93%
Kitchin significant	6	6	12	24
Percentage of present	27%	22%	43%	31%

*SF 49:* The overall average period length of the Juglar spectral peak is approximately 10 years. These fluctuations typically describe 30% of the total business cycle variance (*peak power*). The *modulus* of the peak is 0.9 which indicates a high degree of regularity (periodicity) in the fluctuations. The *coherence* with the national product fluctuations of the same frequency is typically around 80% which indicates a high, phase corrected, correlation<sup>91</sup>.

*SF 50:* The overall average period length of the Kitchin spectral peak is approximately 4.5 years. These fluctuations typically describe 20% of the total business cycle variance (*peak power*). The *modulus* of the peak is 0.8 which indicates a high degree of regularity (periodicity) in the fluctuations, though less than for the Juglar type fluctuations. The *coherence* with the national product fluctuations of the same frequency is typically around 70% which indicates a high, phase corrected, correlation, though again lower than for the Juglar type fluctuations<sup>79</sup>.

In short, the stylized facts *SF 47* through *SF 50* state that Juglar and Kitchin type business cycle fluctuations have been present in all macroeconomic variables analyzed, both from the real and financial sector of the economy, for as long as we can remember. Although these fluctuations are not perfectly periodic, they show a high degree of stochastic regularity. Together these types of fluctuations describe around 50% of all annual macroeconomic fluctuations with a period length shorter than fifteen years. The Kitchin type fluctuations are often encountered in the literature. Based on the method for dating business cycles of Burns and Mitchell (1946), about 80% of the US cycles during the 1854-1933 and 1938-1982 periods fall within the typical Kitchin period length of between 3 and 5 years. For this see tables A.4 and A.5 in Moore and Zarnowitz (1986). Also the existing Real Business Cycle (RBC) literature typically focuses on these types of business cycle fluctuations. Also the Juglar type fluctuations are encountered in the literature more often than one

<sup>91</sup> Based on the results from the EVR VAR models estimated in section 20.5.3, it might be argued that such coherence numbers are too high to be true. They might be the consequence of the types of (unrestricted) VAR models used for estimating the bivariate spectra.

might think. They are also known as *medium* or *major* cycles. For the UK they are reported since at least 1790 by Jevons (1964), Gayer (1953), Matthew (1959) and Klotz (1977). For the US they are reported in long samples of data by Hicks (1950), Hansen (1951) and Howrey (1968) and more recently also by Fan (1992). Finally, Englund et al. (1992) reports evidence of Juglar type cycles in 128 years of annual Swedish data.

The high coherences observed at the Juglar and Kitchin frequencies indicate strong, phase corrected, correlations between these fluctuations in the various variables. This means that the estimated values for the phase differences between the variables can be interpreted with confidence. The following stylized facts relate to these phase differences. It is important to remember that the underlying data is on an annual basis, most of the times also consisting of relatively short (sub-)samples. This means that the reported values of the phase (in years) should not be given a too precise interpretation. Given its “roughness”, the data simply cannot be expected to give very detailed information about phase differences in terms of a number of months. Instead, the reported phases should “merely” be seen as an indication for the distinction between great leads or lags of several years on the one hand and only small phase differences on the other hand.

- SF 51:* The Juglar type fluctuations in the *national product indices* in the UK and the US typically *lead* those in the NL by approximately 1.5 years. At the Kitchin frequencies this *lead* is less than a year. These moderate phase differences are consistent with the positive correlations mentioned in *SF 33*<sup>92</sup>. The fact that the UK and US business cycles move more in parallel also explains why their correlation is higher than those between the NL and each of the other two countries.
- SF 52:* Both the Juglar and the Kitchin fluctuations in the *industrial production indices* show *no significant lead or lag* compared to those in the national product indices. This is consistent with the high correlations and procyclical behavior mentioned in *SF 34*.
- SF 53:* The Juglar fluctuations in the *employment rates lag* those in the national product indices by approximately *half a year*. At the Kitchin frequencies the lag is smaller, even close to zero. This is consistent with the positive correlations and procyclical behavior mentioned in *SF 35*.

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<sup>92</sup> It is easy to verify that two perfect sinusoids of the same frequency / period length show a positive correlation if their phase difference lies between a lead and a lag of a quarter period length. The correlation is negative if the phase difference exceeds a quarter cycle and reaches its maximum negative correlation at a phase difference of half a cycle. For the Juglar cycles with a period length of approximately 10 years this means that leads or lags exceeding 2.5 years cause negative correlations and positive correlations otherwise. For the Kitchin cycles with a period length of approximately 4.5 years this means that leads or lags exceeding 1.1 years cause negative correlations and positive correlations otherwise. These rules of thumb can be used to see the consistency between the reported lead / lag relations in this section and the conventional correlations between the business cycle fluctuations as reported in the previous sub-sections.

Table 17.12 Statistics of the Juglar peak in the estimated spectral densities for the various variables and sub-periods. The missing elements indicate either that the relevant spectrum does not contain a Juglar peak, that no data was available for estimation or that it is non-informative (average phase for the NP). If a period length is printed underlined the corresponding spectral peak is significant at the 95% confidence level. The abbreviations are explained in heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

Pre war	Period length				Peak Power				Modulus				Coherence*				Phase*				
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	
NP	8.5	13.3	9.6	10.5	0.2	0.2	0.2	0.2	0.9	0.9	0.9	0.9	-	0.3	0.9	0.6	-	-0.5	4.7	-	
IP	-	8.5	9.5	9.0	-	0.4	0.3	0.3	-	1.0	0.9	0.9	-	1.0	1.0	1.0	-	-0.2	0.6	0.2	
EM	-	8.4	9.1	8.8	-	0.3	0.3	0.3	-	0.9	0.9	0.9	-	0.9	0.9	0.9	-	-0.1	-3.6	-1.8	
PI	12.5	11.4	8.9	11.0	0.2	0.3	0.4	0.3	1.0	0.9	1.0	1.0	0.8	0.6	0.9	0.8	-4.9	-1.9	-0.6	-2.5	
WI	-	7.5	7.8	7.6	-	0.4	0.5	0.4	-	0.9	0.9	0.9	-	0.6	0.6	0.6	-	-1.1	-3.1	-2.1	
SR	8.9	8.9	13.3	10.4	0.3	0.3	0.5	0.4	0.9	0.9	0.8	0.9	0.6	0.9	0.7	0.7	-4.0	-0.4	1.2	-1.0	
LR	9.4	11.8	9.9	10.4	0.3	0.3	0.2	0.3	0.9	1.0	0.9	0.9	0.5	0.9	0.9	0.8	-3.9	-1.4	-3.4	-2.9	
TS	8.9	9.2	11.3	9.8	0.3	0.3	0.3	0.3	1.0	0.9	0.9	0.9	0.4	0.9	0.7	0.7	-0.1	4.3	-4.6	-0.1	
DS	-	-	11.7	11.7	-	-	0.4	0.4	-	-	0.9	0.9	-	-	0.8	0.8	-	-	0.8	0.8	
TR	9.2	11.9	8.2	9.8	0.4	0.2	0.3	0.3	0.9	0.9	0.9	0.9	0.8	0.4	0.9	0.7	-0.2	-5.5	0.6	-1.7	
Avg	9.6	10.1	9.9	9.9	0.3	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.6	0.7	0.8	0.7	-	-	-	-	
Inter War	Period length				Peak Power				Modulus				Coherence*				Phase*				
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	
	NP	7.6	10.3	-	9.0	0.3	0.3	-	0.3	0.9	0.9	-	0.9	-	0.9	0.9	-	1.4	-	-	
	IP	-	10.0	-	10.0	-	0.3	-	0.3	-	0.9	-	0.9	-	1.0	-	1.0	-	0.6	-	0.6
	EM	10.7	10.3	10.4	10.5	0.4	0.3	0.5	0.4	0.9	0.9	0.9	0.9	0.7	1.0	0.8	0.8	-0.2	0.3	1.6	0.6
	PI	9.5	10.2	10.7	10.1	0.3	0.3	0.4	0.4	0.9	0.9	0.9	0.9	1.0	1.0	0.9	1.0	0.2	0.0	1.8	0.7
	WI	8.0	12.1	8.3	9.5	0.2	0.4	0.3	0.3	0.9	0.9	0.9	0.9	1.0	0.9	1.0	1.0	-3.3	-4.6	0.1	-2.6
	SR	7.5	10.4	8.4	8.8	0.3	0.3	0.4	0.3	0.9	0.9	0.9	0.9	0.9	1.0	0.9	0.9	1.1	-0.8	0.1	0.1
	LR	9.7	10.4	12.4	10.8	0.1	0.4	0.3	0.3	0.9	1.0	0.9	0.9	1.0	1.0	0.8	0.9	-1.6	-1.3	0.0	-1.0
	TS	7.2	11.5	11.4	10.0	0.5	0.3	0.4	0.4	0.8	0.9	0.9	0.9	0.8	1.0	0.8	0.9	-2.7	5.2	-5.2	-0.9
	DS	-	-	11.6	11.6	-	-	0.3	0.3	-	-	1.0	1.0	-	-	0.8	0.8	-	-	0.2	0.2
	TR	9.4	9.1	8.8	9.1	0.5	0.3	0.4	0.4	0.9	0.9	0.9	0.9	0.9	1.0	0.9	0.9	1.1	1.9	1.4	1.5
Avg	8.7	10.5	10.2	9.9	0.3	0.3	0.4	0.3	0.9	0.9	0.9	0.9	0.9	1.0	0.9	0.9	-	-	-	-	
Post War	Period length				Peak Power				Modulus				Coherence*				Phase*				
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	
	NP	9.9	11.1	10.9	10.6	0.4	0.4	0.3	0.4	0.9	1.0	0.9	0.9	-	1.0	0.8	0.9	-	1.4	1.4	-
	IP	8.9	10.1	10.3	9.8	0.2	0.3	0.2	0.2	0.9	0.9	0.9	0.9	0.9	1.0	1.0	0.9	0.2	0.3	-0.2	0.1
	EM	9.9	9.9	9.8	9.9	0.5	0.4	0.3	0.4	0.9	1.0	0.9	0.9	0.8	1.0	1.0	0.9	-0.7	-1.1	-0.1	-0.6
	PI	9.0	9.7	10.1	9.6	0.3	0.2	0.3	0.3	0.9	0.9	1.0	0.9	0.6	0.8	0.9	0.8	-4.2	-2.4	-3.9	-3.5
	WI	10.6	10.0	10.9	10.5	0.4	0.3	0.3	0.3	0.8	0.9	0.9	0.9	0.9	0.7	0.9	0.8	-1.6	1.9	1.2	0.5
	SR	10.1	9.3	8.7	9.4	0.3	0.3	0.4	0.3	0.9	0.9	0.9	0.9	0.9	1.0	0.7	0.9	-1.9	-1.6	-2.4	-2.0
	LR	10.1	7.7	9.3	9.1	0.4	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.9	0.7	1.0	0.8	-2.7	-1.8	-3.0	-2.5
	TS	11.1	9.9	8.0	9.6	0.2	0.3	0.3	0.3	0.9	1.0	0.9	0.9	0.9	1.0	0.8	0.9	4.3	3.2	2.6	3.4
	DS	-	9.9	8.1	9.0	-	0.1	0.2	0.2	-	0.9	0.9	0.9	-	0.9	0.8	0.9	-	-2.9	-2.1	-2.5
	TR	11.5	10.0	9.8	10.5	0.3	0.2	0.2	0.3	0.9	0.9	0.8	0.9	0.2	0.2	0.9	0.4	4.7	3.7	-0.3	2.7
Avg	10.1	9.8	9.6	9.8	0.4	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.8	0.8	0.9	0.8	-	-	-	-	

\* In principle the coherence and phase statistics hold relatively to the national product index of the relevant country. The only exceptions are the national product indices themselves. For the UK and US these hold relatively to the national product for the NL.

Table 17.13 Averages of the statistics of the Juglar spectral peak calculated over the three sub-period statistics as reported in Table 17.12. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

Avg	Period length				Peak Power				Modulus				Coherence*				Phase*			
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg
NP	8.7	11.6	10.2	10.0	0.3	0.3	0.3	0.3	0.9	0.9	0.9	0.9	-	0.7	0.9	0.8	-	0.8	3.0	-
IP	8.9	9.5	9.9	9.6	0.2	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.0	0.2	0.3	0.2	0.3
EM	10.3	9.5	9.8	9.7	0.5	0.4	0.3	0.4	0.9	0.9	0.9	0.9	0.7	0.9	0.9	0.9	-0.4	-0.3	-0.7	-0.6
PI	10.3	10.4	9.9	10.2	0.3	0.3	0.4	0.3	0.9	0.9	1.0	0.9	0.8	0.8	0.9	0.8	-3.0	-1.4	-0.9	-1.8
WI	9.3	9.9	9.0	9.2	0.3	0.4	0.4	0.4	0.9	0.9	0.9	0.9	0.9	0.7	0.8	0.8	-2.4	-1.3	-0.6	-1.4
SR	8.8	9.5	10.1	9.5	0.3	0.3	0.4	0.3	0.9	0.9	0.9	0.9	0.8	0.9	0.8	0.8	-1.6	-0.9	-0.4	-1.0
LR	9.8	10.0	10.5	10.1	0.3	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.8	0.9	0.9	0.8	-2.7	-1.5	-2.1	-2.1
TS	9.1	10.2	10.2	9.8	0.3	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.7	0.9	0.8	0.8	0.5	4.2	-2.4	0.8
DS	-	9.9	10.5	10.8	-	0.1	0.3	0.3	-	0.9	0.9	0.9	-	0.9	0.8	0.8	-	-2.9	-0.4	-0.5
TR	10.0	10.3	8.9	9.8	0.4	0.2	0.3	0.3	0.9	0.9	0.9	0.9	0.6	0.5	0.9	0.7	1.9	0.0	0.5	0.8
Avg	9.5	10.1	9.9	9.9	0.3	0.3	0.3	0.3	0.9	0.9	0.9	0.9	0.8	0.8	0.9	0.8	-	-	-	-

\* In principle the coherence and phase statistics hold relatively to the national product index of the relevant country. The only exceptions are the national product indices themselves. For the UK and US these hold relatively to the national product for the NL.

- SF 54:* The *phase* of the business cycle fluctuations in *consumer prices* has undergone marked *changes* through time. During the prewar period they *lag* by approximately 2.5 and 1 year at respectively the Juglar and the Kitchen frequencies. During the interwar period they show *small leads and lags* of around zero while during the postwar period they show even greater lags than during the prewar period of respectively on average 3.5 and 1.5 years. These lead / lag relations are consistent with the correlation pattern described in *SF 36*. In short, consumer prices lagged national product during both the prewar and postwar period were the lag during the latter period is greater than during the first. During the interwar period consumer prices moved almost in phase with national product.
- SF 55:* In principle the Juglar fluctuations in the *industry wage indices* lag those in the national product indices by approximately 1.5 year. At the Kitchen frequencies the lag is in principle around 1 year. This is consistent with the positive correlations and procyclical behavior mentioned in *SF 34*. Marked exceptions are the UK and the US during the postwar period in which both the Juglar and Kitchen fluctuations in industry wages respectively show a lead of approximately 1.5 year and a lead of approximately 1 year over those in the national product. Also see *SF 68* in section 17.3.1.
- SF 56:* The Juglar fluctuations in the *short term interest rates* lag those in the national product indices by approximately 1 year. At the Kitchen frequencies the lag is smaller at around 0.5 a year. This is consistent with the positive correlations and procyclical behavior mentioned in *SF 39*.

- SF 57:* The Juglar fluctuations in the *long term interest rates lag* those in the national product indices by approximately 2 years which is about a full year more than for the short term interest rates. This means that at the Juglar frequencies the short term interest rates *lead* the long term interest rates by approximately one full year. At the Kitchin frequencies the lag is smaller at around 0.5 a year and comparable to that of the short term interest rates. This is consistent with the positive correlations and the mildly procyclical behavior mentioned in *SF 41*.
- SF 58:* During the postwar period the Juglar fluctuations in the *term spread lead* those in the national product indices by approximately 3.5 years. At the Kitchin frequencies the lead is approximately 1.5 years. This is consistent with the negative correlations and countercyclical behavior mentioned in *SF 43*.
- SF 59:* Both the Juglar and Kitchin fluctuations in “the” *default spread lag* those in the national product indices by approximately 1 to 2 years. This is consistent with the low correlations and rather acyclical behavior mentioned in *SF 44*.
- SF 60:* Both the Juglar and Kitchin fluctuations in the *real equity total return indices* typically *lead* those in the national product indices by approximately 1 year. This is consistent with the positive correlations and procyclical behavior mentioned in *SF 46*.
- SF 61:* In *real equity total return indices* the Kitchin type business cycle fluctuations seem relatively more important than in other macroeconomic variables<sup>93</sup>.
- SF 62:* The Kitchin type business cycle fluctuations are consistent with the slightly *negative* low order *auto correlations* (i.e. *mean reversion*) observed in conventional empirical (*real*) *equity returns*.
- SF 63:* *Nominal equity total return indices* and *returns* show a consistent negative correlation with respectively consumer price indices and inflation rates at the business cycle frequencies (i.e. no *short term inflation hedge*). Here the short run relates to all fluctuations in the variables with a period length shorter than 15 years. The negative short term relation is a direct consequence of the fact that equities lead while consumer prices lag the national product business cycle. Also see *SF 30* for the *long run* relation between equities and consumer prices.

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<sup>93</sup> This is not directly visible from the results in Table 17.14 but comes directly from the analysis in Chapter 16.

The in general very high, phase corrected, correlations (i.e. coherences) between the variables at the business cycle frequencies as described in *SF 49* and *SF 50* together with the lead / lag relations reported in the subsequent stylized facts, illustrate the potential *dangers* of only using conventional contemporaneous correlations to evaluate the relations between macroeconomic variables. For example a low correlation should not necessarily be interpreted as that there is no significant connection between the relevant variables. It might very well be so that the coherence is very high, actually indicating a strong relation, while a significant lead or lag of one of the variables over the other causes the low conventional correlation number. Furthermore, note that the high coherences and the observed lead / lag relations are the principle source of findings that some variables are good *predictors* for other variables. For example the fact that the term spread leads the national product by several years makes that the term spread is a good predictor of future economic growth. Therefore the coherence and lead / lag relations are also the basic elements for the construction of Business Cycle Indicators as described in section 2.2.4. Another example of how the business cycle lead /lag relations “explain” conventional correlations is the often cited empirical positive correlation between equity returns and the term spread which is a direct consequence of the fact that both variables lead the national product business cycle.



Table 17.14 Statistics of the Kitchin peak in the estimated spectral densities for the various variables and sub-periods. The missing elements indicate either that the relevant spectrum does not contain a Kitchin peak, that no data was available for estimation or that it is non-informative (average phase for the NP). If a period length is printed underlined the corresponding spectral peak is significant at the 95% confidence level. The abbreviations are explained in heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

<i>Pre war</i>	<i>Period length</i>				<i>Peak Power</i>				<i>Modulus</i>				<i>Coherence*</i>				<i>Phase*</i>			
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg
NP	-	5.0	4.5	4.8	-	0.2	0.2	0.2	-	0.8	0.8	0.8	-	0.4	0.4	0.4	-	-2.2	0.9	-
IP	-	<u>5.4</u>	3.5	4.5	-	0.2	0.2	0.2	-	0.9	0.8	0.8	-	0.8	0.9	0.9	-	-0.7	0.2	-0.3
EM	-	4.7	3.7	4.2	-	0.2	0.3	0.2	-	0.8	0.8	0.8	-	0.5	0.8	0.6	-	-0.4	1.8	0.7
PI	<u>6.9</u>	<u>5.6</u>	4.9	5.8	0.2	0.3	0.1	0.2	0.9	0.8	0.8	0.9	0.8	0.6	0.5	0.7	-0.2	-1.9	-0.4	-0.8
WI	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
SR	4.3	4.3	3.2	3.9	0.3	0.2	0.2	0.2	0.7	0.8	0.8	0.7	0.4	0.3	0.7	0.5	0.3	0.9	-1.0	0.1
LR	4.8	<u>5.2</u>	5.3	5.1	0.2	0.2	0.2	0.2	0.8	0.9	0.9	0.8	0.5	0.5	0.5	0.5	-0.3	-0.5	-1.6	-0.8
TS	4.8	4.0	4.4	4.4	0.2	0.2	0.1	0.2	0.9	0.8	0.6	0.8	0.7	0.2	0.5	0.5	-1.5	-0.9	1.3	-0.4
DS	-	-	3.6	3.6	-	-	-	-	-	-	0.8	0.8	-	-	0.1	0.1	-	-	0.3	0.3
TR	4.4	<u>5.3</u>	<u>3.5</u>	4.4	0.2	0.3	0.3	0.2	0.8	0.9	0.8	0.8	0.4	0.3	0.9	0.5	2.1	1.1	0.8	1.4
Avg	5.1	4.9	4.1	4.5	0.2	0.2	0.2	0.2	0.8	0.8	0.8	0.8	0.6	0.5	0.6	0.5	-	-	-	-
<i>Inter war</i>	<i>Period length</i>				<i>Peak Power</i>				<i>Modulus</i>				<i>Coherence*</i>				<i>Phase*</i>			
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg
NP	3.3	4.7	<u>5.9</u>	4.6	0.2	0.2	0.5	0.3	0.8	0.8	0.8	0.8	-	0.8	0.8	0.8	-	-1.9	0.1	-
IP	<u>4.8</u>	4.5	<u>6.4</u>	5.2	0.3	0.2	0.3	0.2	0.9	0.8	1.0	0.9	0.4	0.9	1.0	0.8	2.1	0.0	0.3	0.8
EM	4.1	4.3	3.4	3.9	0.1	0.2	0.1	0.1	0.8	0.8	0.6	0.7	0.9	0.9	0.9	0.9	-2.0	0.0	0.3	-0.6
PI	4.1	4.2	4.7	4.3	0.2	0.2	0.1	0.2	0.9	0.8	0.7	0.8	0.7	0.7	0.3	0.5	1.2	-0.7	-1.0	-0.1
WI	<u>4.1</u>	5.2	3.5	4.3	0.2	0.2	0.2	0.2	0.9	0.9	0.7	0.8	0.9	0.9	0.9	0.9	-2.0	-2.2	-0.6	-1.6
SR	3.6	4.5	3.0	3.7	0.2	0.2	0.2	0.2	0.8	0.8	0.8	0.8	1.0	0.9	0.6	0.8	0.9	-1.0	-0.2	-0.1
LR	3.7	5.2	4.8	4.6	0.2	0.1	0.1	0.2	0.9	0.8	0.7	0.8	1.0	0.8	0.6	0.8	1.0	-1.4	0.0	-0.1
TS	-	4.1	4.0	4.0	-	0.2	0.2	0.2	0.0	0.8	0.7	0.5	-	0.8	0.5	0.6	-	0.9	0.1	0.5
DS	-	<u>4.6</u>	6.3	5.4	-	0.2	0.2	0.2	0.0	0.9	0.9	0.6	-	0.9	1.0	0.9	-	-1.1	-2.5	-1.8
TR	3.3	5.4	-	4.3	0.1	0.3	0.0	0.1	0.7	0.8	-	0.8	0.8	0.7	-	0.8	-0.9	0.9	-	0.0
Avg	3.9	4.7	4.7	4.4	0.2	0.2	0.2	0.2	0.7	0.8	0.8	0.8	0.8	0.8	0.7	0.8	-	-	-	-
<i>Post war</i>	<i>Period length</i>				<i>Peak Power</i>				<i>Modulus</i>				<i>Coherence*</i>				<i>Phase*</i>			
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg
NP	4.9	4.8	4.7	4.8	0.2	0.1	0	0.2	0.7	0.9	0.8	0.8	-	0.7	0.6	0.6	-	0.1	0.4	-
IP	<u>4.9</u>	<u>4.9</u>	<u>4.6</u>	4.8	0.3	0.2	0.3	0.2	0.9	0.9	0.8	0.9	1.0	1.0	0.9	0.9	-0.2	0.1	0.0	0.0
EM	3.0	4.9	4.5	4.1	0.1	0.2	0.2	0.1	0.6	0.9	0.8	0.8	0.2	1.0	1.0	0.7	0.0	-0.6	-0.2	-0.3
PI	4.8	<u>4.9</u>	<u>5.3</u>	5.0	0.2	0.3	0.2	0.2	0.8	0.9	0.9	0.9	0.6	0.8	0.9	0.8	-1.4	-1.7	-1.6	-1.6
WI	4.2	<u>5.1</u>	<u>5.6</u>	4.9	0.2	0.4	0.2	0.3	0.7	0.8	0.9	0.8	0.5	0.7	0.9	0.7	-0.9	0.6	1.0	0.2
SR	4.8	<u>5.2</u>	4.6	4.8	0.2	0.3	0.2	0.2	0.8	0.9	0.8	0.8	0.6	0.9	0.8	0.8	-0.6	-1.3	-0.8	-0.9
LR	4.8	<u>5.0</u>	3.3	4.4	0.2	0.3	0.2	0.2	0.8	0.9	0.8	0.8	0.7	0.8	0.5	0.7	-0.6	-1.1	0.2	-0.5
TS	<u>5.5</u>	4.4	4.1	4.7	0.2	0.2	0.3	0.2	0.9	0.8	0.8	0.8	0.4	0.2	0.9	0.5	2.0	0.9	1.3	1.4
DS	-	<u>4.8</u>	4.3	4.6	-	0.2	0.3	0.2	-	0.9	0.8	0.9	-	0.7	0.8	0.7	-	0.1	-1.9	-0.9
TR	<u>4.5</u>	<u>4.4</u>	<u>3.9</u>	4.3	0.2	0.2	0.3	0.2	0.9	0.9	0.9	0.9	0.9	0.9	0.8	0.9	0.7	1.3	0.9	1.0
Avg	4.6	4.8	4.5	4.6	0.2	0.2	0.2	0.2	0.8	0.9	0.8	0.8	0.6	0.8	0.8	0.7	-	-	-	-

\* In principle the coherence and phase statistics hold relatively to the national product index of the relevant country. The only exceptions are the national product indices themselves. For the UK and US these hold relatively to the national product for the NL.



Table 17.15 Averages of the statistics of the Kitchin component calculated over the three sub-period statistics as reported in Table 17.14. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

Avg	Period length				Peak Power				Modulus				Coherence*				Phase*			
	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg	NL	UK	US	Avg
NP	4.1	4.8	5.0	4.7	0.2	0.2	0.3	0.2	0.8	0.8	0.8	0.8	-	0.6	0.6	0.6	-	-1.3	0.5	-
IP	4.8	4.9	4.9	4.8	0.3	0.2	0.3	0.2	0.9	0.9	0.9	0.9	0.7	0.9	1.0	0.9	1.0	-0.2	0.1	0.2
EM	3.5	4.6	3.9	4.1	0.1	0.2	0.2	0.2	0.7	0.8	0.8	0.8	0.5	0.8	0.9	0.8	-1.0	-0.3	0.6	0.0
PI	5.3	4.9	5.0	5.1	0.2	0.2	0.1	0.2	0.9	0.9	0.8	0.8	0.7	0.7	0.6	0.7	-0.1	-1.4	-1.0	-0.9
WI	4.1	5.1	4.5	4.6	0.2	0.3	0.2	0.2	0.8	0.9	0.8	0.8	0.7	0.8	0.9	0.8	-1.4	-0.8	0.2	-0.7
SR	4.2	4.6	3.6	4.2	0.2	0.2	0.2	0.2	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.2	-0.4	-0.7	-0.3
LR	4.4	5.1	4.5	4.7	0.2	0.2	0.2	0.2	0.8	0.8	0.8	0.8	0.7	0.7	0.5	0.7	0.0	-1.0	-0.5	-0.5
TS	5.1	4.2	4.2	4.4	0.2	0.2	0.2	0.2	0.6	0.8	0.7	0.7	0.5	0.4	0.6	0.5	0.2	0.3	0.9	0.5
DS	-	4.7	4.7	4.5	-	0.2	0.2	0.2	0.0	0.9	0.8	0.7	-	0.8	0.6	0.6	-	-0.5	-1.4	-0.8
TR	4.1	5.0	3.7	4.3	0.2	0.3	0.2	0.2	0.8	0.9	0.8	0.8	0.7	0.6	0.9	0.7	0.6	1.1	0.9	0.8
Avg	4.5	4.8	4.4	4.5	0.2	0.2	0.2	0.2	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	-	-	-	-

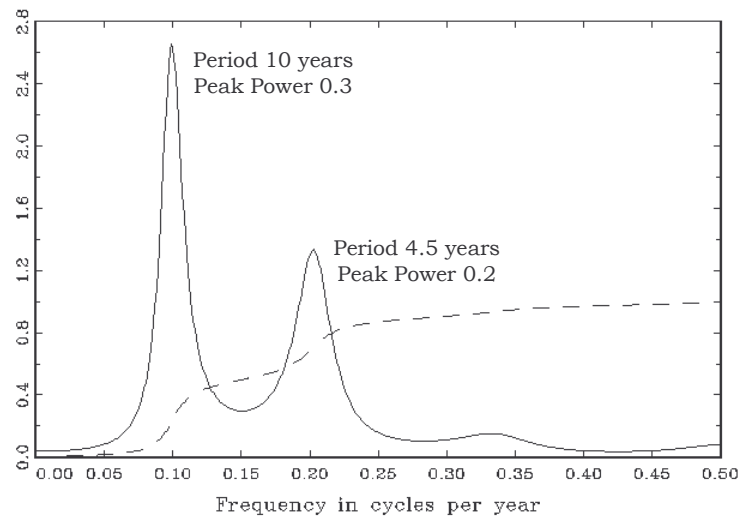
\* In principle the coherence and phase statistics hold relatively to the national product index of the relevant country. The only exceptions are the national product indices themselves. For the UK and US these hold relatively to the national product for the NL.

### 17.2.3 First order differences

The research in the previous chapters of Part III was focussed as much as possible on the raw, not transformed, time series of the variables under investigation. For example employment rates, interest rates and default spreads were analyzed without some sort of prior modification. For filtering reasons explained in section 5.5.6, the (natural) logarithmic transformation was applied on the index variables such as the national product, industrial production, price, wage and equity indices. The stylized facts *SF 47* through *SF 51* describe the typical spectral shape of the stochastic components of these original time series. The corresponding “typical” spectral density is shown in Figure 17.1<sup>94</sup>. It has two important peaks around respectively the Juglar and Kitchin frequencies. The Juglar peak which is the highest, is situated around a period length of some 10 years and describes approximately 30% of the total (business cycle) variance. The Kitchin peak which is in general lower, is situated around a period length of 4.5 years and has a lower peak power of some 20%. Finally, most of the times there is also some spectral mass at period lengths between two and three years describing very high frequency fluctuations (“noise”).

<sup>94</sup> This is the postwar spectral density for the UK industrial production as also shown in Figure D.4 in Appendix D.

Figure 17.1 Typical spectral density of the stochastic component of the original (level) macroeconomic variables, possibly after a logarithmic transformation.



As described in section 2.2.6, in Classical Time Series Analysis one often applies the *first order differencing operator* to render a stationary time series before some time series model is estimated. Table 17.16 summarizes the kind of variables that result after having applied the first order differencing operator on the original time series, possibly after first having applied a logarithmic transformation. In short, for the index time series this gives the annual *growth* or *return* statistics while for the interest rate time series this gives the annual *changes* in interest rates. For some variables, such as the employment rate, the first order differences do not have a familiar interpretation.

Table 17.16 Interpretation of *first order differences* of the original (level) variables, possibly after a logarithmic transformation. For an explanation of the abbreviations see the heading of Table 17.1.

	<i>Transformation</i>	<i>Interpretation</i>
NP	Log	Growth of National Product
IP	Log	Growth of Industrial Production
EM	-	-
PI	Log	Price Inflation
WI	Log	Wage Inflation
SR	-	Short Term Interest Rate changes
LR	-	Long Term Interest Rate changes
TS	-	-
DS	-	-
TR	Log	Real Equity returns

Because the modeling and analysis of these transformed series is often encountered in conventional econometric work it is interesting to see what the typical spectral density of the original (level) time series, as shown in Figure 17.1, implies for the spectral densities of these transformed variables. One possibility would be the *direct approach* of applying the first order differencing operator, filter the transformed series for the stochastic components and estimate separate spectral densities for these filtered transformed series. The theory of spectral analysis and filtering

techniques in the frequency domain, as described in Chapters 4 and 5, however suggest an equivalent, though easier, *indirect approach*. In this indirect approach the estimated (non normalized) spectral densities of the original (level) variables are simply multiplying by the Power Transfer Function (PTF) of the first order differencing operator. After all, as described in sections 5.1 and 5.2.1, the PTF is defined as the squared gain of the (linear) filter and for each frequency it gives the multiplier change of the variance of the component with that frequency in a time series due to the applied filter. That is, the value of the spectrum of the first order differences of some series at some frequency is in theory the same as the value of the spectrum of the original series multiplied by the value of the PTF at that frequency<sup>95</sup>. Note that this indirect approach was also applied in section 16.4.1 in the discussion about mean reversion and autocorrelations in equity returns.

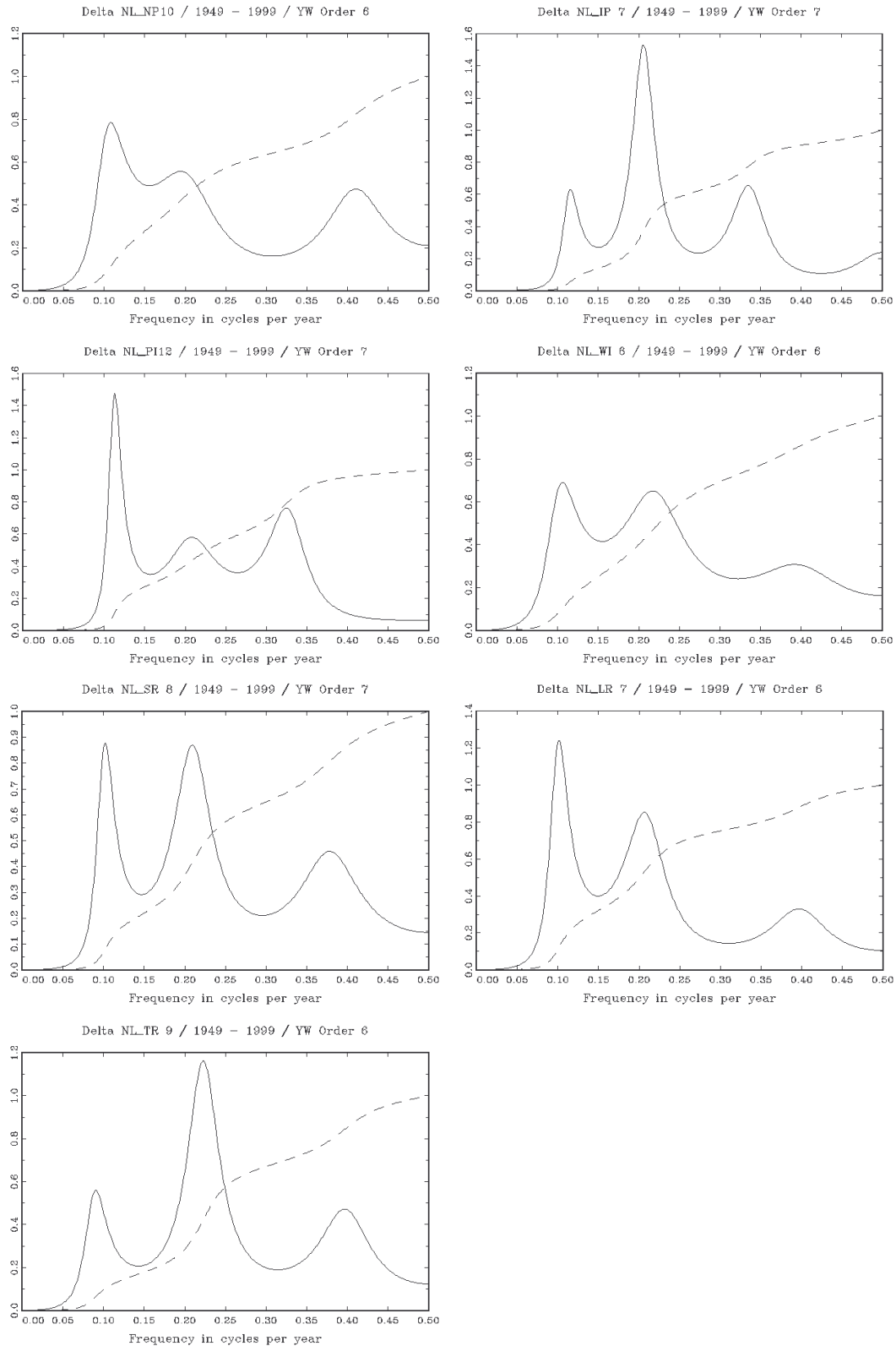
The resulting spectral densities of the stochastic components of the relevant transformed variables, as indicated in Table 17.16, are shown in Figure 17.2 for the postwar period NL data. The postwar results for the UK and the US are shown in respectively Appendix D.11 and E.10. From the PTF of the first order differencing operator in Figure 5.1 in section 5.2.1 we know that in general this transformation amplifies the variance at frequencies above approximately 1/6 cycles per year (i.e. with a period length of less than 6 years) while all lower frequency variance is being reduced. Specifically, the variance contributions at the Juglar frequency of 1/10 and the Kitchin frequency of 1/4.5 are multiplied by respectively a factor 0.38 and 1.65<sup>96</sup>. From this we see that in the resulting spectral densities the Juglar peak will in general be much *lower* while the contribution of the Kitchin peak to the total variance will *increase* because of the transformation. This is confirmed by the displayed spectral densities. Often it is now the Kitchin peak that is the highest instead of the Juglar peak though the latter still plays an important part in the process of the first order differences. It is also clear that the very high frequency fluctuations around a period length of some two to three years are also strongly *amplified* by the transformation. At a frequency of 1/2.5 the value of the PTF is 3.62 implying an almost fourfold multiplication of the variance at this frequency. So besides rearranging the relative importance of the Juglar and Kitchin type business cycle fluctuations, the first order differencing operator also increases the importance of the very high frequency fluctuations (“noise”).

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<sup>95</sup> We checked that the described *direct* and *indirect* approach of obtaining the spectrum of the first order differences of the variables indeed gives very similar results, just as theory predicts.

<sup>96</sup> These factors are simply the value of the squared gain (5.2.4) of the first order differencing filter for the relevant frequencies  $\omega=1/10$  and  $\omega=1/4.5$ .

Figure 17.2 Normalized spectral densities of the first order differences of the stochastic components of some of the original (level) variables for the NL for the postwar period, possibly after a logarithmic transformation. These are obtained by multiplying the estimated (non-normalized) spectral densities of the original (level) series by the PTF (squared gain) of the first order differencing operator. For an explanation of the abbreviations see the heading of Table 17.1.



The first order differencing operator leaves the period lengths of the spectral peaks intact. Furthermore the modulus is not greatly affected and if the transformation is applied on both series also the coherence and phase relations remain unaltered<sup>97</sup>. However because of the reordering of the importance of the frequencies with respect to the total variance, the peak power of the observed peaks *does* change by the first order differencing operator. Table 17.17 summarizes the (normalized) peak powers of the postwar spectral densities shown in Figures 17.2, D.24 and E.21. From this we see that the Juglar peak on average now contributes only 10% to the total variance instead of 30% while the Kitchen contribution has increased from 20% for the original (level) series to 30% for the first order difference series. Together these types of fluctuations therefore describe around 40%, instead of 50%, of all annual macroeconomic fluctuations with a period length shorter than fifteen years in the growth and return rates of the relevant variables.

Table 17.17 Peak Power of (normalized) spectral densities of first order differences of the original (level) variables for the postwar period as shown in Figures 17.2, D.24 and E.21. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

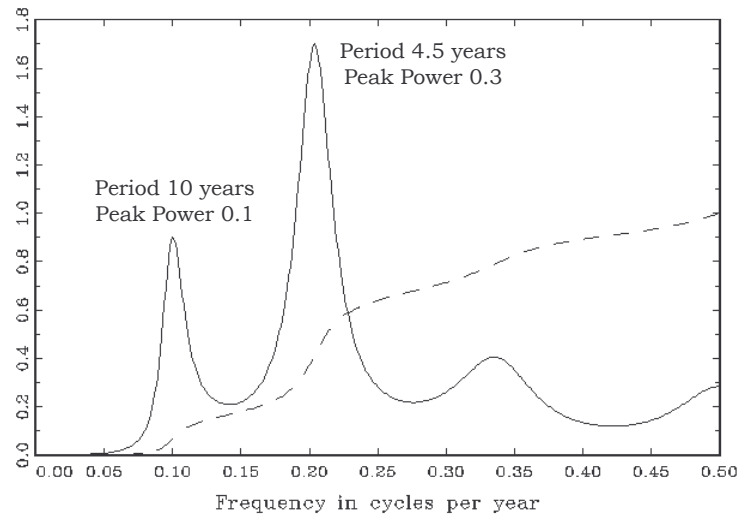
	<i>Juglar</i>				<i>Kitchen</i>			
	<i>NL</i>	<i>UK</i>	<i>US</i>	<i>Avg</i>	<i>NL</i>	<i>UK</i>	<i>US</i>	<i>Avg</i>
NP	0.3	0.1	0.1	0.2	0.3	0.2	0.3	0.3
IP	0.1	0.1	0.1	0.1	0.3	0.3	0.2	0.3
PI	0.2	0.1	0.2	0.1	0.3	0.3	0.3	0.3
WI	0.2	0.1	0.1	0.2	0.4	0.3	0.3	0.3
SR	0.1	0.2	0.3	0.2	0.3	0.3	0.3	0.3
LR	0.2	0.1	0.1	0.1	0.3	0.3	0.3	0.3
TR	0.1	0.1	0.1	0.1	0.3	0.3	0.4	0.3
Avg	0.2	0.1	0.1	0.1	0.3	0.3	0.3	0.3

Based on the previous findings, Figure 17.3 shows the “*typical*” *spectral density* of the stochastic (business cycle) components of the first order differences of macroeconomic variables as indicated in Table 17.16. This spectral density has to be compared to the “*typical*” spectral density of the original (level) variables as shown in Figure 17.1. In general the results are supportive of the following stylized fact.

*SF 64:* Juglar and Kitchen business cycle fluctuations are also present in the *growth* or *return* rates of the relevant macroeconomic variables with very similar properties as described in the previous stylized facts. The major difference is that the Juglar fluctuations typically describe only 10% (instead of 30%) of the total business cycle variance (*peak power*) while for the Kitchen fluctuations this typically is 30% (instead of 20%). Also in these spectral densities very high frequency fluctuations (“noise”) play a more important role in the total variance.

<sup>97</sup> We verified this assertion by direct estimation of spectral densities on the first order differences of the filtered stochastic components.

Figure 17.3 Typical spectral density of the stochastic component of the first order differences of the original (level) macroeconomic variables, possibly after a logarithmic transformation.



## 17.3 Changes

Besides the long run and the business cycle properties as discussed in the previous two sections, the first research objective formulated in section 1.3 also mentions the objective of investigating possible *changes* in the observed stochastic behavior of macroeconomic variables in the course of time. Such changes are the most relevant with respect to the business cycle type fluctuations, that is, with respect to the filtered stochastic components. In the adopted spectral methodology (see section 6.4) the topic of changes is covered in two ways. The first is by estimating separate static spectral densities for the *prewar*, *interwar* and *postwar* periods which may reveal different spectral properties for the various sub-periods. The second is by the *dynamic (rolling window) spectral estimates* for the postwar period. The latter analysis may provide more detailed information about possible changes in the behavior of macroeconomic variables during the most recent decades. Next, separate sub-sections are devoted to the stylized facts coming from these two different and complementary approaches.

### 17.3.1 Between prewar, interwar and postwar periods

The standard deviations of the stochastic components for the prewar, interwar and postwar periods for the three countries analyzed are summarized in Table 17.5. Table 17.12 and 17.14 summarize the properties of respectively the Juglar and the Kitchin peaks in the estimated spectral densities for the three sub-periods. The results in these tables are supportive of the following stylized facts. Some of these facts have already been mentioned in section 17.2.2 but are repeated here to give a complete overview of all relevant changes.

- SF 65:* With the exception of the interest rate related variables (short term interest rates, long term interest rates, term and default spreads), the business cycle fluctuations in all variables analyzed have the greatest *volatility* during the turbulent interwar period. Furthermore the volatility during the postwar period is lower than during the prewar period. For the interest rate related variables the volatility during the prewar and interwar periods is comparable while it is (much) greater during the postwar period.
- SF 66:* With the exception of the absolute volatility, the *business cycle dynamics* of macroeconomic variables have in general been remarkably *stable* when compared for the prewar, interwar and postwar period. This holds both for the *relative volatilities* between the various variables and for the properties of the (normalized) spectral densities.

These two stylized facts suggest that the *absolute* volatility of business cycle fluctuations can change over time but that their *relative* volatility, the type of fluctuations and the business cycle relations between the various variables has been the same for say roughly the past two centuries. This remarkable stability of the business cycle mechanism has also been reported for variables from the real sector of the economy by for example Blackburn and Ravn (1992), Backus and Kehoe (1992) and Englund et al. (1992). The results found here extend their findings to financial variables such as interest rates, default spreads and equity indices. Lucas (1977) summarized this important result as “*though there is absolutely no theoretical reason to anticipate it, one is led by the facts to conclude that, with respect to the qualitative behavior of co-movements among series, business cycles are all alike*”.

The observed pattern in the volatility as described in *SF 65* suggests that when the prewar and postwar volatility are compared, business cycle fluctuations have become more stable. Of course, in part this stabilization may also be caused by the in general poorer quality of the prewar data. More information on the alleged stabilization of the economy is given in point four of section 2.2.4. The high volatility during the interwar period might be seen as an interruption of this pattern of decreasing volatility due to the great historical turbulence during this period. The aberrant behavior of the volatility of the interest rate related variables is very likely caused by the *level effect* as described in sections 12.2.1, 13.2.1 and 15.2.1 for respectively short term interest rates, long term interest rates and the default spread. Note that in section 20.4.2 also indications are found for a level effect in inflation rates. Because during the prewar and interwar periods the average levels of these variables are pretty much the same, also their volatility for these periods is comparable. During the postwar period however their level is in general much higher and therefore also their volatility has increased. Two important exceptions to the general stability in the business cycle behavior are described by the following two stylized facts.



- SF 67:* Consumer prices lagged the national product during both the prewar and postwar period were the lag during the latter period is greater than during the first. During the interwar period consumer prices moved almost in phase with national product.
- SF 68:* In principle *industry wages* lagged the national product indices by a few years. During the postwar period in the UK and the US this has changed into a lead of a few years. Furthermore, the rolling window estimates for the postwar period suggest for each of the three countries a transformation from a lag at the start of the sample into a smaller lag or even a lead towards the end of the sample.

### 17.3.2 During postwar period

The Tables 17.18 through 17.22 summarize the changes in the period length, (normalized) peak power, modulus, coherence and phase of both the Juglar and Kitchin spectral peaks during the postwar period. These results are obtained by estimating *linear trends* on the rolling window sequences of the relevant spectral measures from the dynamic spectral analyses. We then took the values of these linear trends for the first and last of the rolling window sub-samples and computed their *difference*. These differences aim to give an indication of how the relevant spectral measures have changed during the postwar period. Table 17.23 does the same for the volatility of the stochastic components with the only difference that for these the *ratio* between the first and the last value was calculated instead of the difference. Instead of estimating linear trends, an alternative approach could have been to simply compare the first and last values of the rolling window sequences. Although simpler, the disadvantage of this approach is that the outcomes solely depend on, and are therefore very sensitive for, these first and last values. By estimating linear trends also the information from the intermediate values of the rolling window sequences is taken into account to give a broad picture of how the spectral properties have changed during the postwar period. Note that by using linear trends instead of for example higher order polynomials we are only able to say something about whether some spectral measure has increased or decreased and by approximately how much. Higher order polynomials would suggest a level of precision that is not actually there, given the small sample sizes of annual data. The results in the tables mentioned are supportive of the following stylized facts.

- SF 69:* During the postwar period, the typical length of the Kitchin business cycle fluctuations has *lengthened* from approximately 4.5 years to 5 years (+0.5 years). The only variables for which this lengthening is not observed are the default spread and the equity indices. For these variables the Kitchin period length has decreased by a few months.
- SF 70:* During the postwar period, the typical length of the Juglar business cycle fluctuations in the NL has *lengthened* by approximately 2 years
- SF 71:* During the postwar period, the typical length of the Juglar business cycle fluctuations in the US has *shortened* by approximately 1.5 years

These stylized facts with respect to changes in the length of business cycle fluctuations suggest that the alleged lengthening of growth cycles (business cycles), as for example found by the references mentioned in point four of section 2.2.4, is mostly caused by the Kitchin type business cycle fluctuations. The conclusions with respect to the length of the Juglar cycles is mixed. In the NL it has lengthened, in the US it has shortened while for the UK no clear conclusion can be drawn.

*SF 72:* During the postwar period, on average the *volatility* (standard deviation) of the business cycle fluctuations has *increased* by approximately 60%. Decreases in the average volatility have only occurred for industry wages (-10%) and equities (-20%).

*SF 73:* During the postwar period, the contribution of the Juglar fluctuations to the total variance of the business cycle fluctuations (i.e. the Juglar *peak power*) has *increased* from approximately 28% to 33% (+5%).

*SF 74:* During the postwar period, the contribution of the Kitchin fluctuations to the total variance of the business cycle fluctuations (i.e. the Kitchin *peak power*) has *decreased* from approximately 28% to 21% (-7%).

These stylized facts with respect to the business cycle volatility and to what extent it is caused by the Juglar and Kitchin type fluctuations show that during the postwar period volatility has actually increased and not decreased as one might have expected from the literature that suggest stabilization of western economies. Note that *SF 65* indicates that such a stabilization is observed when comparing prewar, interwar and postwar volatilities. *SF 72* can be seen as a refinement to this finding during the postwar period.

*SF 73* and *SF 74* are very interesting. They suggest that the Juglar contribution to the total variance has increased by some 5% while the Kitchin contribution has decreased by approximately the same amount. That is, the relative importance of the Juglar cycle seems to have increased at the cost of the Kitchin cycle. One possible cause for this change could be that monetary policy has gotten a better grip on the latter types of fluctuations. Another could be that economic agents are getting more and more familiar with these short term business cycles and therefore tend to look through them and / or take dampening actions.

*SF 75:* During the postwar period, the *modulus* of the Juglar peak in the business cycle spectral densities has *increased* from approximately 84% to 93% (+9%).

*SF 76:* During the postwar period, the *modulus* of the Kitchin peak in the business cycle spectral densities has *increased* from approximately 80% to 85% (+5%).

- SF 77:* During the postwar period, the *international coherence* between the Juglar business cycle fluctuations in the NL, UK and US as measured by their national product coherence has on average *increased* from 72% to 99% (+27%).
- SF 78:* During the postwar period, the *international coherence* between the Kitchen business cycle fluctuations in the NL, UK and US as measured by their national product coherence has on average *increased* from 72% to 82% (+10%).
- SF 79:* During the postwar period, the *domestic coherence* between the Juglar business cycle fluctuations in the various variables has on average *increased* from 84% to 93% (+9%).
- SF 80:* During the postwar period, the *domestic coherence* between the Kitchen business cycle fluctuations in the *equity indices* and in the national product has on average *decreased* from 98% to 76% (-22%).
- SF 81:* During the postwar period, the *domestic coherence* between the Kitchen business cycle fluctuations in the various variables excluding the equity indices and the “outlier” of employment in the NL has on average *increased* from 83% to 89% (+6%)<sup>98</sup>

These stylized facts show several important changes in business cycle behavior during the postwar period. First of all, *SF 75* and *SF 76* suggest that they have become more regular or, stated otherwise, more periodic during recent decades. Second, from *SF 77* and *SF 78* we see that the international correlations (corrected for the possible phase differences) have increased up to very high levels, especially for the Juglar cycle. This could be because of the increased internationalization of the world economy. Third, *SF 79* and *SF 81* show that the domestic phase corrected correlations between macroeconomic variables has also increased. Finally *SF 80* indicates one important exception to these increased correlations. The correlation between the Kitchen cycle in equities, which as we know plays an important role in the total equity dynamics, with those in the national product has strongly decreased. This could “explain” the reduced predictive qualities of stocks with respect to the business cycle. For this see for example point three in section 2.2.4 and section 16.3.1. What remains are possible changes in the lead / lag relations between the variables. The relevant stylized facts are the following.

- SF 82:* During the postwar period the *lead* of the UK and the US over the NL, as measured at the Juglar frequencies for the national product indices, has *decreased* by approximately 1 year.

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<sup>98</sup> Not directly visible from the average numbers in Table 17.21.

- SF 83:* During the postwar period the *phase* of the *industry wages* compared to the national product and measured at the Juglar frequencies has *increased* by approximately 3 years. For the NL and the US this means that an initial lag has been transformed into a substantial lead while for the UK the initial small lead has further increased.
- SF 84:* During the postwar period the *lead* of the *term spread* compared to the national product and measured at the Kitchin frequencies has *increased* by approximately 0.5 year.
- SF 85:* All phases (variables  $\times$  spectral peak) other than those mentioned in the previous stylized facts *SF 82* through *SF 84* have remained approximately the same during the postwar period.

Generally speaking the lead / lag relations between macroeconomic variables measured at the business cycle frequencies do not seem to have changed very much during the postwar period. The first exception to this finding are the indications for reduced international phase differences as described by *SF 82*. So besides the increased international phase corrected correlation described in *SF 77* and *SF 78*, apparently international business cycles are also moving more in phase nowadays<sup>99</sup>. Perhaps the most surprising change is described in *SF 83* for the wage indices. That real wages nowadays tend to lead the business cycle seems intuitively illogical given the often cited lagging behavior of price variables such as for example observed for consumer prices. The increased lead of the term spread described in *SF 76* implies an improvement in its predictive quality of future business cycle fluctuations in for example the national product.

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<sup>99</sup> Of course such a statement is only based on a study of the international spectral relations of the national product indices and may therefore not be very robust. Studying the international spectral relations for the other variables could give more information on the issue of the international business cycle relations.





Table 17.20 First and last values of *linear trends* estimated on the rolling window sequences of the *modulus* of both the Juglar and the Kitchin peak from the dynamic spectral estimates for the postwar period. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts. The bottom row shows the percentage of variables for which an increase is found (difference > 0).

	Juglar										Kitchin									
	NL					UK					NL					UK				
	Start	End	Diff	Start	End	Diff	Start	End	Diff	Avg	Start	End	Diff	Start	End	Diff	Start	End	Diff	Avg
NP	0.87	0.93	0.06	0.91	0.94	0.02	0.84	0.93	0.09	0.06	0.86	0.86	0.00	0.89	0.80	-0.09	0.80	0.82	0.02	-0.02
IP	0.91	0.95	0.05	0.81	0.97	0.16	0.83	0.90	0.07	0.09	0.89	0.86	-0.03	0.91	0.90	-0.01	0.80	0.83	0.03	0.00
EM	0.69	1.00	0.31	0.73	0.95	0.22	0.75	0.97	0.22	0.25	0.38	0.69	0.32	0.86	0.85	-0.02	0.81	0.81	0.00	0.10
PI	0.91	0.97	0.06	0.88	0.86	-0.02	0.89	0.95	0.06	0.03	0.62	0.93	0.31	0.86	0.91	0.05	0.86	0.90	0.04	0.14
WI	0.81	0.91	0.09	0.88	0.92	0.04	0.84	0.94	0.10	0.08	0.68	0.88	0.21	0.79	0.90	0.11	0.84	0.89	0.05	0.12
SR	0.88	0.94	0.06	0.81	0.89	0.08	0.63	1.01	0.38	0.17	0.81	0.91	0.10	0.82	0.91	0.09	0.69	0.82	0.13	0.11
LR	0.85	0.93	0.09	0.88	0.89	0.01	0.86	0.91	0.05	0.05	0.86	0.83	-0.03	0.83	0.86	0.03	0.65	0.89	0.24	0.08
TS	0.90	0.90	0.00	0.92	0.92	0.00	0.73	0.99	0.26	0.09	0.84	0.92	0.08	0.80	0.73	-0.07	0.74	0.84	0.10	0.03
DS	-	-	-	0.76	0.96	0.20	0.88	0.94	0.06	0.13	-	-	-	0.80	0.93	0.12	0.89	0.78	-0.11	0.01
TR	0.92	0.88	-0.04	0.96	0.78	-0.18	0.68	0.90	0.22	0.00	0.92	0.77	-0.14	0.95	0.66	-0.29	0.85	0.86	0.00	-0.14
Avg	0.86	0.93	0.08	0.85	0.91	0.05	0.79	0.94	0.15	0.09	0.76	0.85	0.09	0.85	0.84	-0.01	0.79	0.84	0.05	0.04
Perc > 0			89%			70%			100%	90%			56%			50%			80%	70%





Table 17.22 First and last values of *linear trends* estimated on the rolling window sequences of the *phase* of both the Juglar and the Kitchin peak from the dynamic spectral estimates for the postwar period. The reported phase is formulated in years and as the lead of the relevant variable over the national product. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

	Juglar											Kitchin									
	NL			UK			US			Avg		NL			UK			US			Avg
	Start	End	Diff	Start	End	Diff	Start	End	Diff			Start	End	Diff	Start	End	Diff	Start	End	Diff	
NP*	-	-	-	1.9	1.5	-0.4	2.1	0.3	-1.9	-1.1	NP*	-	-	-	-0.3	1.5	1.8	0.7	0.6	0.0	0.9
IP	0.7	-0.7	-1.4	-0.8	0.1	0.9	-0.5	0.3	0.9	0.1	IP	-0.1	0.2	0.3	0.1	0.1	0.0	-0.1	0.1	0.2	0.2
EM	0.4	-0.9	-1.3	-1.0	-1.6	-0.6	-1.1	0.1	1.2	-0.2	EM	-0.4	0.1	0.5	-0.5	-0.7	-0.3	-0.3	-0.2	0.0	0.1
PI	-3.2	-6.3	-3.0	-2.1	-2.5	-0.4	-3.9	-3.7	0.2	-1.1	PI	-1.9	-1.6	0.4	-1.7	-2.0	-0.3	-1.7	-1.7	0.0	0.0
WI	-2.0	0.7	2.7	0.5	3.3	2.8	-0.3	2.5	2.8	2.8	WI	-0.4	-2.1	-1.8	0.3	0.0	-0.3	0.8	1.1	0.3	-0.6
SR	-1.4	-2.4	-1.0	-2.1	-2.1	0.0	-1.9	-2.3	-0.4	-0.5	SR	-0.5	-0.4	0.1	-0.5	-1.5	-1.0	-1.0	-0.7	0.3	-0.2
LR	-3.1	-2.6	0.6	-1.6	-2.8	-1.2	-3.7	-2.6	1.1	0.2	LR	-1.5	-0.8	0.7	-0.6	-1.4	-0.8	-0.1	0.3	0.4	0.1
TS	4.7	4.2	-0.5	1.9	3.1	1.2	1.9	2.2	0.3	0.3	TS	2.2	2.5	0.3	-0.2	0.5	0.6	1.1	1.3	0.2	0.4
DS	-	-	-	-4.5	-2.7	1.8	1.6	-3.5	-5.2	-1.7	DS	-	-	-	-0.9	0.2	1.1	-1.9	-1.2	0.7	0.9
TR	-2.7	4.7	7.4	3.4	0.4	-3.0	1.0	-0.6	-1.6	0.9	TR	0.6	1.1	0.6	1.3	1.4	0.1	0.9	1.1	0.2	0.3

\* For the UK and the US relatively to the NL national product.

Table 17.23 First and last values of *linear trends* estimated on the rolling window sequences of the *standard deviations* of the stochastic components for the postwar period. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts. The bottom row shows the percentage of variables for which an increase is found (ratio > 1).

	<i>NL</i>			<i>UK</i>			<i>US</i>			<i>Avg</i>
	<i>Start</i>	<i>End</i>	<i>Ratio</i>	<i>Start</i>	<i>End</i>	<i>Ratio</i>	<i>Start</i>	<i>End</i>	<i>Ratio</i>	
NP	1.8%	1.4%	0.8	1.5%	2.6%	1.8	2.1%	2.6%	1.3	1.3
IP	2.0%	1.8%	0.9	2.3%	3.5%	1.5	4.6%	4.9%	1.1	1.1
EM	0.5%	2.2%	4.1	0.4%	1.7%	4.6	0.8%	1.1%	1.4	3.4
PI	1.8%	1.5%	0.8	2.8%	2.9%	1.0	2.0%	2.6%	1.3	1.1
WI	2.2%	1.4%	0.7	2.5%	2.0%	0.8	2.1%	2.8%	1.4	0.9
SR	1.6%	2.2%	1.4	1.1%	2.2%	2.1	1.0%	2.0%	2.0	1.8
LR	0.4%	1.2%	2.7	1.1%	1.4%	1.3	0.6%	1.1%	1.9	2.0
TS	1.2%	1.4%	1.1	0.7%	1.9%	2.6	0.8%	1.4%	1.9	1.9
DS	-	-	-	0.4%	0.6%	1.7	0.3%	0.4%	1.4	1.6
TR	17.5%	14.7%	0.8	25.8%	17.6%	0.7	11.8%	10.5%	0.9	0.8
Avg	-	-	1.6	-	-	1.8	-	-	1.5	1.6
Perc > 1			44%			80%			90%	80%

## 17.4 Variance decomposition

The previous sections of this chapter already provide scattered information about the volatilities of the various frequency components in the variables analyzed. Table 17.2 describes the standard deviation of the periodic and long wave components. Table 17.5 describes the standard deviation of the stochastic component. Table 17.12 and 17.13 describe the relative Peak Power of the Juglar peak in the auto-spectrum of the stochastic component while Table 17.14 and 17.15 do the same for the Kitchin peak. Finally Table 17.17 describes the Peak Power of the Juglar and Kitchin peaks in the auto-spectrum of the first order differences of the relevant variables. This final section aims to give a complete overview of how the total variance of the variables is constituted from the variance of the various components. Here the *total variance* of a variable is defined as the sum of the variance of the *periodic component* and the variance of the *stochastic component*. Hence, the variance of the trend component is left out of consideration. For index variables this total variance is therefore the variance around its trend while for level variables it is typically the variance around the average value. For the variance of the stochastic component, the prewar, interwar and postwar filtered components are considered together. Note that simply adding the variance of the two components is a valid approach because they are uncorrelated by the definition of the applied frequency domain filters. Next, the total variance is decomposed into the variance of the long wave component, the remainder of the periodic component, the Juglar and Kitchin peaks in the auto-spectra of the stochastic components and the remainder of the stochastic component. The variance of the Juglar and Kitchin peaks is determined as the product of the variance of the stochastic component and the average Peak Power (per variable) from respectively Table 17.13 and 17.15. The results of this *variance decomposition* of the *level* of the variables are shown in Table 17.24. Table 17.25 is similar, but holds for the *first order differences* of the variables. For this, the originally filtered series are transformed before calculating the variances. Furthermore, the Peak Powers from Table 17.17 for the postwar period are assumed to be representative for the entire sample. Finally, note that such a variance decomposition is closely related to the basic idea of spectral analysis: attributing variance to various frequencies. The results in Table 17.24 and 17.25 are supportive of the following stylized facts.

- SF 86:* The *long wave* or *Kondratieff* type fluctuations, as described in *SF 17*, on average describe around 25% of the (detrended) total variance of the level of macroeconomic variables. For the first order differences this reduces to around 5%.
- SF 87:* The *Juglar* type business cycle fluctuations, as described in *SF 47* and *SF 49*, on average describe around 10% of the (detrended) total variance of the level of macroeconomic variables. For the first order differences this remains around 10%.
- SF 88:* The *Kitchin* type business cycle fluctuations, as described in *SF 48* and *SF 50*, on average describe around 5% of the (detrended) total variance of the level of macroeconomic variables. For the first order differences this increases to around 25%.

Table 17.24 A decomposition of the variance of the sum of the periodic and stochastic components of the *level* of the variables. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

				Periodic component		Stochastic component			
		Stdev	Variance	Long wave	Remainder	Juglar	Kitchin	Remainder	Total
NP	NL	12.8%	1.6%	0.24	0.72	0.01	0.01	0.02	1.00
	UK	8.3%	0.7%	0.08	0.82	0.03	0.02	0.05	1.00
	US	10.0%	1.0%	0.02	0.88	0.03	0.03	0.04	1.00
Avg				0.11	0.80	0.02	0.02	0.04	1.00
IP	NL	15.3%	2.3%	0.93	0.04	0.01	0.01	0.01	1.00
	UK	7.8%	0.6%	0.12	0.56	0.10	0.06	0.16	1.00
	US	14.1%	2.0%	0.23	0.52	0.07	0.07	0.12	1.00
Avg				0.43	0.37	0.06	0.05	0.10	1.00
EM	NL	3.9%	0.2%	0.56	0.24	0.09	0.02	0.09	1.00
	UK	3.4%	0.1%	0.35	0.33	0.11	0.05	0.15	1.00
	US	5.0%	0.2%	0.37	0.33	0.10	0.06	0.14	1.00
Avg				0.43	0.30	0.10	0.04	0.13	1.00
PI	NL	18.5%	3.4%	0.22	0.64	0.04	0.03	0.07	1.00
	UK	63.0%	39.6%	0.03	0.96	0.00	0.00	0.01	1.00
	US	20.3%	4.1%	0.46	0.49	0.02	0.01	0.02	1.00
Avg				0.23	0.70	0.02	0.01	0.04	1.00
WI	NL	16.0%	2.6%	0.99	-0.01	0.01	0.00	0.01	1.00
	UK	13.0%	1.7%	0.35	0.54	0.04	0.03	0.04	1.00
	US	20.2%	4.1%	0.03	0.85	0.04	0.03	0.05	1.00
Avg				0.46	0.46	0.03	0.02	0.03	1.00
SR	NL	2.0%	0.039%	0.14	0.53	0.11	0.08	0.14	1.00
	UK	2.8%	0.076%	0.31	0.50	0.05	0.04	0.10	1.00
	US	2.6%	0.067%	0.20	0.27	0.22	0.10	0.22	1.00
Avg				0.22	0.43	0.13	0.07	0.15	1.00
LR	NL	1.1%	0.012%	0.52	0.23	0.07	0.05	0.12	1.00
	UK	1.9%	0.035%	0.27	0.64	0.03	0.02	0.05	1.00
	US	1.9%	0.038%	0.20	0.74	0.02	0.01	0.03	1.00
Avg				0.33	0.54	0.04	0.03	0.07	1.00
TS	NL	1.1%	0.013%	0.07	0.26	0.22	0.14	0.32	1.00
	UK	1.4%	0.020%	0.06	0.42	0.15	0.11	0.26	1.00
	US	2.1%	0.043%	0.03	0.37	0.18	0.11	0.31	1.00
Avg				0.05	0.35	0.18	0.12	0.30	1.00
CS	NL	-	-	-	-	-	-	-	-
	UK	0.5%	0.003%	0.25	0.25	0.07	0.10	0.34	1.00
	US	0.5%	0.003%	0.23	0.17	0.19	0.13	0.29	1.00
Avg				0.24	0.21	0.13	0.11	0.31	1.00
TR	NL	34.7%	12.1%	0.37	0.42	0.08	0.03	0.09	1.00
	UK	39.2%	15.4%	0.17	0.73	0.03	0.03	0.05	1.00
	US	31.0%	9.6%	0.04	0.77	0.06	0.04	0.10	1.00
Avg				0.19	0.64	0.06	0.03	0.08	1.00
Total	NL			0.45	0.34	0.07	0.04	0.10	1.00
	UK			0.20	0.58	0.06	0.05	0.12	1.00
	US			0.18	0.54	0.09	0.06	0.13	1.00
Avg				0.27	0.48	0.08	0.05	0.12	1.00

Table 17.25 A decomposition of the variance of the sum of the periodic and stochastic components of the *first order differences* of the variables. For an explanation of the abbreviations see the heading of Table 17.1. In gray are important parts of the table that are referred to in the stylized facts.

				<i>Periodic component</i>		<i>Stochastic component</i>			<i>Total</i>
		<i>Stdev</i>	<i>Variance</i>	<i>Long wave</i>	<i>Remainder</i>	<i>Juglar</i>	<i>Kitchin</i>	<i>Remainder</i>	
NP	NL	3.7%	0.1%	0.05	0.09	0.22	0.25	0.39	1.00
	UK	3.0%	0.1%	0.01	0.04	0.14	0.20	0.61	1.00
	US	4.0%	0.2%	0.00	0.13	0.12	0.26	0.49	1.00
Avg				0.02	0.09	0.16	0.24	0.50	1.00
IP	NL	4.0%	0.2%	0.19	0.10	0.07	0.19	0.44	1.00
	UK	4.7%	0.2%	0.00	0.03	0.09	0.27	0.61	1.00
	US	9.1%	0.8%	0.01	0.02	0.07	0.20	0.69	1.00
Avg				0.07	0.05	0.08	0.22	0.58	1.00
PI	NL	7.2%	0.5%	0.02	0.11	0.14	0.27	0.46	1.00
	UK	8.1%	0.7%	0.02	0.15	0.09	0.26	0.48	1.00
	US	5.2%	0.3%	0.08	0.24	0.12	0.20	0.37	1.00
Avg				0.04	0.17	0.12	0.24	0.44	1.00
WI	NL	2.9%	0.1%	0.46	0.04	0.12	0.19	0.19	1.00
	UK	4.8%	0.2%	0.05	0.07	0.10	0.33	0.45	1.00
	US	6.9%	0.5%	0.00	0.04	0.10	0.24	0.61	1.00
Avg				0.17	0.05	0.11	0.25	0.42	1.00
SR	NL	1.2%	0.015%	0.01	0.01	0.13	0.27	0.59	1.00
	UK	1.4%	0.018%	0.02	0.01	0.15	0.15	0.67	1.00
	US	2.2%	0.048%	0.00	0.03	0.28	0.33	0.36	1.00
Avg				0.01	0.02	0.19	0.25	0.54	1.00
LR	NL	0.5%	0.003%	0.04	0.02	0.19	0.30	0.46	1.00
	UK	0.7%	0.005%	0.03	0.07	0.13	0.24	0.52	1.00
	US	0.6%	0.004%	0.03	0.08	0.08	0.29	0.53	1.00
Avg				0.03	0.06	0.14	0.28	0.50	1.00
TR	NL	16.0%	2.5%	0.04	0.06	0.10	0.27	0.54	1.00
	UK	15.5%	2.4%	0.01	0.10	0.06	0.25	0.57	1.00
	US	16.5%	2.7%	0.00	0.09	0.08	0.34	0.50	1.00
Avg				0.02	0.08	0.08	0.28	0.54	1.00
Total	NL			0.11	0.06	0.14	0.25	0.44	1.00
	UK			0.02	0.07	0.11	0.24	0.56	1.00
	US			0.02	0.09	0.12	0.27	0.51	1.00
Avg				0.05	0.07	0.12	0.25	0.50	1.00





## *Part IV*

# Scenarios and Stylized Facts

*“All model economies are abstractions and are by definition false.”*

Finn E. Kydland and Edward C. Prescott (1996)

Part IV presents the results on the second research objective as formulated in section 1.3. Chapter 18 describes the basic methodology used for the research presented here. Chapter 19 describes the results of testing existing (scenario) models on being consistent with the empirical knowledge about the behavior of macroeconomic variables as represented by the stylized facts that are obtained by applying the specific methodology of Part III and are summarized in Chapter 17. Chapter 20 presents an alternative scenario framework which resolves most of the shortcomings of existing models as found in Chapter 19. While the stylized facts in Chapter 17 provide the “answer” to the first research question, the summary of the testing results in section 19.3 together with the new scenario framework described in section 20.1 provide the “answer” to the second research question.



## 18 Methodology Part IV

Part IV contains the results with respect to the second research objective as posed in section 1.3. The objective is in the first place to test whether ALM scenario applications of Vector AutoRegressive (VAR) models indeed lead to scenarios that are *consistent* with empirical knowledge about the behavior of macroeconomic variables as the scenario definition of Bunn and Salo (1993) from section 1.3 prescribes. If necessary, the objective is in the second place also to find out what modifications can be made to the conventional VAR model applications to resolve possible shortcomings. Note that although this research question is originally derived from and placed in the specific context of ALM scenario models, the results have relevance in many other lines of economic research and applications. The results found here are for example also relevant for applications in the field of optimal investment portfolio choices.

In the research presented here, the testing objective is implemented by confronting various types of scenario applications of VAR models with the empirical knowledge as obtained in Part III and summarized in terms of the list of stylized facts in Chapter 17. Although the methodology used here is not as specific and elaborate as the methodology used in Part III, because of the confrontation with those results it is indirectly also based on the relevant theory and experiments with respect to three techniques: the VAR models, Spectral Analysis and filtering techniques from Part II. As we will see, the testing of the models will result in a number of shortcomings in terms of showing consistency with the empirical knowledge. Therefore an alternative scenario framework is presented which resolve most of these shortcomings.

The approach followed for testing the existing models is described in section 18.2 while section 18.3 does the same for the developed alternative scenario models. This chapter however starts with some preliminary methodological considerations in section 18.1 that are relevant to all subsequent sections. Chapter 19 and Chapter 20 contain the results corresponding to the methodology and objectives described in respectively section 18.2 and 18.3.

### 18.1 Preliminary considerations

Before commencing with a description of the specific approach followed for testing existing scenario models on the one hand and for developing alternative models on the other hand, some preliminary methodological considerations are in order that are relevant to all subsequent sections. Consecutively, these issues concern how the research presented here deals with the empirical knowledge and what it implicitly assumes about the Data Generating Process (DGP), why the testing takes place in terms of scenario properties instead of policy consequences and finally some remarks about the non-automated character of generating scenarios and testing the related models.

### 18.1.1 Empirical knowledge and the DGP

Both for the testing of existing scenario models and for the development of alternative models, the empirical knowledge obtained in Part III and summarized in terms of the list of stylized facts in Chapter 17 is considered as the primary reference. As should be clear from the description of the specific methodology in Chapter 6 together with the description and thorough testing procedures of its components in Part II, the approach followed for obtaining the stylized facts is very complete, consistent and as pure and correct as possible in a methodological sense. Nevertheless, in the end the empirical knowledge obtained in Part III is still conditional on the specific methodology and data used to conduct the empirical research. By testing scenario models on being consistent with this specific empirical knowledge, we implicitly raise its status to being the “true” general empirical behavior of macroeconomic variables. Although we are aware of the still specific nature of the results, we do consider the empirical knowledge obtained in Part III as being the “benchmark” knowledge which current and new scenario models should be able to describe or adjust properly. Hence, every time that in following chapters we speak of “empirical knowledge” this refers to the empirical knowledge obtained in Part III of the research presented here using the specific methodology. Furthermore, when we speak of “consistency” this specifically refers to whether the scenario models replicate this benchmark empirical knowledge correctly or not. Consistency in terms of for example theoretical economic relations is not considered here.

Because of the spectral analysis approach, most of the empirical knowledge concerns the second order moments (covariance and autocovariance matrices) of the historical data, although also some higher order properties are included, for example the level effect in interest rates. This confinement to second order moments can also be seen from the prominent role of VAR models in the methodology of the Part III research. This should however *not* be interpreted as that the empirical knowledge is obtained by estimating VAR models in a conventional sense. Instead, by coincidence, the VAR models enter the spectral analysis methodology through the Maximum Entropy information concept (see section 4.6.4). For the univariate dynamic properties of the economic variables (i.e. their auto-spectra), the results from univariate AR models are considered superior over the implied univariate processes from multivariate VAR models because in univariate AR models disturbing effects in the estimated auto-spectra as caused by the *feed-across effect* (see section 4.7.7) by definition cannot occur. Cross-spectra can of course only be estimated on multivariate VAR models but for the same reason these are based on two dimensional models, the lowest dimension possible in this case. In a sense, the following two chapters are about the extent to which models estimated on some set of time series data are capable of describing the same stochastic properties as found in these time series data using a consistent methodology for both obtaining the empirical knowledge and for testing the model behavior.

Furthermore, the dominant presence of the VAR models in the research presented here should *not* be interpreted as that a Data Generating Process (DGP) is assumed to exist and that this DGP is of the VAR model type. Even if one does believe in the existence of some true DGP, than this would very likely be much more complex than a simple static linear Normally distributed VAR model describing only the first and second order moments of a process. Note that some other modeling approaches, such as for example the London School of Economics approach as

described in section 2.2.8, do assume the existence of some true DGP within the class of models that is considered. On the one hand, VAR models enter the spectral analysis methodology used in Part III through the Maximum Entropy information concept (see section 4.6.4). On the other hand, because of their simple and flexible structure and because understanding and describing the second order moments of the macroeconomic process correctly is already a substantial step, VAR models are often used for generating economic scenarios. Understanding and describing the second order moments as well as possible is the key issue here and VAR models in principle happen to be suited well for this purpose.

The fact that this research does not assume the existence of some true GDP and that if it would exist it would certainly not be of the VAR model type, also implies that there will always be some specification error in the models used. This is especially relevant with respect to the alternative scenario methodology described in Chapter 20. This proposes, amongst other things, to use different samples and different observation frequencies (5 years, annual, monthly, daily,...) for the scenario properties in different frequency ranges (long term fluctuations, business cycle fluctuations, seasonal fluctuations, daily fluctuations,...). The results in Den Butter (1976) indicate that if one *does not* assume that the data are generated from a model within the class of models considered in the specification analysis, using different observation frequencies may actually result in better predictions and hence also better scenario properties. For a monthly and a quarterly model on the same set of monthly data, he finds that the quarterly model is to be preferred for predictions more than one quarter ahead. More generally stated, he finds that one does not invariably need to use the data available for the shortest observation interval between the observations when the objective is to predict many (instead of one) periods ahead. Apparently, the specification error can be smaller with lower observation frequencies than with the highest available observation frequency. For example, using a VAR model to obtain information about business cycle fluctuations from monthly time series data will be more difficult than from annual time series data. In the case of the monthly time series data, VAR models of very high orders, and hence with a large number of parameters to be estimated, will be needed to be able to capture the business cycle dynamics. With respect to this *temporal aggregation* issue, Lütkepohl (1987) finds exactly the opposite in case one *does* assume that the data are generated from a model within the class of models considered in the specification analysis. His results suggest that in that case when forecasting a temporally aggregated flow variable which is the sum of consecutive disaggregate values, it is always preferable to use the disaggregate data if available. The relative forecast efficiency will in general increase if disaggregate data is used for estimating the process. Intuitively, this means that if we are considering the right class of models, the chance of finding the true DGP will be largest if we have as much data as possible. With respect to *contemporaneous aggregation*, Lütkepohl (1987) finds that it is preferable to directly predict the aggregate instead of the multivariate components which are then aggregated. In scenario analysis applications however there is not much choice here. Which variables need to be simulated is dictated by the application of the scenarios. In case for example the consumption, investments and government expenditure components of the national product are needed, they need to be explicitly present in the scenario model. The results of Lütkepohl do imply that one should not simulate the separate components in case they are not needed

on an individual bases. Finally note that in practical scenario applications the choice of data to use is limited, both in terms of sample length and observation frequency and in terms of the available variables.

### 18.1.2 Scenario properties versus policy consequences

Both the testing of existing scenario models as well as the development of new scenario models is completely confined to the statistical (dynamic) properties of the economic scenarios themselves in terms of their observed behavior in the frequency domain. By this we mean that for example scenario properties in terms of their consequences for strategic policy decisions are not considered here, despite the fact that, as we know from the example in section 1.2, such effects can be enormous. The main reason for this is that any strategic ALM policy decision is in the end *conditional* on the properties of the scenarios that are used. Of course, knowing in itself the impact different scenario properties can have on strategic policy choices is very relevant. This might even give us clues about which scenario properties are the most relevant ones. However, at a fundamental level, policy outcomes cannot give us information on what good quality economic scenarios should look like, simply because the policy outcomes depend on the scenarios and there is no such thing as the “true” policy outcome on which to evaluate or develop a scenario model.

### 18.1.3 Art versus science - II

At various places in both the research presented in the following and in the previous chapters, results are obtained by non-automated procedures. In those cases, either implicitly or explicitly, external knowledge (expert opinion) is brought into the different procedures. Examples of non-automated procedures in the context of the empirical research of Part III, such as the altering of samples, changing the pass-bands in the filtering procedures and choosing the model order in the Maximum Entropy spectral analysis, are described in section 6.1.1.

An example of a non-automated procedure in the context of conventional ALM scenario models that are tested in chapter 19 is that the expected values of the variables involved (interest rates, inflation rates, equity risk premium, etc.) in the models are changed from the historical estimates into averages based on forward looking assumptions about the variables. As an example think of expected inflation rates that because of the inflation targeting policy of the European Central Bank will typically have a lower expected value (say 2%) than the average postwar inflation rates (say 4%). Another example is the use of dummy variables to filter out the effects of specific historical observations because these observations are assumed to deviate from the underlying process to be modeled in the scenarios. More information on conventional scenario model applications and the use of expert opinion can be found in section 18.2.1.

Finally, it is important to note that also the new scenario framework presented in Chapter 20 is not meant to result in a fully automated procedure that result in a set of economic scenarios with the desired specifications by simply running some computer program. Instead, the new framework provides a set of optimal tools that can be used by an expert to construct high quality economic scenarios according to the desired specifications. Once the expert has constructed a basic scenario set, it is



possible for non-experts to change some of the parameters according to their own needs. An obvious split between what an expert should do and what a non-expert could do is between taking care of the appropriate dynamic stochastic behavior and adjusting the simpler expected values and (unconditional) volatilities. This standpoint is supported by Morgan and Magnus (1999) who perform several experiments to illustrate that econometrics is more an art than a set of simple textbook recipes that can be applied successfully by everyone.

## 18.2 Testing existing models

Following the general methodological considerations from the previous section, this section describes the specific methodology that is used to test to what extent conventional VAR scenario models are able to describe the “benchmark” empirical knowledge from Part III as summarized in terms of the list of stylized facts summarized in Chapter 17. Despite their great flexibility and their capability to describe many types of stochastic processes, there are various reasons why estimating a VAR model does not automatically lead to a stochastic process, and hence scenarios, that are consistent with the empirical behavior of the time series that are used for the estimation. Reasons for this are the following.

### *Imprecise estimates*

It has been mentioned several times before that the advantage of the great flexibility of VAR models may at the same time be their greatest drawback. A general and well recognized practical problem are the imprecise estimates of VAR model parameters caused by the fact that the number of parameters is often quite substantial relative to the available sample size or time series length.

### *Data representation*

There are choices that have to be made with respect to the representation of the data used for estimation that can effect the properties of the final model. A prerequisite is that the data are stationary in the sense that they show stable statistical properties over time during the sample period, such as means, variances and (auto) correlations. Although formal statistical testing procedures are available, for example unit root tests, for assisting in these choices, the final representation used will always depend on the judgement of the model builder. He or she may choose to use the data in an unchanged format, apply first order differencing or apply another type of filtering technique or transformation.

### *Model order*

Once the data representation has been chosen, the next choice concerns the order of the VAR model. This choice can be based on the various order selection criteria as described in section 3.5 but again the final choice will depend on the judgement of the model builder. Too low an order can be harmful because potential higher order dynamics are ruled out. Too high an order can be harmful because of the problem of imprecise estimates.



*Estimation procedure*

Finally there is the choice of the method used for the actual estimation of the model. Various estimation methods are available such as those described in section 3.4. While the imprecise estimates are a general problem, the choices of data representation, model order and estimation procedure can also have a potentially large impact on the properties of the stochastic process described by a VAR model and hence on the extent to which the model describes variable behavior that is consistent with the empirical knowledge. We want to test the way VAR models are typically applied in both academic and practical ALM scenario applications. Instead of selecting several *actual* VAR applications, here we will test several combinations of

- (a) data representation
- (b) model order
- (c) estimation procedure

These combinations aim to give a *stylized* representation of the typical applications encountered. Reasons for following this approach are the following.

*Consistency with Part III*

By performing estimates on the same data as used in Part III, the model results can be directly compared with the empirical knowledge obtained there. In this way we can rule out that possible inconsistencies found are the consequence of the use of different data in terms of historical time period, country and the actual time series used.

*Model information*

In quite a few cases the information of the models used in specific applications is insufficient to perform the required testing. For example, the estimated parameter matrices or covariance matrix of the residuals are not always given. By performing new estimates, all the required information is automatically available.

*Flexibility*

By performing new estimates, we are flexible in for example testing the impact of changing the model order or estimation procedure while remaining consistent with the other results. The specific applications do not cover the complete desired range of choices for the data representation, model order and estimation procedure.

*Relevance*

Finally, the use of stylized instead of actual applications is without loss of information on the consistency question under investigation. On the contrary, in this way we are able to adjust the available results to our specific needs, thereby actually increasing the relevance of the information.

From here on we proceed as follows. Section 18.2.1 first gives an overview of some actual applications of VAR models for generating scenarios. Based on these actual applications, section 18.2.2 then describes which stylized applications are tested in terms of specific combinations of data representation, order selection and estimation method. Section 18.2.3 summarizes the empirical knowledge from Part III that is included in the testing procedure and presents arguments why other parts of the

empirical knowledge are not taken into account. A part of the testing is done in terms of comparing spectral densities. Section 18.2.4 describes how this is done.

### 18.2.1 Actual applications

This section presents a short description of some actual applications of VAR models for generating scenarios from the literature. The stylized combinations of data representation, model order and estimation procedure from section 18.2.2 are meant to represent these types of actual applications. The examples not only illustrate the types of scenario applications of VAR models in a technical sense, but also the wide range of problems they can be applied to.

Dert (1995) uses a VAR model to generate a tree-type structure of scenarios for determining an optimal chance constrained investment and funding policy for a pension fund by adopting a multistage stochastic programming approach. It is a seven dimensional VAR model simulating Dutch wage inflation, price inflation, short term interest rates, Gross National Product growth and total rate of returns on an internationally diversified equity, real estate and bond portfolio. The model is estimated on annual data for the 1956-1994 period and has order  $p=1$  without mentioning the use of order selection criteria. The model parameters are estimated by means of an iterative procedure that selects parameters until all remaining estimated parameters are statistically significant different from zero at the 1% confidence level. Because of the zero constraints, the SUR estimation procedure is applied. Besides the vector of constants, only five out of the remaining forty-nine parameters are included in the final model because of the rather strict confidence level of 1%. The expected values of the model are similar to the historical averages.

Boender et al. (2000) use a VAR model to generate path-type of scenarios for investigating optimal dynamic asset allocation rules for a pension fund. It is a four dimensional VAR model used for simulating Dutch wage inflation, price inflation, short and long term interest rates and total rate of returns on an equity portfolio. The model is estimated on annual data for the 1966-1997 period and has order  $p=1$  without using order selection criteria. The model parameters are estimated by means of an iterative procedure selecting parameters in such a way that the statistical properties of the model like correlations and autocorrelations are as close as possible to the same statistics as observed for the sample period. The model is estimated using the OLS estimation technique. Another difference with the Dert (1995) model is that for the expected values of the model, the historical averages are overruled by averages based on forward looking assumptions about the variables involved.

A closely related application is Boender et al. (2001) in which the potential risk reducing effects of investing in index linked bonds for a pension fund are investigated. Besides the fact that the VAR models involved are estimated by the Yule-Walker procedure, the most important difference with the previous applications is that here a model is used in which a correction is applied for the high inflation rates that occurred during the 1970's by first regressing the price and wage inflation time series on a constant term and a dummy variable for the 1971-1975 time period. One of the effects of such a correction is a lower volatility of the simulated inflation

rates. A virtually identical model is used by Boender (2001) for investigating the benefits of solidarity and risk sharing in a pension system. He motivates overruling the sample period (1976-1999) arithmetic equity risk premium of 12.5% by 3.5% by the fact that many very long term empirical studies point to an equity risk premium of around this number. For this, also see Tables 16.9 and 16.10 while noting that the “typical” 3.5% risk premium is probably a geometric instead of an arithmetic value.

In various aspects the applications of VAR models in Damm et al. (1994), Damm (1995) and Steehouwer (1996) are rather different than the previous ones. They apply VAR models for simulating yield curve scenarios to examine the risk and return effects of bond strategies and bank ALM in general. Instead of directly observable time series variables, these models simulate three of the four parameters of the Nelson and Siegel (1987) yield curve model. The combination of the VAR time series model and the parsimonious Nelson and Siegel yield curve model proves to be a very powerful and transparent combination for simulating yield curve scenarios with realistic yield curve shapes and dynamics. Another difference is that these models are estimated on (in terms of time) shorter samples but at a higher frequency. Steehouwer (1996) for example estimates various models on monthly data for the period February 1979 - March 1996. Partly because of the higher frequency of the data, they also consider models of orders higher than  $p=1$  and use order selection criteria to help selecting the appropriate model order. Finally, Steehouwer (1996) also uses unit root tests to determine whether differencing is required for rendering stationary time series or not, and if so, whether there exist co-integration relations between the variables involved. Related applications of VAR models in combination with the Nelson and Siegel yield curve model are Diebold and Li (2003) and Diebold et al. (2003).

Kim et al. (1999) present a methodology for long term forecasting and scenario generation for risk management purposes. While most of the previous applications consider the long run as being somewhere around twenty years, Kim et al. (1999) define the long run as ranging from three months up to two years. They use a VAR model in error correction format for modeling the uncertainty in exchange rates, interest rates, equity indices and commodity prices. Hence the model is specified in terms of the first order differences of (the natural logarithm of) the variables while including co-integrating relations for modeling long term dependencies between the (index) levels of the variables. The parameters of the model are chosen based on economic theory while their values are estimated on long term monthly data. The Akaike Information Criterion is used for an automatic selection of the model order.

The final application of VAR models is Campbell and Viceira (2001). They use VAR models for describing investment opportunities in an optimal investment portfolio choice problem by modeling a six dimensional process of ex-post real interest rates, excess stock returns, excess bond returns, short term nominal interest rates, equity dividend-price ratios and the yield spread between long term bonds and Treasury Bills. Again, the estimated models have order  $p=1$  for the determination of which no order selection criteria are mentioned. Using virtually an unrestricted OLS estimator they estimate both an annual model based on a very long term 1890-1998 sample of US data and a quarterly model based on a (in terms of time) shorter 1952:I-1999:IV

sample. Because VAR models are reduced form models, the estimates of the unrestricted models are used to obtain the parameters of underlying theoretical models for the behavior of asset returns. Furthermore, the dynamic properties of the estimated models are used to assess the conditional uncertainty in multi-period asset returns for various investment horizons.

### 18.2.2 Stylized applications

Based on the range of actual applications of VAR models for generating scenarios as described in the previous sub-section, we actually tested the following stylized combinations of data representation, order selection and estimation method on their ability to describe the empirical knowledge from Part III.

#### *Data representation*

All models tested are estimated on Dutch annual data for the postwar 1949-1999 period. The same data is used as in Part III. Section 6.1 gives a description of the data while the time series themselves and a description of their sources can be found in Appendix F. Table 18.1 shows the variables described by the six dimensional VAR models. This table also shows the two representations of the data that are used.

The first representation is a traditional representation as used in all the applications described above. The short and long term interest rates are used in an unchanged level format. For the non-stationary index variables the relative annual growth is used which is approximated by the first order differences of the natural logarithm of the time series. This approximation is used instead of the actual growth percentages for two reasons. First, this is consistent with the transformations used in Part III. Second, it allows us to apply the Power Transfer Function (PTF) of the first order differencing filter or its inverse to the spectral densities of the models to obtain the spectral densities of the first order differences or of the levels of the variables in cases where a model has been estimated on the opposite representation of the data. More information on the PTF of these filters can be found in sections 5.2.1 and 5.2.5.

The second representation uses the same time series as the first representation, however after they have been filtered with a pass-band representing the frequency range  $[1/15, 0.5]$ . This results in all fluctuations in the (relative growth of the) time series with a period length of between 15 and 2 years. These are equal to the *stochastic components* of the (relative growth of the) time series as described in section 6.3 and used in the analysis of the time series in the relevant chapters of Part III. The filtering there was implemented by the *zero phase frequency filter* technique described in section 5.4. Note that in case of the index variables, here the filtering was not performed again on the first order differences of the natural logarithms of the series. Instead, the first order differences of the already filtered natural logarithms are used which in theory are identical. Section 6.3.3 explains why this is so. The reason for using these filtered representations is as follows. The results will show that using the first representation of the data results in a poor reproduction of the spectral densities in the business cycle frequency range  $[1/15, 0.5]$  as found in Part III. This holds no matter what order of the model or estimation method is used. By estimating models on the filtered time series we are able to determine to what extent these inconsistencies are the consequence of using non-filtered data, with a possible dominance of the low frequency fluctuations, or of using the VAR models.

Table 18.1 Data representations used for the testing of the VAR models for the 1949-1999 sample period for the Netherlands.

	<i>Representation 1</i>	<i>Representation 2</i>
National Product Index	$\Delta\log(\text{NP})$	$[1/15, 0.5] \Delta\log(\text{NP})$
Consumer Price Index	$\Delta\log(\text{PI})$	$[1/15, 0.5] \Delta\log(\text{PI})$
Real Industry Wage Index	$\Delta\log(\text{WI})$	$[1/15, 0.5] \Delta\log(\text{WI})$
Short Term Interest Rate	SR	$[1/15, 0.5]$ SR
Long Term Interest Rate	LR	$[1/15, 0.5]$ LR
Real Equity TRR Index	$\Delta\log(\text{TR})$	$[1/15, 0.5] \Delta\log(\text{TR})$

### Order selection

The descriptions of the actual applications in the previous section show that the typical construction of VAR models makes little use of order selection criteria. Furthermore the Monte Carlo experiment in section 4.7 already gives information on the effects of using the various types of order selection criteria. Because of these two reasons the effects of using the orders indicated by the various order selection criteria are not investigated here. Instead, the order selection criteria are calculated to give some indication for the order these criteria point to. Next however, for both data representations and all estimation procedures, models are estimated for the orders  $p=1, \dots, 4$ . Because of the six dimensions, a maximum order of four should provide more than enough potential dynamics of the implied univariate processes. Zellner and Palm (1974) show that a VAR( $p$ ) model for  $n$  variables implies univariate ARMA( $np, (n-1)p$ ) models for each of the individual variables. For  $p=4$  and  $n=6$  this comes down to high order univariate ARMA(24,20) models. In case the results of some order appear to be similar to those of another order these results will not be explicitly presented.

### Estimation procedure

Table 18.2 shows the estimation procedures applied to the models of order  $p=1, \dots, 4$  for both data representations. The Top-Down procedure based on the Akaike Information Criterion (AIC) is used as a proxy for a procedure such as the one in Dert (1995) where parameters are successively constraint to zero based on some statistical criterion. The specific method and criterion chosen here performed best in the Monte Carlo experiment described in section 4.7.5. The Yule-Walker estimation method in combination with applying dummy variables to the price inflation series as in Boender et al. (2001) can only be sensibly applied for the traditional representation of the data. Note that, contrary to Boender et al. (2001), here the dummy variables are not applied to the wage inflation series because these are now in real instead of nominal terms and therefore no longer have the high inflation levels during the 1970's.

Table 18.2 Estimation procedures.

	<i>Representation 1</i>	<i>Representation 2</i>
Unrestricted OLS	✓	✓
Top-Down AIC	✓	✓
Yule-Walker	✓	✓
Yule-Walker with dummy	✓	



### *Other types of VAR models*

In section 2.2.7 several other types of (restricted) VAR models were mentioned, such as *Bayesian VAR models*, *index models*, *Reduced Rank VAR (RR VAR) models* and *Structural VAR (SVAR) models*. All these types of VAR models are not tested here. The main reason is that these models are not used in the actual applications of VAR models for generating (ALM) scenarios from the literature as described in section 18.2.1. Hence, these types of VAR models fall out of the scope of the second research objective as described in section 1.3.

### 18.2.3 Empirical knowledge

As stated in section 18.1.1, for both the testing of existing scenario models and for the development of alternative models, the empirical knowledge obtained in Part III and summarized in terms of the list of stylized facts in Chapter 17 is considered as the primary reference. That is, we consider the empirical knowledge obtained in Part III as being the “benchmark” knowledge which current and new scenario models should be able to describe or adjust properly. In order to test their consistency with this empirical knowledge, the in total 28 six dimensional VAR models (the 7 models from Table 18.2 times the 4 model orders  $p=1,\dots,4$ ) as described in the previous subsection are confronted with the stylized facts found in Part III. To ensure a fair comparison, the specific results for the Netherlands as derived from the identical time series are used instead of the formal stylized facts presented in Chapter 17. Not all aspects of the stylized facts and empirical knowledge are taken into account. Instead the focus is on the long wave and business cycle stochastic behavior which is the most important element of applied VAR models. These aspects are also the most difficult to estimate and model correctly. What follows is a discussion of the various aspects of the stylized facts and why these are or are not included in the testing of the models. This discussion follows the classification used in the sections of Chapter 17.

#### *Trends*

The stylized facts with respect to the long term averages or trends are given in section 17.1.1. In the applications discussed in section 18.2.1, two approaches for determining the long term averages described by the models are applied. The first is to adopt the expected values as given by the model estimates. This gives averages that are very similar to the averages from the sample used for estimation. The second is to overrule these model expected values by averages based on forward looking information or averages emerging from other research or time periods. The trend or average values of the models are not included in the testing of the models. The main reason is that model averages can be easily adjusted without changing the other stochastic properties of a model by applying (3.2.34). The only remark we want to make here is that *if* one chooses to simply use the averages from the sample period it would be logical to ensure that the model expected values are indeed identical to these sample averages. By using the unrestricted model estimates this will often not be the case because (3.2.34) shows that the expected values of a VAR model both depend on the constant vector  $v$  and on the estimated parameter matrices  $A_1, \dots, A_p$ .

*Long waves and periodic components*

The stylized facts with respect to the long term periodic components and more specifically to the long wave type fluctuations are given in section 17.1.2. The periodic components are left out of consideration because they did not lead to any stylized facts in Part III. The consistency with respect to the long wave fluctuations is only analyzed for the first set of data representations because in the second set the fluctuations in this frequency range are filtered out of the time series and should therefore not be present in the models. The (relative) volatilities of the long wave component are not analyzed because in the first set of data representations all frequencies of the series are modeled together in one single model. It is therefore difficult to obtain information on the volatilities of the long waves implied by the models. One possibility would be to determine the variance implied by the spectral densities of the series in the frequency range  $[1/70, 1/30]$ . If required, the total variance of the variables in the models can be adjusted without changing the other stochastic properties by using the procedure described with point 3 in section 3.7.5. Note that a potential shortcoming of modeling all frequencies together in one single model is that in this way it is not possible to change the variances for specific frequency ranges. Besides checking that they are present in the models, the most important information that is included in the testing of the models with respect to the long waves are the phase relations between the variables in the relevant frequency range. Following SF 17 from section 17.1.2, a long wave period length of 50 years is assumed. The phase relations as indicated in Table 18.3 are based on the specific results for the Netherlands in terms of the peaks and troughs of the long wave components from the tables in the relevant chapters of Part III. In principle, the numbers in the table use the original phase results in terms of the (logarithmic) levels of the time series. In case of first order differencing both variables involved, these phase relations can be used without change. In case of first order differencing only one of the variables involved, the phase is changed according to the phase of the first order differencing filter. At a frequency of  $1/50=0.02$ , Figure 5.1 shows a phase shift of approximately 0.25 times the period length of 50 years, that is 12.5 years, backwards in time. Section 18.1.4 provides more information on the comparison of the spectral densities from the stylized facts and the ones from the tested multivariate VAR models.

Table 18.3 Phase in years of long wave for the data representations from Table 18.1 based on estimates for the Netherlands from Part III. A positive number indicates a lead and a negative number indicates a lag. All phase numbers are relative to the representation of the National Product Index.

	<i>Representation 1</i>	<i>Representation 2</i>
National Product Index	-	-
Consumer Price Index	-12.5*	-
Real Industry Wage Index	0	-
Short Interest Rates	-17.5	-
Long Interest Rates	-17.5	-
Real Equity TRR Index	15.0	-

\* Based on stylized fact across NL, UK and US results instead of specific NL results because of aberrant results for NL.



### *Business cycles*

Besides the long waves, the testing of the various applications of the VAR models focuses on the properties of the spectral densities in the business cycle frequency range  $[1/15, 0.5]$ . Just as with the long waves, the (relative) volatilities of the business cycle fluctuations are not included in the testing of the models. In case of the second data representation these volatilities can easily be adjusted without changing the other stochastic properties of the models by using the procedure described at point 3 in section 3.7.5. Note again that a potential shortcoming of modeling all frequencies together in one single model, as in the case of the first data representation, is that in this way it is impossible to change the variances for specific frequency ranges. The correlations at the business cycle frequencies are taken into account by means of comparing the more informative coherence and phase spectra. The stylized facts of the business cycle type fluctuations as described in section 17.2 are clearly dominated by the presence of Juglar and Kitchin type fluctuations. Tables 18.4 summarize the properties of the peaks at the Juglar and Kitchin frequencies in the spectral densities as estimated for the postwar period in the Netherlands in the relevant chapters of Part III. Where necessary (Peak Power and Phase for the non interest rate variables) these properties are adjusted for the data representations from Table 18.1. The interpretation of the presented Peak Period, Peak Power (PP), Coherence and Phase numbers can be found in Section 6.4.1. Section 18.1.4 provides more information on the comparison of the spectral densities from the stylized facts and the ones from the multivariate VAR models.

Table 18.4 Properties of Juglar and Kitchin peaks in spectral densities of stochastic component for the 1949-1999 period based on estimates for the Netherlands from Part III adjusted for the data representations from Table 18.1.

	<i>Juglar</i>				<i>Kitchin</i>			
	<i>Period</i>	<i>PP</i>	<i>Coh</i>	<i>Phase</i>	<i>Period</i>	<i>PP</i>	<i>Coh</i>	<i>Phase</i>
National Product Index	9.9	0.3	-	-	4.9	0.3	-	-
Consumer Price Index	9.0	0.2	0.6	-4.2	4.8	0.3	0.6	-1.4
Real Industry Wage Index	10.6	0.2	0.9	-1.6	4.2	0.3	0.5	-0.9
Short Interest Rates	10.1	0.3	0.9	-3.9	4.8	0.2	0.6	-1.3
Long Interest Rates	10.1	0.4	0.9	-4.7	4.8	0.2	0.7	-1.3
Real Equity TRR Index	11.5	0.1	0.2	4.7	4.5	0.3	0.9	0.7
Average	10.2	0.3	0.7	-	4.7	0.3	0.6	-

### *Changes*

The final part of the stylized facts found in Part III and described in section 17.3 concerns the possible changes of the stochastic behavior of economic variables in the course of time. The results on such changes are not included in the testing of the VAR models. Testing for the changes between the prewar, interwar and postwar period is not possible because the models here are based on the postwar 1949-1999 sample only. Changes during the postwar period are not tested because all applications described in section 18.1.1 use static models. By this we mean that the models are estimated once on the full sample and are kept constant during simulations.

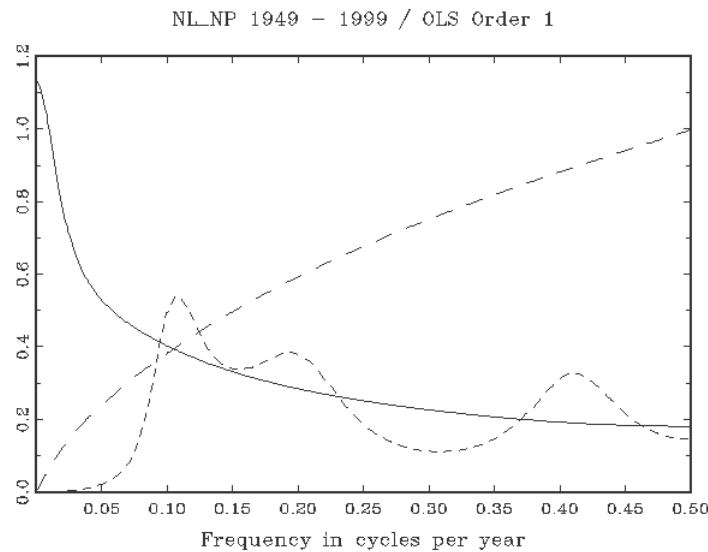
#### 18.2.4 Comparing spectral densities

As the previous section indicates, the testing of the VAR models focuses on the properties of the models at the long wave and business cycle frequencies. In the context of this research it is the most obvious to compare the model properties to the stylized facts in terms of multivariate spectral densities. Note that in this way it is not necessary to compare correlations and autocorrelations. All relevant information is provided by the spectral densities. Furthermore note that this method of assessing multivariate spectral densities can be used for many other types of models than VAR models alone. The only requirement is that the auto- and crosscorrelation structure of the variables in the model can be calculated, either analytically or numerically by means of simulation, from which the spectral densities can be calculated by means of the Fourier transform as described in Chapter 4. For the various components of a multivariate spectral density as described in section 4.3 and 4.4 we proceed as follows.

##### *Auto-spectra*

Here we will separately plot the six normalized auto-spectra of the variables as described by the parameters of the VAR model together with the corresponding integrated auto-spectra to be able to judge the contribution of various frequency ranges to the total variance. These are plotted together with the relevant normalized auto-spectra derived from the univariate Maximum Entropy estimates for the stochastic components for the Netherlands for the postwar period from Part III. Where necessary, the latter (non-normalized) spectra are adjusted for the data representations from Table 18.1 by applying the PTF of the first order differencing transformation. In the case of the first data representation, the latter spectra are also multiplied by the ratio between the variance of the filtered series and the variance of the unfiltered series to correct for the differences in volatility. In this way, both auto-spectra are in fact normalized by the same variance and are therefore presented on the same scale. In fact, this is very similar to comparing non-normalized spectra. The overlapping parts of these spectral densities can be directly compared because theoretically speaking the spectrum obtained by applying the PTF of the ideal  $[1/15, 0.5]$  high-pass filter to the total spectrum is identical to the spectrum of the identically filtered time series. Figure 18.1 gives an example of a comparison of the auto-spectra.

Figure 18.1 Example comparison of auto-spectrum from one of the estimated VAR models (solid line) together with (transformed) auto-spectrum for the stochastic component from Maximum Entropy estimates from Part III (short dashes).



#### *Peak Power*

The Peak Power of the Juglar and Kitchin spectral peaks are not explicitly compared between the spectra from Part III and the spectra from the tested VAR models. For the latter, the Peak Power is often not well defined because of the lack of peaks in the estimated spectra. Instead, the contribution to the total variance can also be judged by directly comparing of the shape of the spectral densities.

#### *Coherence and Phase*

Plotting and comparing the complete coherence and phase spectra over the entire frequency range would be too much compared to the available information from Part III. Instead, a table is included for each estimated model with for each of the five variables the coherence and phase in years relative to the National Product Index at the long wave, Juglar and Kitchin frequencies. These fixed frequencies are assumed to be respectively  $1/50=0.02$ ,  $1/10=0.10$  and  $1/4.5 = 0.22$ . These coherence and phase numbers can then be compared to the corresponding numbers in Table 18.3 and 18.4.

Just as in the chapters of Part III, the results of the testing of existing models in Chapter 19 are presented in figures and tables which are complemented by a discussion of the results in a bulleted format.

### 18.3 Developing alternative models

As we will see, a number of shortcomings will emerge from the testing of the typical applications of VAR models as described in the previous section. By shortcomings we mean that the conventional VAR model applications are not able to adequately describe the “benchmark” empirical knowledge as summarized in section 18.2.3. The shortcomings will be summarized in section 19.3. Therefore, alternative models have been developed to resolve most of these shortcomings. To develop these alternative models, we did not follow the approach of first constructing a detailed and complete specification of the models and then testing them on actual data. Instead, we started off with a broad idea of what these models should look like and then quickly went on to experimenting with these ideas on actual data and modifying the original ideas based on the results of these experiments. This resulted in what is more of a new framework for scenario analysis consisting of various model components than one formal model specification. As already noted in section 18.1.3, the new framework is therefore not meant to result in a fully automated procedure that result in a set of economic scenarios with the desired specifications by simply running some computer program. Instead, the new framework aims to provide a set of optimal tools that can be used by experts to construct high quality economic scenarios according to their needs.

The final framework together with its components will be presented in section 20.1. Where possible, the associated theory is already included in the chapters of Part II to which the appropriate references will be made. In general, one can say that the alternative scenario framework presented here stays as close as possible to the conventional use of VAR models. The current applications can even be seen as special case of the new framework. The main differences are that VAR models are used in a different manner and special restricted VAR models are being used. Furthermore, all elements of the new framework have a strong foundation in thinking about scenarios in the frequency domain. It can therefore suitably be described as “*A frequency domain Vector AutoRegressive scenario framework*”.

After the presentation of the formal framework and its components in section 20.1, in the subsequent sections, four example applications of the new framework are presented to illustrate the various components of the new framework. Most of these examples are strongly related to the experiments performed to actually develop the framework. Each example illustrates one or more of the specific components of the new framework in isolation. An example that combines all components is not included. However, combining the various components is straightforward.

## 19 Testing Existing Models

This chapter describes the results of the testing of a great number of typical but stylized applications of VAR models for generating scenarios on their consistency with the empirical knowledge obtained in Part III and represented by the stylized facts of Chapter 17. That is, as described in section 18.1.1, we consider the empirical knowledge obtained in Part III as being the “benchmark” knowledge which current and new scenario models should be able to describe or adjust properly. Section 18.2.3 formalizes the empirical knowledge against which the models are tested. The other details of the approach followed for testing the models are described in the remaining parts of section 18.2. Section 19.1 and 19.2 present the results for the various combinations of data representation, estimation procedure and model order. Section 19.1 covers all models for the first (traditional) data representation from Table 18.1, hence all models (estimation procedures) from the left column of Table 18.2 for orders  $p=1,\dots,4$ . Section 19.2 covers all models for the second (filtered) data representation from Table 18.1, hence all models (estimation procedures) from the right column of Table 18.2 for orders  $p=1,\dots,4$ . Just as in the chapters of Part III, the results here are presented in figures and tables which are complemented by a discussion of the results in a bulleted format. Finally, a summary in terms of the most important findings from the testing results is presented in section 19.3 which thereby provides the “answer” to the first part of the second research question as formulated in section 1.3.

## 19.1 Traditional data representation

This section presents the testing results of all models for the first (traditional) data representation from Table 18.1, hence all models (estimation procedures) from the left column of Table 18.2 for orders  $p=1, \dots, 4$ . The sub-sections are divided according to the estimation procedures from Table 18.2. In case the results of some order appear to be similar to those of another order, these results are not explicitly presented. Instead we will show for example only the results for order  $p=1$  and  $p=4$ .

### 19.1.1 Ordinary Least Squares

Table 19.1 Ordering of model orders according to the lowest values of the various order selection criteria based on unrestricted OLS estimates for the first data representation.

	$FPE(p)$	$AIC(p)$	$SC(p)$	$CAT(p)$	$HQ(p)$
1	3	4	1	3	1
2	4	3	2	1	3
3	1	1	3	4	4
4	2	2	4	2	2

- The FPE, AIC and CAT order selection criteria point to the use of an order  $p=3$  or  $p=4$ . The SC and HQ criteria point to using a low order of  $p=1$ . These results are consistent with the Monte Carlo results from section 4.7.3.
- Practical applications of VAR models for scenario analysis, as those described in section 18.1.1, make little to no use of order selection criteria. Almost without exception they use an order of  $p=1$ . From the results of the Monte Carlo experiment in section 4.7 we know that using too low an order can be more harmful than using too high an order. Too low an order can be seen as an invalid model restriction which can never be corrected by a subsequent estimation technique.
- For all models in this chapter we do not follow the order selection criteria. Instead, we estimate all models for orders  $p=1, \dots, 4$ .

Figure 19.1 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(1) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

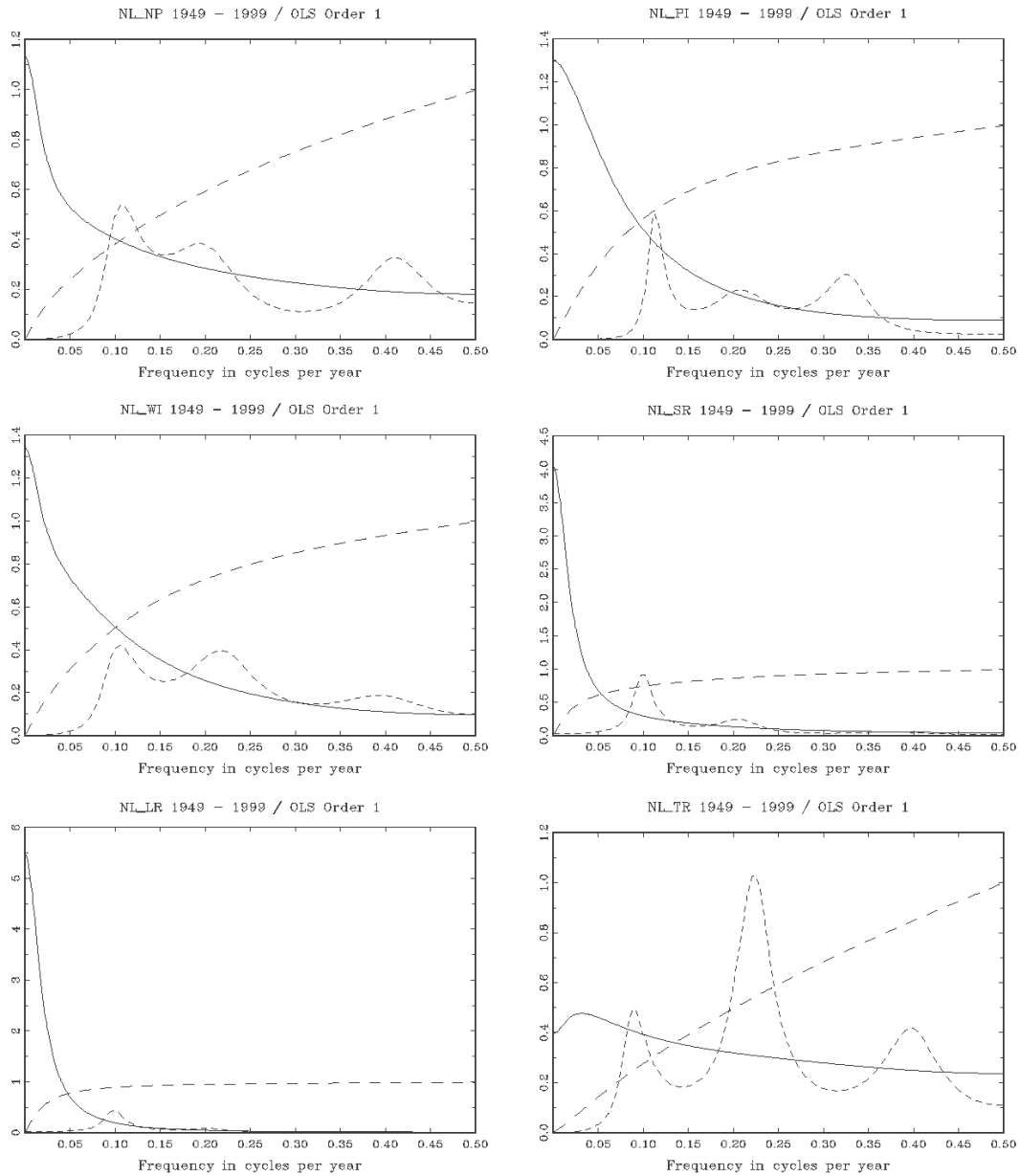


Table 19.2 Coherence and phase in years relative to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(1) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.3	-17.9	0.4	-2.9	0.4	-1.1
WI	0.8	-1.0	0.7	-0.6	0.5	-0.6
SR	0.8	-23.3	0.6	-3.6	0.6	-1.2
LR	0.7	-24.1	0.5	-4.1	0.6	-1.4
TR	0.3	10.7	0.5	2.1	0.5	1.2
Average	0.6		0.5		0.5	



- The *auto-spectra* of the OLS VAR(1) model for the first data representation, as shown in Figure 19.1, are clearly not consistent with the empirical knowledge in the business cycle frequency range. They are an approximation of the Maximum Entropy spectra but do not show any pseudo-periodic behavior in terms of spectral peaks. The spectrum of the real equity Total Return (TR) is even close to that of a random walk model. This is a surprising result because a six dimensional VAR(1) model allows for three pairs of complex conjugated eigenvalues which should be enough to describe a spectral peak at each of the long wave, Juglar and Kitchin frequencies. Also, the univariate models implied by the VAR(1) model are ARMA(6,5) models which have a similar AR order as the Maximum Entropy estimates. Apparently a sufficient number of eigenvalues is not enough to guarantee the expected use of the eigenvalues by the model to produce the expected model dynamics. A reason for this might be that a major difference with univariate models with the same number of eigenvalues is that in a VAR model the *same* eigenvalues have to be used for each variable instead of a (slightly) *different* set for each variable in separate univariate models.
- Furthermore, with exception of the spectrum for the real equity Total Return (TR), all spectra are dominated by *low frequency variance* caused by the real eigenvalues of the model. This is especially the case for the Short and Long interest Rate variables (SR and LR). The likely reason is that for minimizing the residual sum of squares, the model prefers to fit the long wave fluctuations as present in the interest rates for the postwar sample period instead of the higher frequency fluctuations that contribute relatively less to the total variance of the levels of the interest rates. Figure 12.1 and 13.1 show the behavior of the interest rates during the postwar sample period. This model is therefore not capable to describe both the low and high frequency fluctuations well at the same time. This is a serious defect because in other representations or applications of the model, such as the first order differences, the high frequency instead of the low frequency fluctuations may describe the greater part of the variance and hence, be the most important. Therefore the dynamics at all frequencies are in principle equally important. Conventional models focus on the frequencies that are by chance the most important in the data representation chosen for the estimation of the model.
- As we know from its Power Transfer Function, plotted in Figure 5.1, the first order differencing operator does not completely suppress the low frequency fluctuations in the time series of the first data representation. The auto-spectra therefore also show (a lot of) variance at the low frequencies. From a frequency domain point of view it is strange to say the least that the model behavior at these low frequencies, which may have period lengths of for example 50 years for the long wave type of fluctuations, is based on such a *short sample* as the 1949-1999 period. For these type of fluctuations, it would be more logical and even necessary to use a sample of several centuries to at least allow for several completed cycles within the sample period instead of having in fact only one observation of a completed cycle to base the model on. On the other hand, for higher frequency fluctuations, such as for example in the business cycle frequency range, the use of such long samples may be undesirable or even impossible because of the quality and relevance of the data. This holds even more in case a model should also describe for example monthly or daily fluctuations. This data problem calls

for models that allow for different samples and observation frequencies (5 years, annual, monthly, daily,...) per frequency range.

- Comparing the *phase* numbers in Table 19.2 to those in Table 18.3 and 18.4 shows that, contrary to the auto-spectra, the OLS VAR(1) model estimates the phase relations between the variables rather well. Note that this even holds for the phase relations at the long wave frequency.
- With an average of around 50%, the *coherence* numbers for the model are somewhat lower than the average 70% and 60% for respectively the Juglar and Kitchin peaks from the Maximum Entropy estimates. Note that, a coherence number for the long wave frequency is not available from Part III.

Figure 19.2 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(2) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

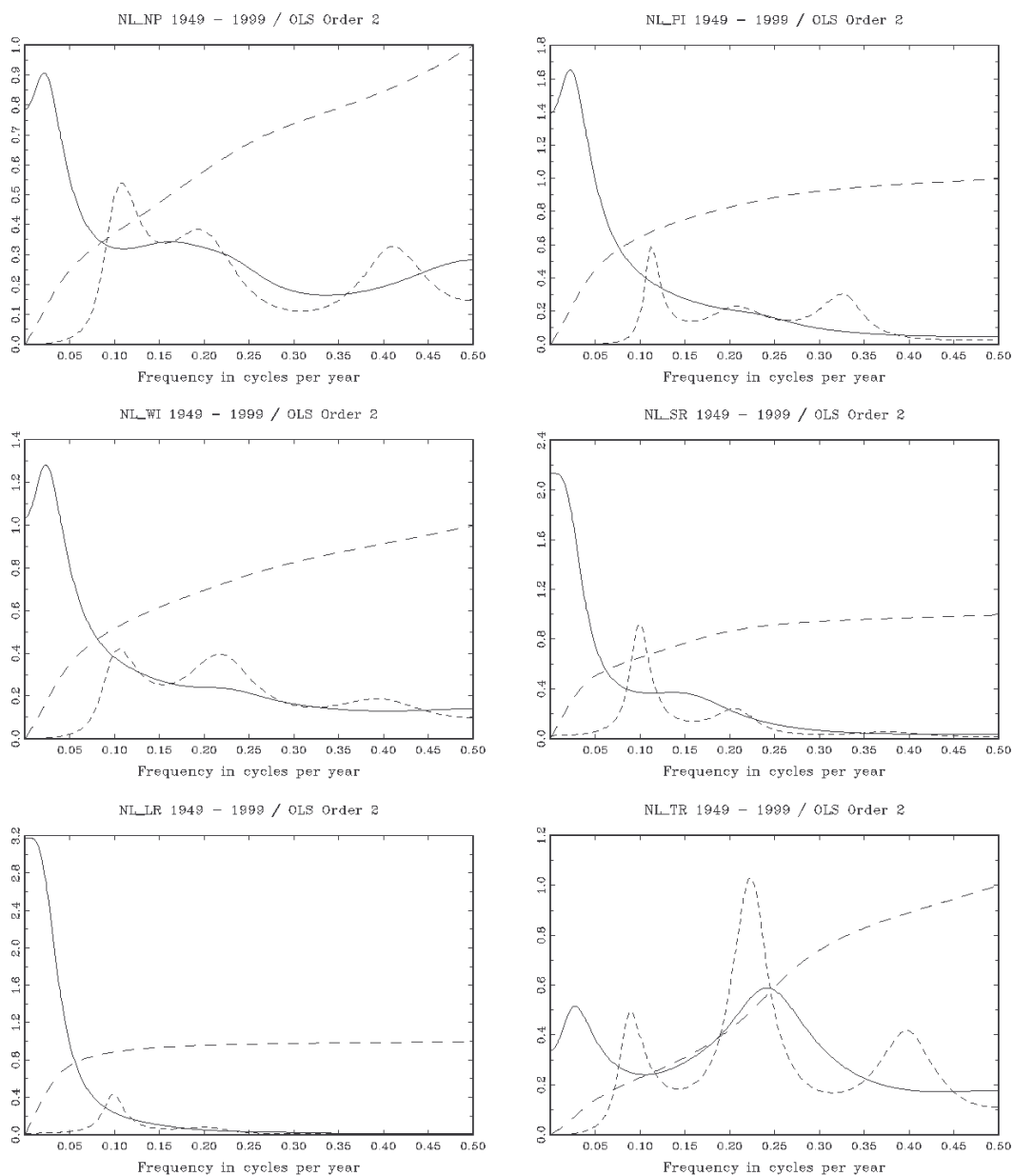


Table 19.3 Coherence and phase in years relative to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(2) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.6	-11.8	0.5	-3.4	0.5	-1.2
WI	0.9	-1.7	0.7	-0.7	0.3	-1.5
SR	0.8	-21.0	0.6	-3.4	0.5	-1.5
LR	0.8	-20.7	0.5	-4.2	0.3	-1.4
TR	0.6	15.5	0.6	2.2	0.6	0.6
Average	0.8		0.6		0.4	

- The *auto-spectra* of the OLS VAR(2) model for the first data representation, as shown in Figure 19.2, show more pseudo-periodic behavior than the VAR(1) model because of the apparent introduction of complex eigenvalues in the model. For the non interest rate variables an expected peak starts to emerge in the long wave frequency range. Especially also the Kitchin peak is introduced in the spectrum for the real equity Total Return (TR), probably because for this variable the long wave fluctuations are relatively less important for the total variance than for the other variables. However, as a whole, the auto-spectra remain inconsistent with the empirical knowledge in the business cycle frequency range.
- The *phase* numbers in Table 19.3 have changed very little from those of the VAR(1) model and therefore remain very similar to the phase relations between the variables as given by the Maximum Entropy estimates.
- The greatest change with respect to the *coherence* is the coherence at the long wave frequency which has increased from an average of 60% to an average of 80%. The coherence at the Juglar and Kitchin peaks remains low compared to the results from the Maximum Entropy estimates.

Figure 19.3 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(3) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

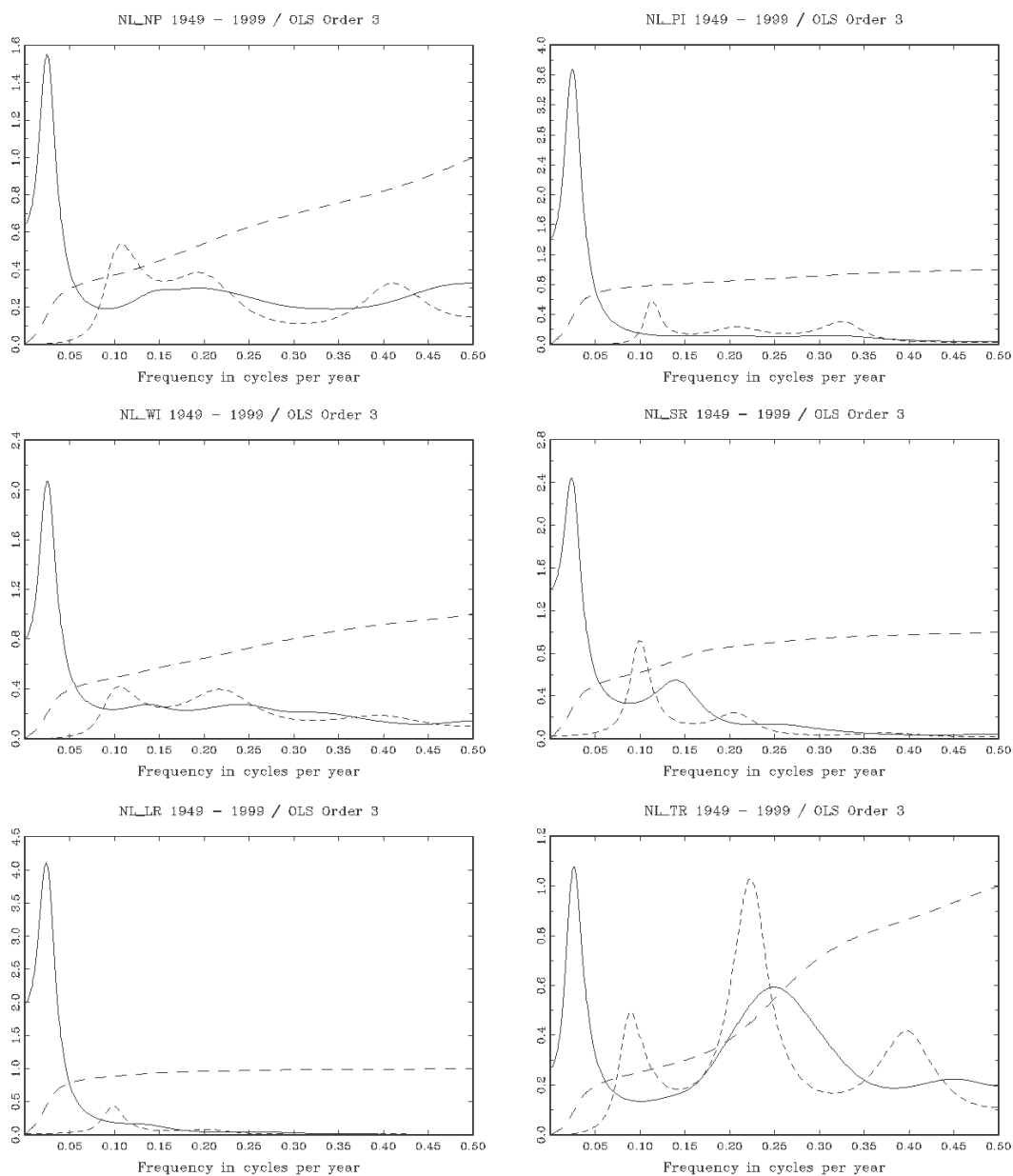


Table 19.4 Coherence and phase in years relative to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(3) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.8	-12.6	0.6	-3.4	0.6	-1.4
WI	1.0	-2.1	0.5	-0.8	0.6	-1.6
SR	0.9	-18.9	0.6	-4.2	0.5	-1.4
LR	0.9	-18.9	0.5	-4.9	0.4	-1.4
TR	0.9	15.9	0.5	2.5	0.8	0.8
Average	0.9		0.5		0.6	

- The *auto-spectra* of the OLS VAR(3) model for the first data representation, as shown in Figure 19.3, show a further increase of the pseudo-periodic behavior, especially in the long wave frequency range. The latter has a peak frequency of around 0.02 which is consistent with a long wave period length of around 50 years. Intuitively it seems logical that, picking up these long wave type fluctuations with such a long period length, requires a high model order, including lags for several years.
- The *phase* numbers in Table 19.4 have again changed very little compared to the lower order models.
- The *coherence* at the long wave frequency has further increased from an average of 80% to an average of 90% while the coherence for the Juglar and Kitchin peaks remains low.

Figure 19.4 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(4) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

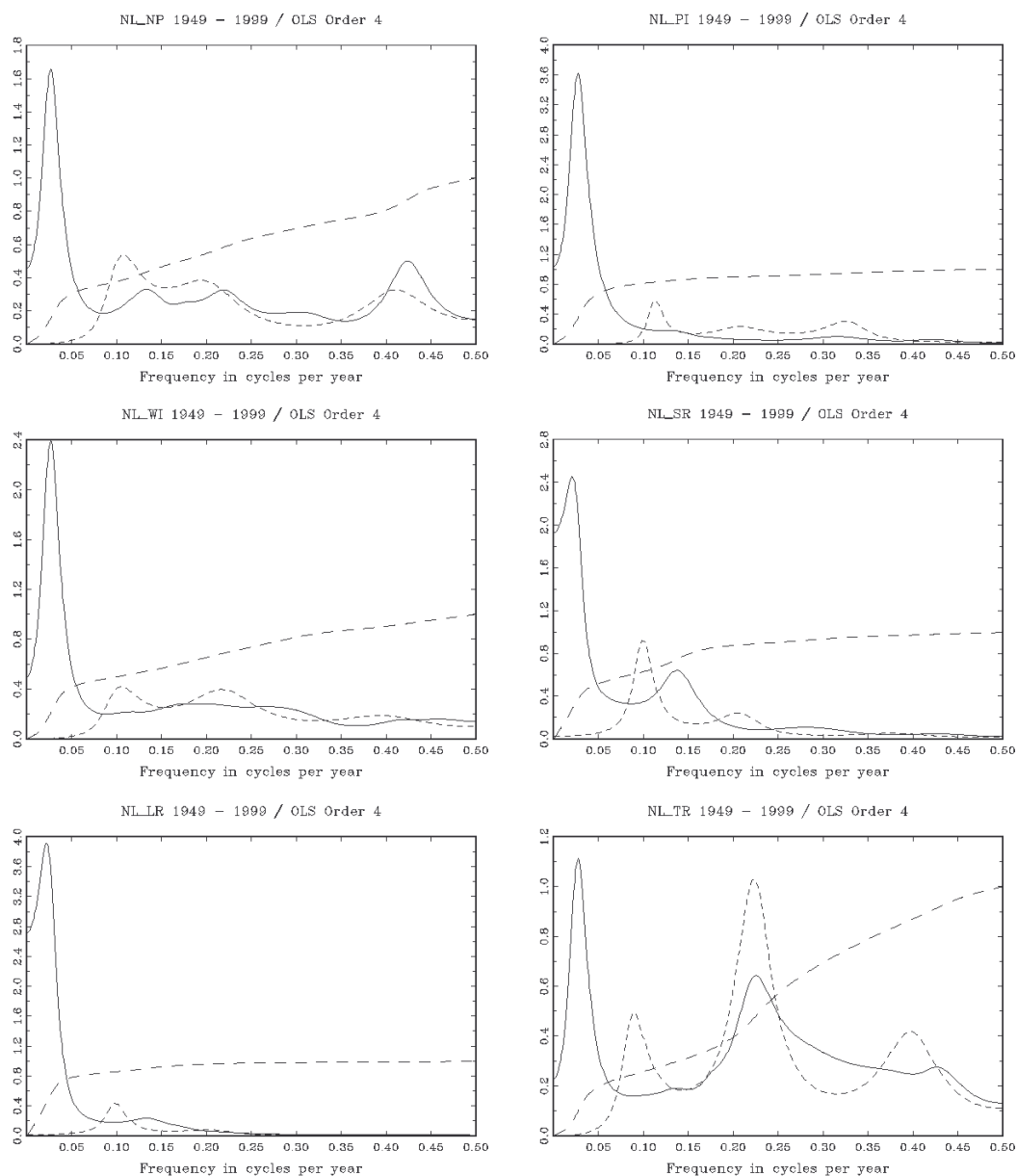


Table 19.5 Coherence and phase in years relative to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(4) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.9	-11.3	0.7	-3.8	0.6	-1.1
WI	1.0	-1.4	0.6	-1.3	0.5	-1.4
SR	0.9	-17.4	0.6	-4.3	0.5	-1.1
LR	0.9	-17.7	0.7	-4.9	0.7	-1.2
TR	0.9	16.8	0.5	2.3	0.9	0.9
Average	0.9		0.6		0.6	



- Although the *auto-spectra* of the OLS VAR(4) model for the first data representation, as shown in Figure 19.4, especially for the National Product growth (NP) and the real equity Total return (TR), show the best resemblance with the Maximum Entropy estimates thus far, the inconsistencies remain. The spectrum of the Short and Long interest Rate (SR and LR) for example shows a “spurious” peak right in the middle of the Juglar and Kitchin peaks. Apparently the inconsistencies found are not resolved by increasing the model order. Note that a six dimensional VAR(4) model implies very high order univariate ARMA(24,20) models for the separate variables so that not much further improvement is to be expected for even higher model orders.
- The *phase* numbers of the model in Table 19.5 are virtually identical to those of the Maximum Entropy estimates in Table 18.3 and 18.4.
- Of the four orders considered, the *coherence* numbers of the OLS VAR(4) models for the Juglar and Kitchin peaks are closest to those of the Maximum Entropy estimates, but still relatively low.
- Although important inconsistencies remain, a high order VAR model performs best in terms of consistency with the empirical knowledge. This is consistent with the results from the order selection criteria in Table 19.1 that indicate to use a high model order.

## 19.1.2 Top-Down strategy

Figure 19.5 Auto-spectra from Top-Down AIC estimates for six dimensional VAR(1) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

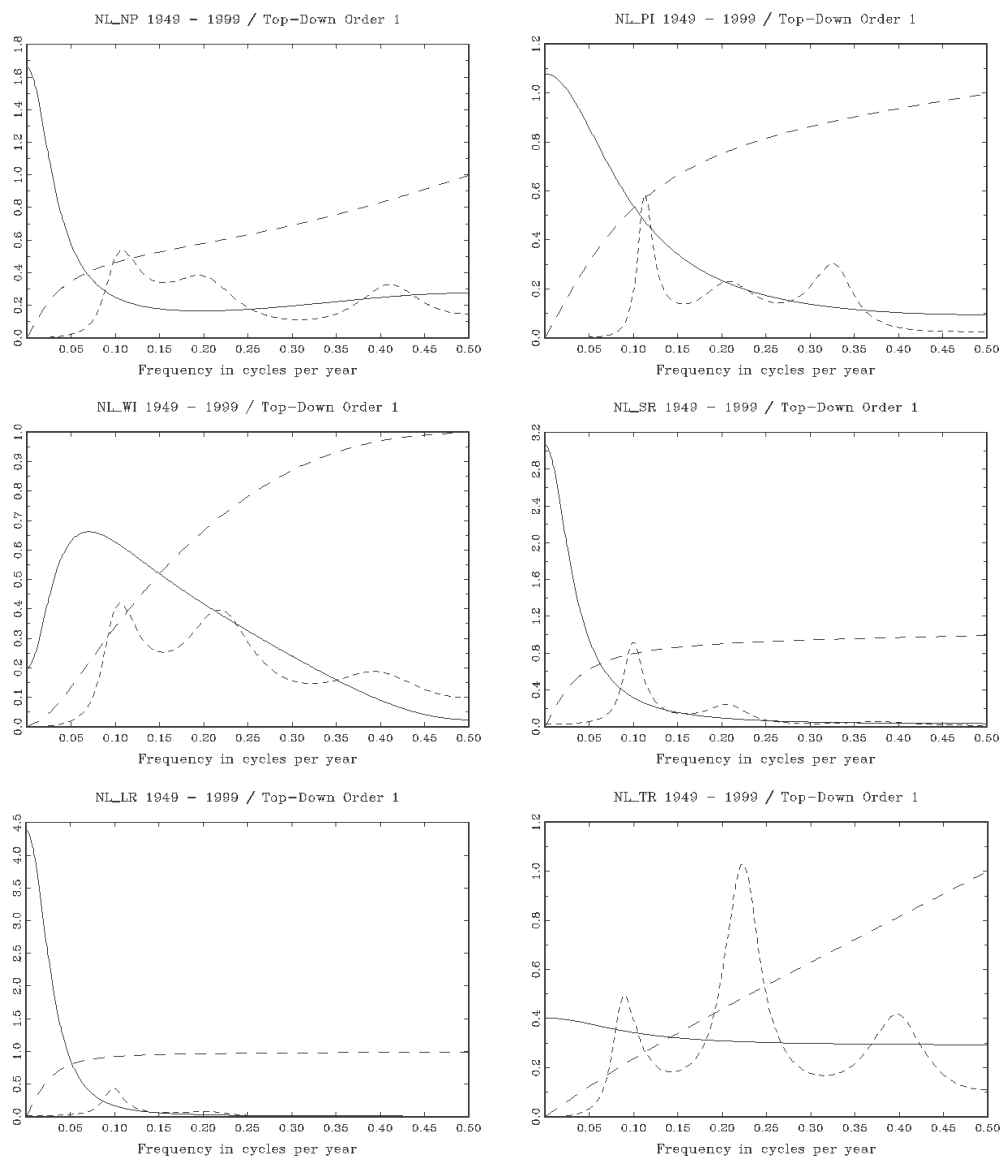


Table 19.6 Coherence and phase in years relative to the National Product index for three frequencies from Top-Down AIC estimates for six dimensional VAR(1) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.9	-21.5	0.9	-3.5	0.9	-1.3
WI	0.9	-8.3	0.9	-2.5	1.0	-1.2
SR	1.0	-24.2	1.0	-4.2	1.0	-1.4
LR	1.0	24.3	1.0	-4.9	1.0	-1.6
TR	1.0	4.4	1.0	2.2	1.0	1.3
Average	1.0		1.0		1.0	

- With the exception of that of the real Wage Inflation(WI), the *auto-spectra* of the Top-Down AIC VAR(1) model for the first data representation, as shown in Figure 19.5, are not very different from those of the OLS VAR(1) model, as shown in Figure 19.1.
- The *phase* numbers in Table 19.6 for the Juglar and Kitchin peaks are still consistent with those from the bivariate Maximum Entropy estimates. However, most phase numbers at the long wave frequency are about 1/5-th period length ( $50/5 = 10$  years) lower than those from Table 18.3. Here lower means a greater lag or a smaller lead. It seems as if the long wave in the National Product growth has been shifted backwards in time by approximately 10 years.
- The *coherence* numbers show aberrant and unrealistic high values of 90% or even 100%.
- For this low order model the consistency with the empirical knowledge in terms of the auto-spectra has neither improved nor worsened much because of the optimization of the parameter constraints. However, the coherence and phase consistency has actually worsened instead of improved by this estimation procedure.

Figure 19.6 Auto-spectra from Top-Down AIC estimates for six dimensional VAR(4) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

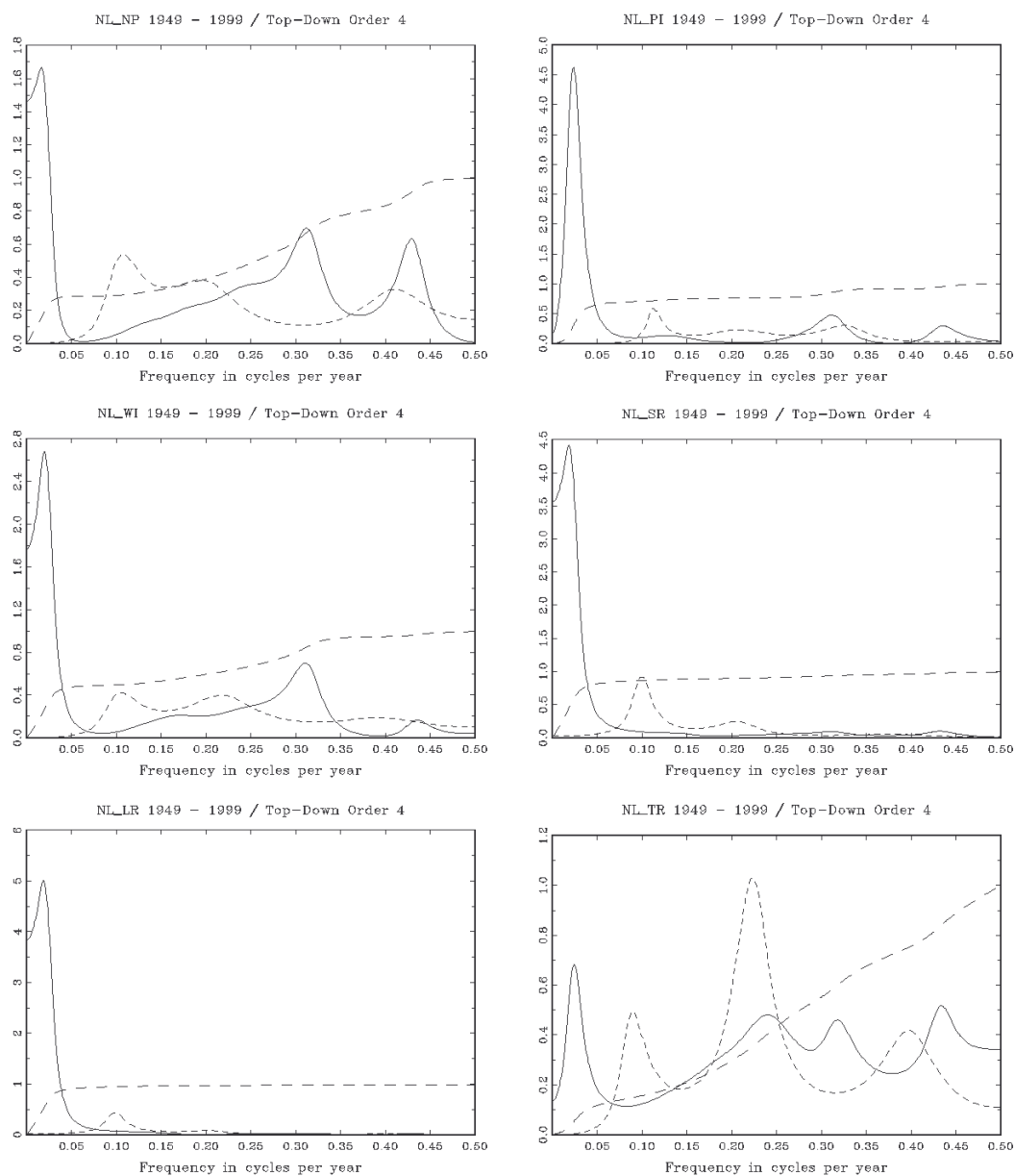


Table 19.7 Coherence and phase in years relative to the National Product index for three frequencies from Top-Down AIC estimates for six dimensional VAR(4) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	1.0	-14.9	0.9	3.5	1.0	-1.7
WI	1.0	-2.9	0.8	-4.0	1.0	-1.3
SR	1.0	-21.2	0.9	2.9	1.0	-1.7
LR	1.0	-21.8	0.9	2.4	1.0	-1.8
TR	1.0	11.1	0.9	-0.3	1.0	0.9
Average	1.0		0.9		1.0	

- Contrary to the previous low order model, the *auto-spectra* of the high order Top-Down AIC VAR(4) model for the first data representation, as shown in Figure 19.6, have worsened compared to those of the OLS VAR(4) model, as shown in Figure 19.4. For the OLS estimates, the VAR(4) model gave the most consistent results compared to the empirical knowledge. Now, the VAR(4) model results are even worse than for the VAR(1) case. For example, the auto-spectra of the National Product growth (NP), real Wage Inflation (WI) and real equity Total Return (TR) show too much variance at the very high frequencies and too little variance in the intermediate business cycle frequency range. Furthermore, the auto-spectra for the Short and Long interest Rate (SR and LR) virtually show no other variance than in the long wave frequency range.
- Compared to the Top-Down AIC VAR(1) model, the *phase* numbers in Table 19.7 for the long wave frequency have pretty much returned to their expected values. Now however, the phase numbers for the Juglar peak are no longer consistent with the results from the Maximum Entropy estimates.
- All *coherence* numbers have increased to unrealistic values of 100%.
- Based on these results, apparently the potential damage done, in terms of the consistency of the models with the empirical knowledge, by estimation procedures that in some way optimize parameter constraints, increases with the model order. Intuitively this seems logical since a higher order means a greater parameter-space that can be disturbed. The fact that optimization procedures such as these actually worsen results instead of improving them was also found in the Monte Carlo experiment described in section 4.7.5.

## 19.1.3 Yule-Walker

Figure 19.7 Auto-spectra from Yule-Walker estimates for six dimensional VAR(4) model for the first data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

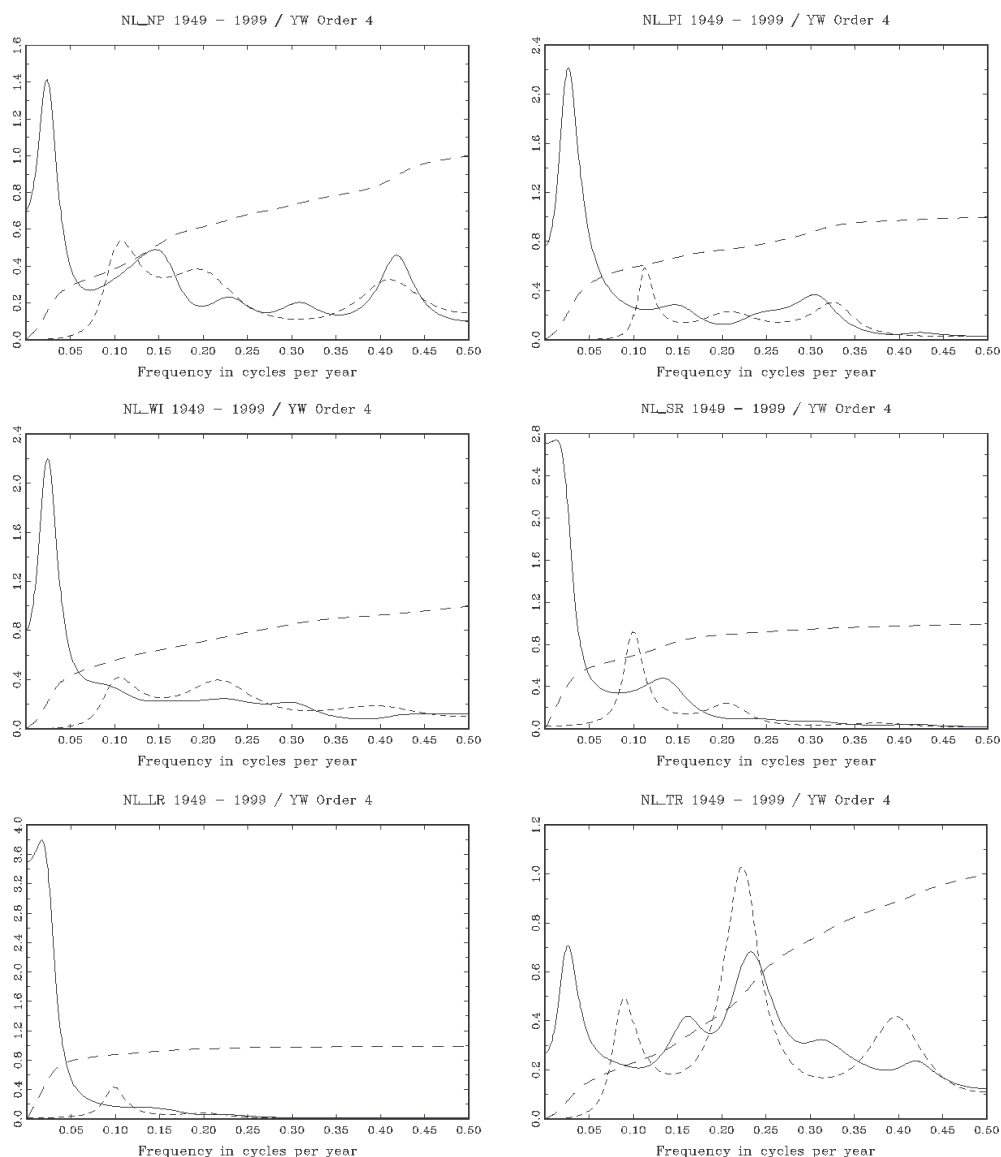


Table 19.8 Coherence and phase in years relative to the National Product index for three frequencies from Yule-Walker estimates for six dimensional VAR(4) model for the first data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.8	-10.8	0.6	-4.5	0.4	-0.6
WI	1.0	-2.0	0.8	-1.2	0.4	-1.8
SR	0.9	-18.4	0.7	-3.4	0.6	-1.3
LR	0.9	-18.6	0.7	-3.7	0.6	-1.3
TR	0.8	18.8	0.5	0.6	0.9	0.9
Average	0.9		0.7		0.6	

- Especially for the low orders, the YW VAR models lead to virtually identical auto-spectra and coherence and phase numbers as for the corresponding OLS VAR models from section 19.1. The results of these models are therefore not shown here.
- With respect to the *phase* and *coherence* numbers in Table 19.8 this is also the case for the YW VAR(4) model when compared to the OLS VAR(4) numbers in Table 19.5.
- With respect to the *auto-spectra* of the YW VAR(4) model the results are mixed. For some variables, such as the consumer Price Inflation (PI), the auto-spectrum, as shown in Figure 19.7, has a better match with the variance distribution from the Maximum Entropy estimate than was the case for the OLS VAR(4) model, shown in Figure 19.4. This is consistent with the results from the Monte Carlo experiment described in section 4.7.2 which found the Yule-Walker method to produce the best estimation results in terms of the relative spectral error (4.7.1). For other variables however, such as for the real equity Total Return (TR), the results have worsened in terms of the number and location of (spurious) peaks in the auto-spectrum.

#### 19.1.4 Dummy variables

- Figure 19.8 shows the effect of applying dummy variables for the consumer Price Inflation (PI) series. The solid line represents the dummy corrected series that is in fact input for the estimation of the standard VAR model. The main effect of the dummy variables is the reduction of the variance of the series by reducing the very high inflation rates of around 10% in the early 1950's and mid 1970's. This is exactly what they are meant to do in the relevant scenario applications. In terms of the frequency domain, the dummy variables have "only" a variance or amplitude effect. Because there are no shifts in time, they cause no phase effects. However, from Figure 19.8 it is clear that the dummy variables brutally change the course of the fluctuations in the series and hence also the dynamics in terms of autocorrelations and spectral densities. The first oil crisis is often used as a reason for the 1970's to be considered as an extraordinary episode, thereby justifying the use of a dummy variable for this period. With the empirical knowledge from Part III we know now that the high inflation rates during that period may very well be caused by the long wave fluctuations as observed in all analyzed economic variables and therefore reported as a stylized fact in Chapter 17. Table 17.3 reports a peak of the long wave type fluctuations around 1972, precisely in the middle of the oil crisis. So, even without having seen the effects on the spectral densities, the justification for this specific use of dummy variables seems dubious.
- Table 19.9 shows that the dummy corrected average inflation rate is 2.9% while during the 1950-1951 and 1971-1975 periods the inflation rate is on average respectively 6.3% and 5.7% higher. The coefficients for both dummy variables are highly significant, hence providing a statistical justification for applying the dummy variables.



Figure 19.8 Results from regressing Dutch consumer Price Inflation (PI) on a constant term and dummy variables for the 1950-1951 and 1971-1975 periods.

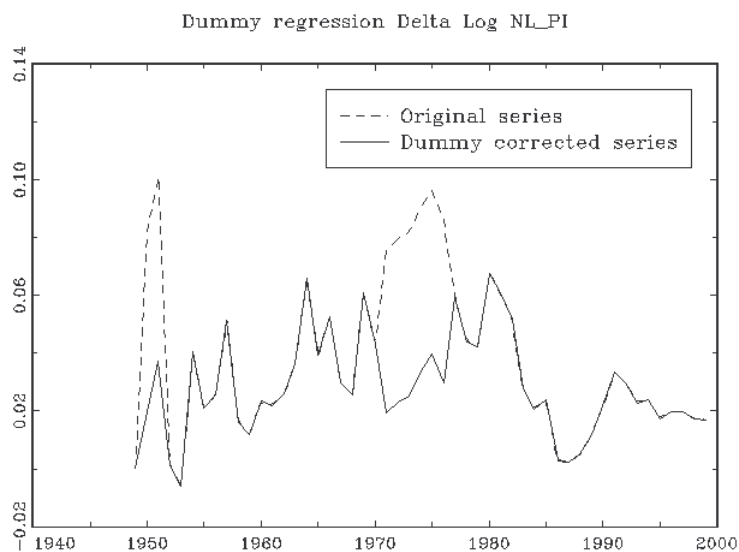


Table 19.9 Estimation results from regressing Dutch consumer Price Inflation for the 1949-1999 period on a constant term and two dummy variables.

<i>Variable</i>	<i>Coefficient</i>	<i>Standard error</i>	<i>t-value</i>
Constant	2.9%	0.003	10.488
Dummy 1950-1951	6.3%	0.013	4.911
Dummy 1971-1976	5.7%	0.008	7.267

Figure 19.9 Auto-spectra for the consumer Price Inflation (PI) from Yule-Walker estimates for six dimensional VAR(1), VAR(4) and VAR(5) model for the first data representation *with* (right) and *without* (left) dummy variable regression compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

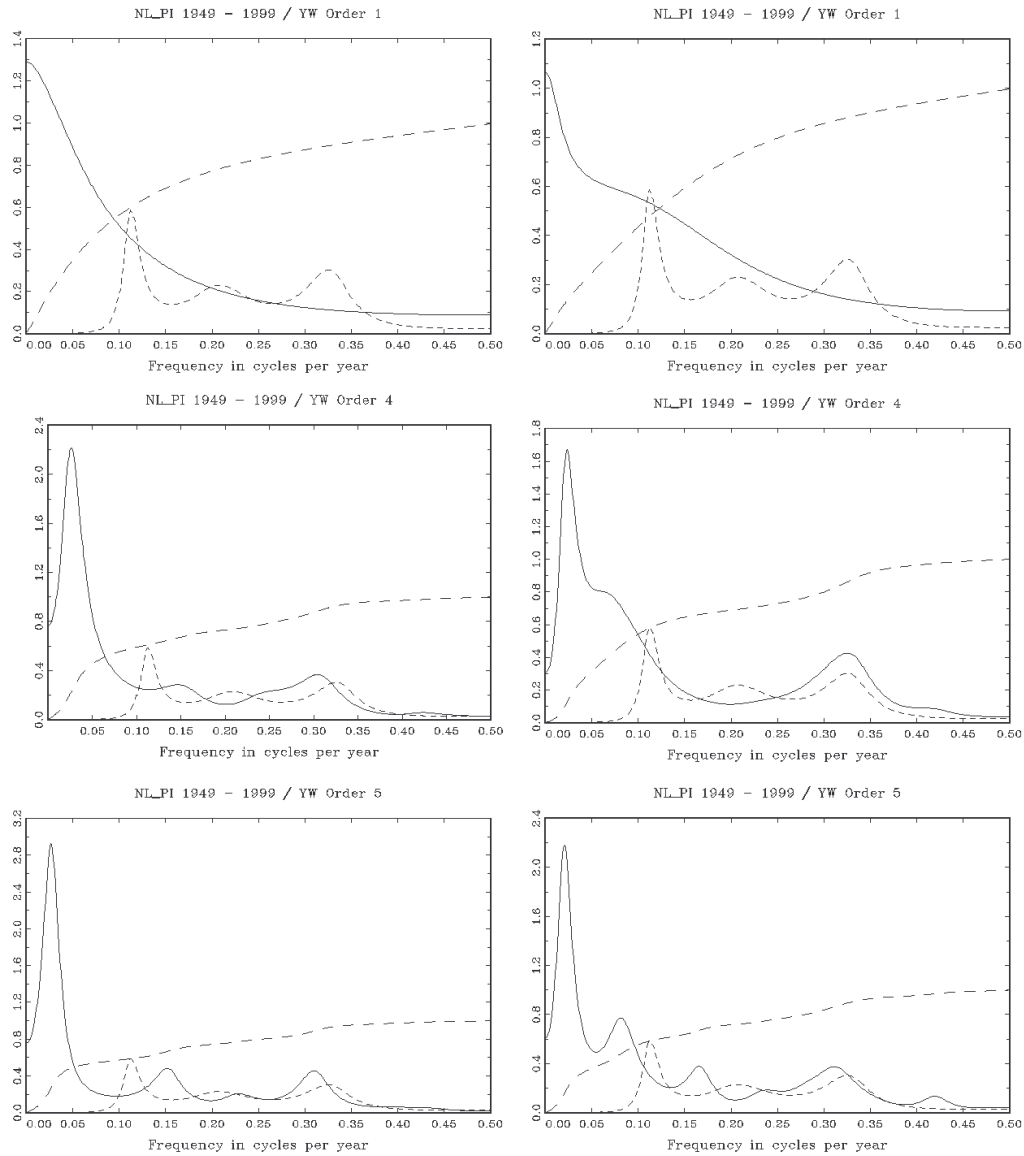


Table 19.10 Coherence and phase in years relative to the National Product index for three frequencies from YW estimates for six dimensional VAR(4) model for the first data representation with dummy variable regression for the Price Inflation (PI).

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.9	-11.4	0.5	-4.0	0.8	-0.9
WI	1.0	-2.0	0.8	-1.0	0.2	-1.9
SR	0.9	-18.2	0.7	-3.7	0.6	-1.4
LR	1.0	-18.4	0.7	-4.0	0.6	-1.4
TR	0.8	18.6	0.5	0.7	0.9	0.8
Average	0.9		0.6		0.6	

- The *auto-spectra* of the other variables then the consumer Price Inflation (PI) itself, are virtually unchanged by applying the dummy variables indicated in Table 19.9 for the consumer Price Inflation. When estimating the models with the Yule-Walker procedure, the spectra are very similar to the ones for the Yule-Walker estimated models without applying the dummy variables as described in the previous section. Figure 19.9 shows that the auto-spectra for the consumer Price Inflation *do* change because of the application of the dummy variables. Especially for order  $p=4$  and  $p=5$ , the variance in the frequency range 0.05-0.10 has increased. For order  $p=5$  a clear spurious peak is introduced in this frequency range that is not present in the normal Yule-Walker estimates. Note that these types of effects on the auto-spectra were to be expected from the effects of the dummy variables on the course of the fluctuations in the time series as shown in Figure 19.8 with peaks of a cycle around 1964 and 1980, 16 years apart, hence with a frequency of  $1/16=0.0625$ .
- The *phase* numbers, including those for the consumer Price Inflation, hardly change when applying the dummy variables. Compare for example the results for the YW VAR(4) model *with* dummy variables in Table 19.10 to those for the YW VAR(4) model *without* the dummy variables in Table 19.8. The fact that the phase numbers are unaffected by applying dummy variables seems logical given the earlier remark that the dummy variables have only a variance or amplitude effect. Because there are no shifts in time, they cause no phase effects.
- Just as the auto-spectra, the *coherence* numbers can change because of the dummy variables. Consider for example the coherence for the consumer Price Inflation for the Kitchin peak in Tables 19.10 and 19.8. This coherence has doubled from 40% to 80% because of the application of the dummy variables.

## 19.2 Filtered time series

The results from the previous section show that models based on the first data representation result in a poor reproduction of the spectral densities in the business cycle frequency range  $[1/15, 0.5]$  as found in Part III. This holds no matter what order of the model or estimation method is used. By estimating models on the filtered time series we are able to determine to what extent these inconsistencies are the consequence of using non-filtered data, with a possible dominance of the low frequency fluctuations, or of using the VAR models. Therefore in this section most models from the previous sections are re-estimated on the filtered second data representation of the time series as described in section 18.2.2. The models with the dummy variables are excluded here because using the dummy variables on the filtered consumer Price Inflation is not meaningful because the filtering has already eliminated all the low frequency components from the time series. In total, this section presents the testing results of all models (estimation procedures) from the right column of Table 18.2 for orders  $p=1, \dots, 4$ . The sub-sections are divided according to the estimation procedures from Table 18.2. In case the results of some order appear to be similar to those of another order these results will not be explicitly presented. Instead we will show for example only the results for order  $p=1$  and  $p=4$ . Finally, note that the experiments in this section show a strong resemblance with the Monte Carlo experiment from section 4.7. A difference is however that there the true spectra were known with certainty while here the reference spectra are estimates themselves, based on the same data but obtained from other models and estimation procedures.

### 19.2.1 Ordinary Least Squares

Table 19.11 Ordering of model orders according to the lowest values of the various order selection criteria based on unrestricted OLS estimates for the second data representation.

	$FPE(p)$	$AIC(p)$	$SC(p)$	$CAT(p)$	$HQ(p)$
1	4	4	1	4	4
2	3	3	2	3	3
3	2	2	4	2	2
4	1	1	3	1	1

- The order selection criteria results from Table 19.11 point even stronger to using a high order model than was the case for the first data representation in Table 19.1. Only the SC criterion still indicates a low order. However, when including order  $p=5$ , also this criterion prefers this high order instead of the low order  $p=1$ .
- Because of the information from the order selection criteria and because the results from the previous sections confirm that the high order models indeed perform best in terms of the consistency with the spectral densities from Part III, the following sections primarily discuss the results for the highest order considered, order  $p=4$ .

Figure 19.10 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(1) model for second data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

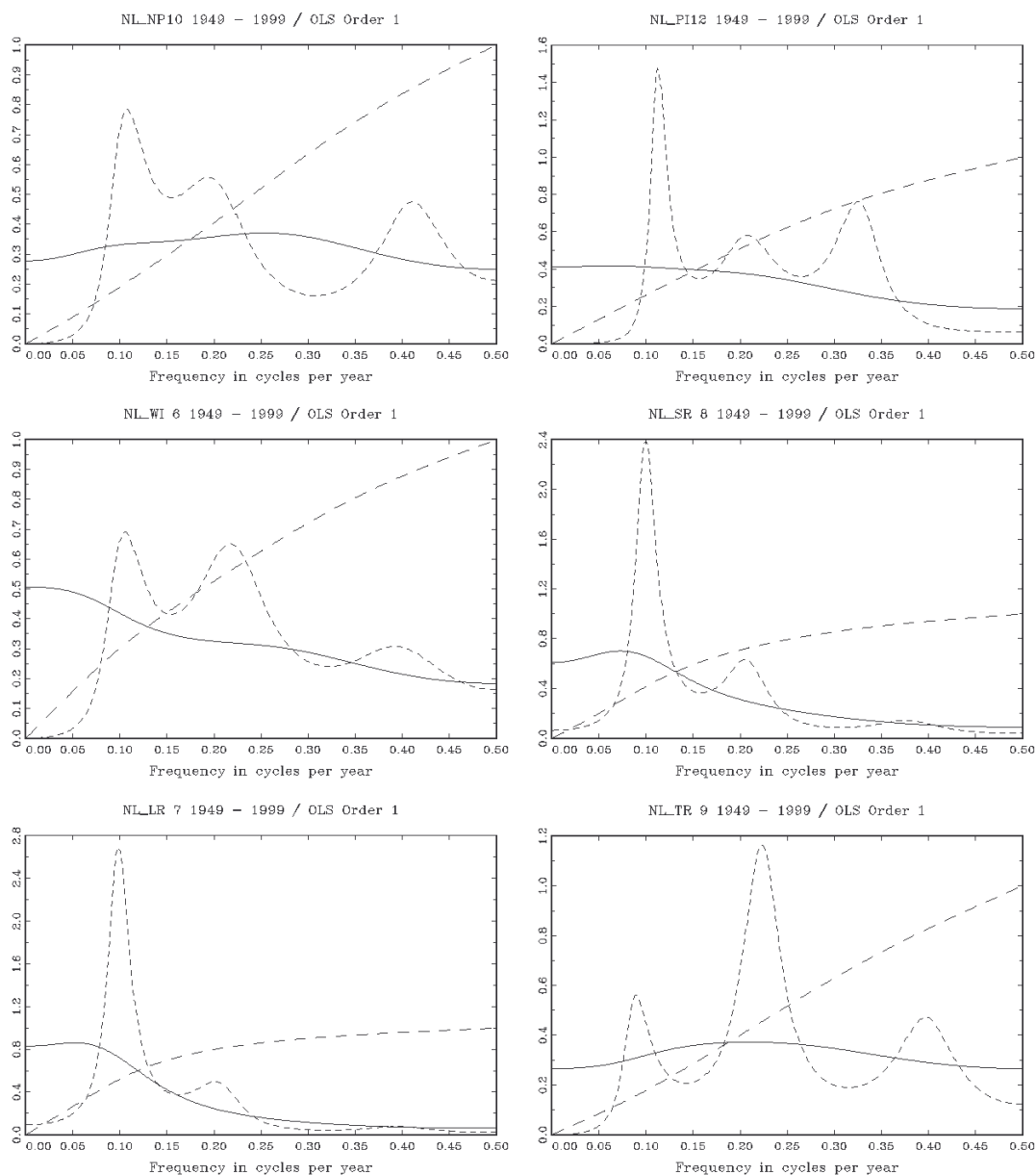


Table 19.12 Coherence and phase in years relative to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(1) model for the second data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.6	-24.5	0.6	-4.3	0.6	-1.4
WI	0.5	-0.6	0.4	-0.5	0.4	-0.9
SR	0.6	-24.1	0.6	-4.2	0.6	-1.3
LR	0.6	24.6	0.6	4.9	0.5	-1.7
TR	0.6	0.7	0.7	0.8	0.6	0.8
Average	0.6		0.6		0.5	

- Just as for the unfiltered data representation, the *auto-spectra* for the OLS VAR(1) model for the filtered second data representation, as shown in Figure 19.10, are still very much inconsistent with the empirical knowledge in the business cycle frequency range. The spectra hardly show any of the expected pseudo-periodic behavior. A sufficient number of eigenvalues compared to the expected dynamics is still no guarantee for an estimated model to actually describe these dynamics. The biggest difference with the results for the first data representation from Figure 19.1 is of course that the low frequency fluctuations no longer dominate the spectra. However, because all fluctuations in the frequency range  $[0, 1/15]$  have been completely filtered out of the time series, another clear error in these auto-spectra is that most of them do show significant variance mass in this frequency range that we know is not really there.
- Comparing the *phase* numbers in Table 19.12 to those in Table 18.3 and 18.4 shows that these are again described pretty well by the model. A somewhat strange result is that even the phase numbers for the long wave frequency are not that wrong, even though these fluctuations are not present in the time series.
- The *coherence* numbers remain relatively low compared to those from the Maximum Entropy estimates.

Figure 19.11 Auto-spectra from unrestricted OLS estimates for six dimensional VAR(4) model for second data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

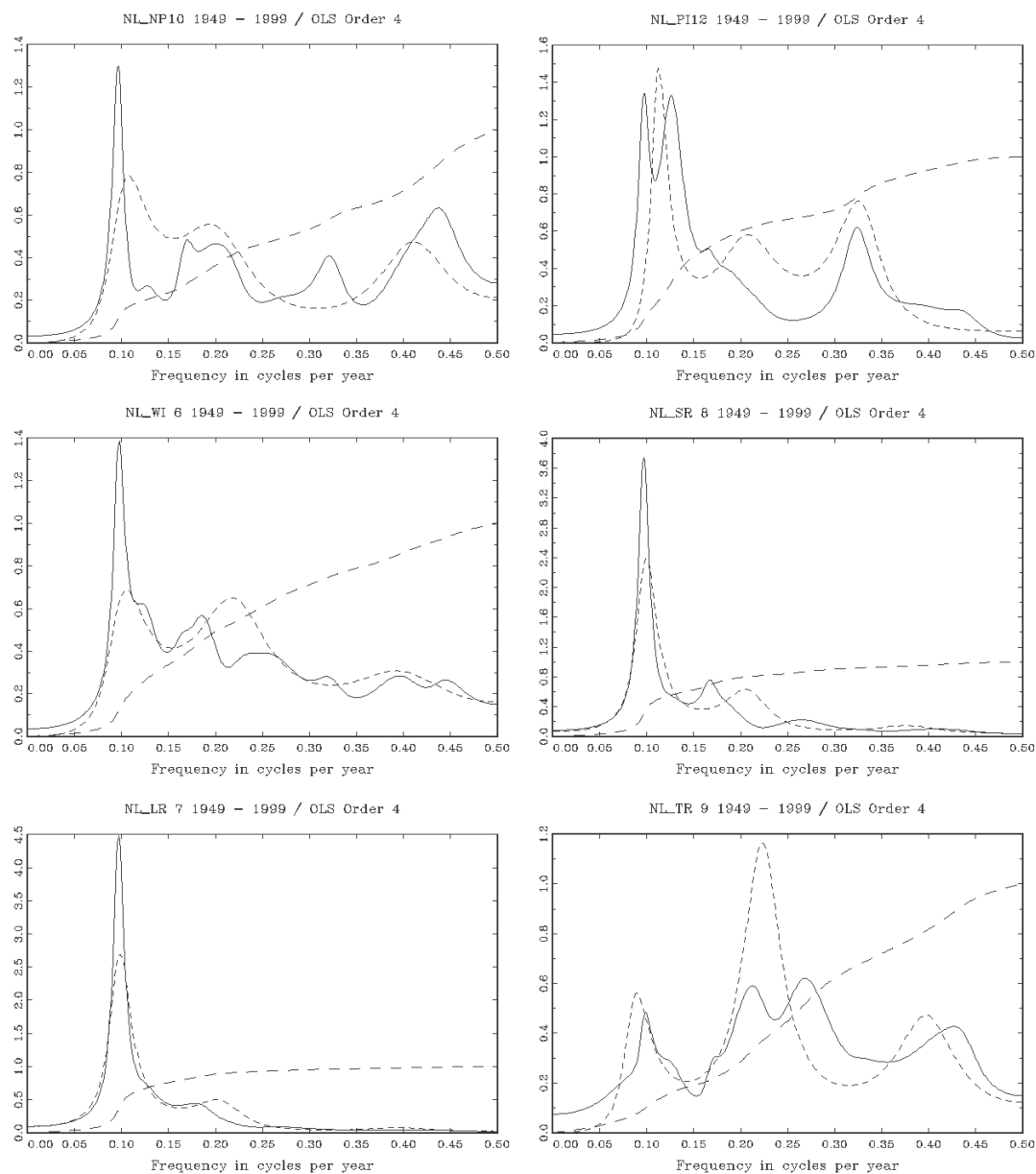


Table 19.13 Coherence and phase in years relatively to the National Product index for three frequencies from unrestricted OLS estimates for six dimensional VAR(4) model for the second data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.4	23.3	0.8	2.5	0.7	-1.0
WI	0.6	-2.3	0.9	-0.5	0.1	-0.7
SR	0.6	-24.4	1.0	-4.2	0.5	-0.5
LR	0.7	24.7	1.0	-4.9	0.7	-1.3
TR	0.5	-0.3	0.8	2.0	0.9	0.8
Average	0.6		0.9		0.6	



- The *auto-spectra* from the OLS VAR(4) model for the filtered second data representation, as shown in Figure 19.11, are clearly more consistent with the Maximum Entropy estimates from Part III than the ones for the same model estimated on the unfiltered first data representation, as shown in Figure 19.4. This holds especially for the Short and Long term interest Rates (SR and LR). This confirms that in the unfiltered representation the dominance of the low frequencies causes the poor estimates at the higher frequencies. This only holds if the model order is sufficiently high. After all, the results for the OLS VAR(1) model for the filtered series were still of poor quality. From this we can learn that by filtering time series and estimating models on these filtered time series, better estimates can be obtained for the relevant frequency range. However three other problems are visible from these auto-spectra.
- The first is that, especially the spectrum of the consumer Price Index (PI) and to a lesser extent also the spectrum of the real equity Total Return (TR), show the *spectral line splitting* phenomenon as described in section 4.7.7. Note that contrary to what is stated there, this shows that spectral line splitting can actually also occur when using the OLS estimation method.
- The second problem is that the OLS estimates have a rather strong tendency to show *too strong periodic behavior*. This can be seen from the fact that peaks at the Juglar frequency in some of the spectra are sharper than those in the Maximum Entropy estimates.
- Although both the problem of spectral line splitting and too sharp spectral peaks are not present in the results for the Yule-Walker estimator, as described in section 19.2.3, the following and third problem holds for high order models in general. The auto-spectra for the National Product growth (NP) and the real Wage Index (WI) for example show that high order models for high dimensional processes may offer too much flexibility in terms of the number of eigenvalues, thereby producing spectra with many peaks that blur the “true” underlying spectrum. The many peaks in the auto-spectra are partly caused by the feed-across effect as described at the end of section 4.7.7. Note that now a difficult problem of conflicting model requirements has emerged. We have seen that on the one hand, a low order model with in principle enough eigenvalues is not enough to produce the right model dynamics. In this case unconstrained models prefer to use the available eigenvalues for “averaging” across the univariate spectral densities by using spectral peaks at intermediate frequencies with a low modulus. On the other hand, a high order model performs better on this aspect but offers too much flexibility in the dynamics of the model. The only way out of this problem seems to be to apply the appropriate restrictions in order to have the best of both model orders. We return to this in the next chapter on alternative scenario models.
- The results in Table 19.13 show that also the *phase* and *coherence* numbers can be disturbed by the problem of having too much flexibility in the model. The coherence and phase spectra from which the numbers are taken have many peaks as well, just as the auto-spectra in Figure 19.11. Some coherence numbers have an unrealistically high value of 100% or a very low value of only 10%. The phase for example for the Juglar peak for the consumer Price Inflation (PI) shows a lead of 2.5 years while a lag of a little more than 4 years is expected.

## 19.2.2 Top-Down strategy

Figure 19.12 Auto-spectra from Top-Down AIC estimates for six dimensional VAR(4) model for second data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

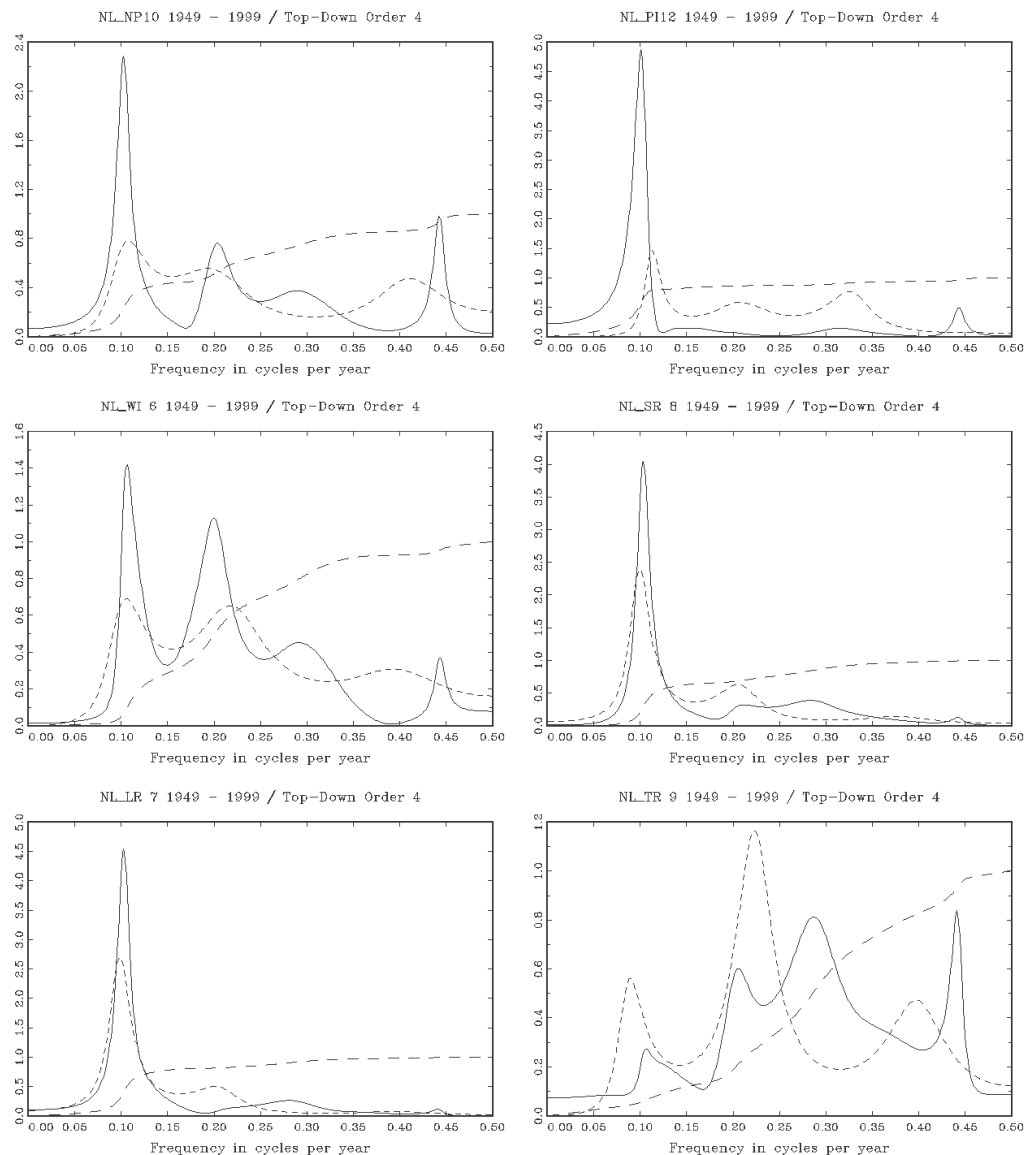


Table 19.14 Coherence and phase in years relatively to the National Product index for three frequencies from Top-Down AIC estimates for six dimensional VAR(4) model for the second data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	1.0	23.5	1.0	2.4	0.9	-0.9
WI	1.0	-3.1	1.0	-0.7	1.0	-1.4
SR	0.9	-22.3	1.0	-4.1	1.0	-1.0
LR	1.0	24.7	1.0	4.8	1.0	-1.1
TR	1.0	-0.7	1.0	1.2	1.0	1.0
Average	1.0		1.0		1.0	

- Just as was the case for the results of the first data representation in section 19.1, the *auto-spectra* of the Top-Down AIC VAR(1) model for the filtered second data representation are not very different from those of the OLS VAR(1) model, as shown in Figure 19.10. The *phase* numbers for this low order model are consistent with the empirical knowledge. Just as in section 19.1.2 however, the *coherence* numbers have practically all become unrealistically high at 100% because of the Top-Down estimation. Because these results are not new, they are not presented here.
- Also the results for the Top-Down AIC VAR(4) model for the second data representation, as shown in Figure 19.12 and Table 19.14, are not very different from those for the first data representation from section 19.2. That is, optimization procedures can disturb both auto-spectra and phase and coherence numbers. These problems do not disappear by using filtered data. What is more clear from the results here than from those in section 19.2, is that the zero constraints applied by the estimation procedure limit the number of peaks in the spectra. The auto-spectra show a more smooth pattern indicating the presence of less different eigenvalues. In itself this is a desirable property in the light of the problem of too much flexibility in the dynamics of the high order model described in the previous section. Unfortunately, because as we have seen the consistency actually worsens, the resulting spectra are apparently restricted in the wrong way. A more appropriate way of restricting the model will be applied in section 20.5.

## 19.2.3 Yule-Walker

Figure 19.13 Auto-spectra from Yule-Walker estimates for six dimensional VAR(4) model for second data representation compared to univariate auto-spectra derived from the relevant univariate Maximum Entropy estimates from Part III.

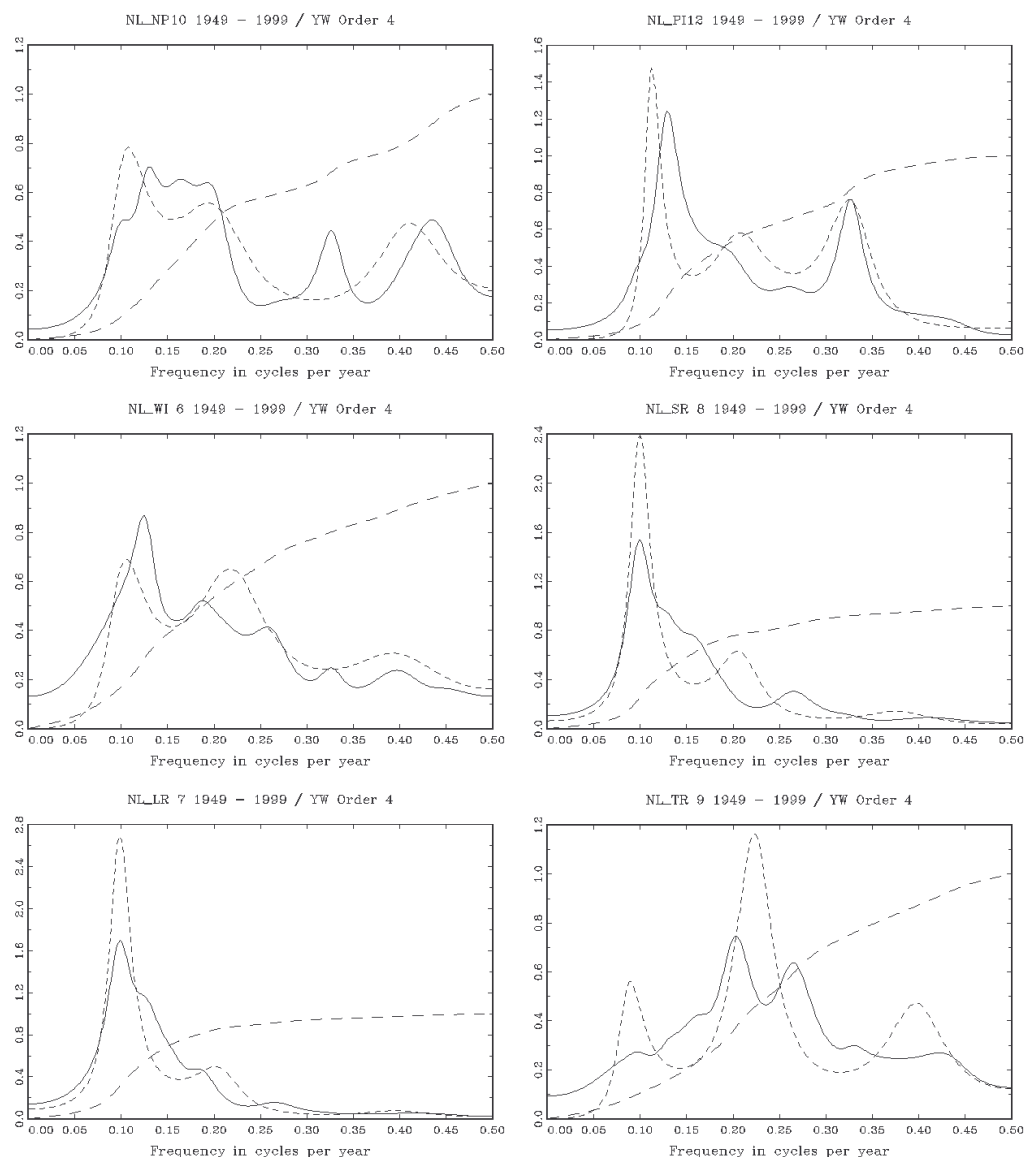


Table 19.15 Coherence and phase in years relatively to the National Product index for three frequencies from Yule-Walker estimates for six dimensional VAR(4) model for the second data representation.

	$\omega=1/50$		$\omega=1/10$		$\omega=1/4.5$	
	Coh	Phase	Coh	Phase	Coh	Phase
PI	0.2	15.3	0.6	4.0	0.7	-0.9
WI	0.4	-5.2	0.7	-1.2	0.2	0.5
SR	0.6	24.5	0.8	-3.7	0.2	-1.0
LR	0.8	24.0	0.8	-4.5	0.6	-1.4
TR	0.6	-0.7	0.2	0.9	0.9	0.7
Average	0.5		0.6		0.5	

- When compared to the results for the OLS VAR(4) model in section 19.2.1, the *auto-spectra* for the YW VAR(4) model, as shown in Figure 19.13, suffer less from the problems of spectral line splitting and too sharp spectral peaks with the associated too strong periodic behavior. Because of this, these auto-spectra are slightly more consistent with the Maximum Entropy estimates. A problem that is not resolved by the use of the Yule-Walker estimation method is that of too much flexibility in the dynamics of the model in terms of the large number of eigenvalues. The auto-spectra still show many peaks that blur the “true” underlying spectrum.
- Probably because of the large number of eigenvalues, inconsistencies in the *phase* and *coherence* numbers, as shown in Table 19.15, also remain present.

### 19.3 Summary of testing results

The objective of the analysis described in section 18.2 and applied in the sections 19.1 and 19.2, was to test whether conventional ALM scenario applications of Vector AutoRegressive (VAR) models indeed lead to scenarios that are *consistent* with the “benchmark” empirical knowledge about the behavior of macroeconomic variables as obtained in Part III and summarized in section 18.2.3. This objective constitutes the important first part of the second research question as formulated in section 1.3. Based on the results found here, the overall “answer” to this question consists of the following points.

1. Conventional applications of VAR models in many respects do *not* produce results that are consistent with the empirical knowledge found in Part III.

This is supported by the following list of main findings from the analysis. Some of these findings have not been explicitly reported in the previous sections but have been encountered during the testing procedures.

2. Applications of VAR models for scenario analysis make little to no use of order selection criteria. Almost without exception they use an order of  $p=1$  which can result in poor results, especially in low dimensional models, because applying too low an order is in fact an invalid model restriction which can never be corrected for by a subsequent estimation procedure. Indeed, the high order models analyzed here produce the best results as also indicated by the order selection criteria.
3. Partially because spectral representations are rarely or never used for analyzing the dynamic properties of estimated VAR models, one is often not well aware of these model properties which are of course also present in the scenarios that are generated with these models. Examples are the unawareness of the presence of long wave type fluctuations in models of a sufficiently high order and of the model properties in the important business cycle frequency range. More in general, both the (spectral) dynamics of the variables and the (coherence and phase) relations between

the variables at the various frequencies are rarely known and cannot be modeled separately.

4. From a frequency domain point of view, it is strange to say the least that, implicitly, model behavior at low frequencies, which may have period lengths of for example 50 years for the long wave type fluctuations, are often based on samples covering only a few decades at the most. In a sense one could say that this part of the model behavior is based on only one single observation of the relevant phenomenon. It would be more appropriate to use different sample and different observation frequencies for different frequency ranges.
5. Relations between the model behavior at the various frequencies cannot be modeled well by conventional VAR model applications. Think of the level effect in interest rates that describes a positive relation between the volatility in the business cycle frequency range and the underlying level of the interest rates.
6. Because of the thin tails of the often assumed Normal distribution, using conventional VAR models it is problematic to truncate (transformations of) variables at predefined levels. Think of interest rates or the relative changes of a consumer price index (i.e. price inflation) which should or could be wanted to stay strictly positive. Using a Normal distribution is simply not always a good modeling assumption.
7. Estimating a low order (multivariate) VAR model that, because of its dimension, in principle has a sufficient number of eigenvalues to describe the expected (univariate) variable dynamics is not enough to guarantee that the model will actually also describe these dynamics. In this case, unconstrained models prefer to use the available eigenvalues for “averaging” across the univariate spectral densities by using spectral peaks at intermediate frequencies with a low modulus. A reason for this might be that a major difference with univariate models with the same number of eigenvalues is that in a VAR model the *same* eigenvalues have to be used for each variable instead of a (slightly) *different* set for each variable in separate univariate models.
8. Using conventional representations of the data (interest rate variables in levels and index variables in terms of growth rates) conventional VAR models are often not capable of describing both the low and high frequency fluctuations well at the same time because of the dominance of the low frequency fluctuations in terms of the variance of the variables. However, the dynamics at all frequencies are in principle equally important. In other representations or applications of the same models, for example the first order differences, the relative importance of the frequencies in describing the total variance can be very different. Conventional models focus on the frequencies that are by chance the most important in the data representation chosen for the estimation of the

model. By first filtering time series and then estimating models on those filtered times series, better estimates can be obtained for the relevant frequency range.

9. The phase relations in the spectra from multivariate models are consistent with the empirical knowledge for most combinations of data representation, model order and estimation method and are robust in that sense.
10. The coherence relations in the spectra from multivariate models are often 10 to 20% lower than those from the empirical knowledge. The coherence numbers slightly increase with the model order but remain relatively low. An exception are the models estimated by procedures such as the Top-Down procedure that apply optimal zero constraints on the models, based on some statistical criterion. These models produce unrealistically high coherence numbers of around 100%. It might be argued that the coherence numbers from the empirical knowledge from Part III are too high to be true. They might be the consequence of the types of (unrestricted) VAR models used for estimating the bivariate Maximum Entropy spectra.
11. Consistency of the important auto-spectra from multivariate models with the spectra from the univariate empirical knowledge is difficult to achieve. Consistent with the information from the order selection criteria, high order models show the best performance on this aspect.
12. Models estimated by Ordinary Least Squares suffer from spectral line splitting and too sharp spectral peaks indicating too strong periodic behavior. Models estimated by the Yule-Walker procedure suffer less from the problems of spectral line splitting and too sharp spectral peaks and thereby produce (slightly) more consistent spectral estimates.
13. Also models estimated by the Yule-Walker procedure are troubled by the fact that high order models for high dimensional processes may offer too much flexibility in terms of the number of different eigenvalues, thereby producing spectra with too many peaks that blur the “true” underlying spectrum. Also the phase and coherence relations can be disturbed by having this type of too much flexibility in a model. One could also say that some of the great number of eigenvalues available in a high order model are used to model specific peaks in the auto-spectra of specific variables. Because of the feed-across effect, the specific spectral peaks from one variable also turn up in the spectra of other variables. In total, this causes spectral densities with too many peaks. Together with finding number 7, this causes a difficult problem of conflicting model requirements. On the one hand, a low order model with in principle enough eigenvalues is not enough to produce the right model dynamics while on the other hand, a high order model performs better on this aspect but offers too much flexibility in the dynamics of the model.



14. A possible way out of this problem is to apply appropriate restrictions on a model in order to have the best of both model orders. One possibility for this is to apply zero constraints on a model, based on some statistical criterion, for example by applying a Top-Down estimation procedure. This indeed results in smoother auto-spectra with fewer peaks. Unfortunately, especially for high order models, the consistency of the spectra with the empirical knowledge actually worsens instead of improves and the resulting spectra are therefore apparently restricted in the wrong way by such a procedure.
15. In the light of the empirical knowledge about the long wave fluctuations as observed in all analyzed economic variables, and therefore reported as a stylized fact in Chapter 17, the justification for applying dummy variables on inflation rates during the 1970's as being an extraordinary episode becomes dubious.
16. The application of dummy variables brutally changes the course of the fluctuations in time series and can therefore disturb model spectra by for example introducing spurious peaks in the auto-spectra and altering coherence numbers. The effects are limited to the variables to which the dummy variables are applied. Because dummy variables only have a variance or amplitude effect and cause no shifts in time, applying them has no effect on the phase relations.
17. Especially for high order unrestricted VAR models, the model standard deviations can be very different from the standard deviations directly estimated on the sample data.

## 20 Developing Alternative Models

The results from the testing of existing applications of VAR models for generating scenarios, as described in the previous chapter, indicate a number of shortcomings in terms of their consistency with the “benchmark” empirical knowledge obtained in Part III and represented by the stylized facts of Chapter 17. These shortcomings are summarized in section 19.3. This chapter describes alternative models that resolve most of these shortcomings. Together these model constitute what can be called “A *frequency domain Vector Autoregressive scenario framework*”. Section 18.3 describes the approach that was followed to develop this framework. First, section 20.1 gives a formal description of this new scenario framework and its components. Next, in the subsequent sections 20.2 through 20.5, four example applications of the new framework are presented to illustrate the various components of the new framework. Most of these examples are strongly related to the experiments performed to actually develop the framework and are meant to gain experience with the new framework. Each example illustrates one or more of the specific components of the new framework in isolation. An example that combines all components is not included. However, combining the various components is straightforward. To provide a reference for studying the workings of the new framework, section 20.1.8 presents an overview of the steps that need to be taken in the new scenario framework for constructing a new scenario set, how these steps relate to the various components and in which of the examples these steps and components are illustrated.

### 20.1 A frequency domain VAR scenario framework

This first section of this chapter formally describes a new framework for generating and analyzing economic scenarios. This new framework is more of a combination of various model components than one detailed model specification. As already noted in section 18.1.3, the new framework is therefore not meant to result in a fully automated procedure that results in a set of economic scenarios with the desired specifications by simply running some computer program. Instead, the new framework provides a set of optimal tools that can be used by experts to construct high quality economic scenarios according to their needs. The framework especially offers a flexible scenario modeling environment in which the complex behavior of economic variables can be adequately described and analyzed. At some points in the framework, difficult choices have to be made. This is especially difficult because in most conventional scenario models these choices are already implicitly made once the model has been chosen. In that case however, the scenario properties related to these choices, for example the variance and correlations of the long term trends of economic variables, are difficult to adjust and sometimes even unknown. Obviously this does not contribute very much to generating scenarios based on a “clear set of assumptions” from the scenario definition of Bunn and Salo (1993) in section 1.3. That some difficult, but not impossible, choices need to be made in the new framework is caused by the simple fact that the behavior of economic variables is extremely complex. The models from the framework merely focus the attention on the various elements of this complex behavior. The sections 20.1.1 through 20.1.7

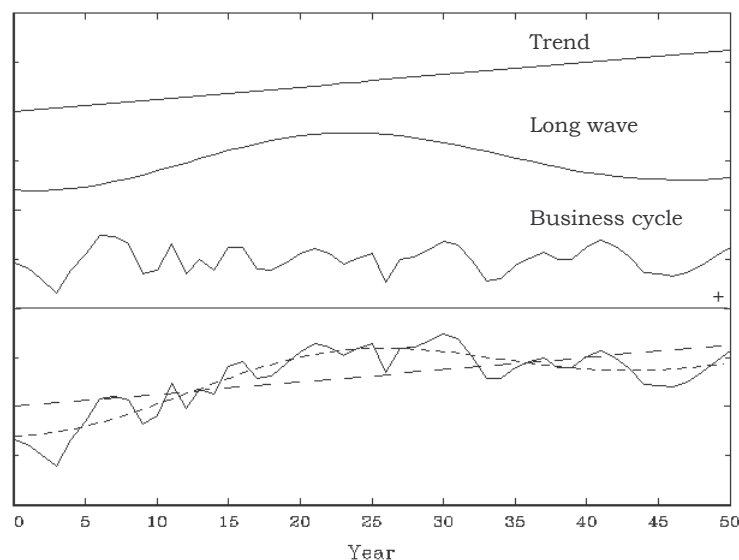
describe the various components of the new scenario framework and the shortcomings of the current models they are meant to resolve. Where possible, the relevant theory is contained in Part II to which the appropriate references will be made. Finally, section 20.1.8 presents an overview of the consecutive steps that need to be taken to construct a scenario set according to the newly proposed scenario framework and how these steps are related to the various components and examples. This overview can serve as a reference for studying the workings of the new framework.

### 20.1.1 Multifactor VAR (MVAR) models

The first important and most fundamental component of the new framework are what we will call *Multifactor VAR (MVAR) models*. The basic idea of these models is that they do not use one stochastic factor to describe the evolution of a variable on the entire frequency range  $[0, 0.5]$ , as is done in conventional applications of VAR and many other types of models, but several stochastic factors, each for a different frequency range, which together cover all frequencies. This idea can be put into practice by first filtering historical time series into various non-overlapping frequency components using the *zero phase frequency filter* described in section 5.4 with the required pass-bands. Next, a separate VAR model is estimated for each set of identically filtered components of variables. Finally, for simulation or predicting purposes, the individual VAR models, each describing a distinct set of factors for a specific frequency range, can be modeled separately and added up to obtain the total model. This simple separate modeling and addition rule is possible because of the fact that in case of applying ideal filters with *non-overlapping* pass-bands, the resulting filtered components of the time series are *uncorrelated* in the time domain and can simply be added to reconstruct the original time series, hence it is an *additive* decomposition. The zero correlation property is the result of the continuous analogue of the orthogonal property (4.1.7) of cosine functions. For this fundamental frequency domain property, also see the end of section 5.4. Note that the same frequency components from different variables may very well be correlated and modeled accordingly. Just as with the empirical research in Part III, the filtering needs to be done with perfect pass-bands and without phase shifts to leave the fundamental behavior of the time series undisturbed and to actually realize a zero correlation between the resulting filtered time series for the different frequency ranges which justifies to construct a separate model for each frequency range. In case that different underlying time series are used for different frequency ranges, the zero correlation property will no longer hold exactly but will most of the times still hold approximately. For more information on the filtering process also see sections 6.3.1 and 6.3.2. Note that the principle of first filtering time series and then constructing separate models for each filtered component, is not restricted to the use of VAR models alone. Any other type of model can be used to model the filtered components, even a mixture of different models for different components can be used.

As an illustration of the fundamental mechanism of MVAR models, consider Figure 20.1. This shows one simulated scenario from a trend model, a long wave model and a business cycle model for the (natural) logarithm of a National Product index. The trend model describes the ultra low frequency fluctuations, hence the trends. The long wave model describes the low frequency fluctuations, hence the long waves with period lengths of several decades. Finally, the business cycle model describes the middle and high frequency fluctuations, hence the business cycles with a maximum period length of 15 years. The three models and hence there simulations are uncorrelated. The total National Product scenario is then simply obtained by adding the three separate simulations.

Figure 20.1 Illustration of simulation from an MVAR model with separate models for the trend (ultra low frequencies), long waves (low frequencies) and business cycles (middle to high frequencies).



The potential benefits of MVAR models can perhaps best be illustrated by the following extreme example. Suppose one wants to simulate realistic scenarios of variables with an horizon of several decades but on a daily basis. Such a model should give a good description of both the long wave type fluctuations, the business cycle type fluctuations, the seasonal fluctuations and the daily fluctuations, each with their very specific properties. Obviously this can never be achieved by using one single VAR model estimated on for example a daily sample of data. With MVAR models however this is exactly what is possible. Important properties of MVAR models that cannot be modeled by conventional VAR models are the following.

#### *Separate properties per frequency range*

Because of the separate models for the various components of the variables, the properties of these components can be analyzed and modeled separately. It is possible for example to change the volatility of low frequency, long wave type, fluctuations while leaving the model behavior at the high frequency, business cycle type, fluctuations unchanged. Besides the simple variance, also the dynamic properties in the various frequency ranges can be modeled separately. These

dynamic properties need to be analyzed and set as much as possible in the frequency domain in terms of their spectral densities (instead of in terms of the conventional autocorrelation matrices) to ensure a clear understanding and better awareness of the final model behavior.

#### *Better estimation per frequency range*

Because of the separate estimation per frequency range, problems of poor quality estimates in one frequency range because of the dominance of some other frequency range, while in another data representation the former may very well be the dominant frequency range, are strongly reduced. What matters here is that, because of variable transformations, the importance of frequencies in describing the total variance can change. Therefore the dynamics at all frequencies are in principle equally important. As an example think of modeling an interest rate variable. For the long term uncertainty and total variance of the level of the interest rate, the *low* frequency fluctuations may be the most important. Hence, these low frequency components will dominate (OLS) estimates on the historical time series. However for the year to year changes of the interest rate, the *high* frequency fluctuations will dominate the total variance. For this also see the Power Transfer Function of the first order differencing transformation in section 5.2.1. The annual changes of interest rates must also be modeled correctly if they are used for example for the modeling of market value returns on a bond portfolio. In short, different frequencies may be important for different applications of one and the same model. Therefore all frequencies need to be modeled correctly.

#### *Different sample and observation frequencies*

The MVAR models allow for using different samples and observation frequencies for various frequency ranges. This is best illustrated by the following example. Consider an MVAR model consisting of three layers of models, each for a separate frequency range: long term fluctuations, business cycle fluctuations and seasonal fluctuations. For the long term fluctuations model, comprising period lengths of several decades, it is desirable, if not necessary, to use very long term data with for example a five year observation frequency, for example for the entire 20<sup>th</sup> century or even longer. For the business cycle fluctuations one may want to use annual data for the postwar period. Finally, the model for the seasonal fluctuations may be estimated on monthly data for the last couple of years. In this way, also different sample periods, or even countries, may be chosen that are considered to be the most representative or to provide the best quality data for the model behavior in specific frequency ranges.

#### *Factor dependencies*

Because appropriately filtered frequency components of a variable are by definition uncorrelated, thereby allowing for the simple addition of the VAR models for these components, it may come as a surprise that some complex relations between the frequencies can actually only be modeled by using an MVAR approach. An example of such *factor dependencies* is the level effect observed in the evolution of interest rates. This effect is described in sections 12.2.1, 13.2.1 and in stylized facts *SF 40* and *SF 42* in section 17.2.1. This effect describes the dependence of the volatility of fluctuations in the business cycle frequency range on the underlying low frequency level of interest rates. High levels of interest rates imply more volatile interest rates

and vice versa. In an MVAR model such a dependency can easily be modeled by specifying a relation between the variance of the model for the business cycle factor and the level of the underlying low frequency factor. Similar principles can be used to model more complex types of factor dependencies. Think for example of state dependent correlations in a multivariate MVAR model or even state dependent dynamics in terms of autocorrelations. An example of the first are the empirically observed extreme joint co-movements of asset returns while in more normal market circumstances correlations are lower. Instead of using Copula functions as in for example Romano (2002) to describe these complex dependency structures, one might very well use an MVAR model with a level effects in asset correlations.

#### *Awareness of model properties*

MVAR models force the model builder to take an explicit stand on the properties of the model at the various frequencies, ranging from the trends and long term fluctuations all the way up to possibly the daily fluctuations and perhaps even the relations between the properties at the various frequencies. Specifying these properties can be very difficult or sometimes even impossible, especially when this situation is compared to a traditional VAR model estimated on one unfiltered historical time series which models all frequencies at once without having to make all these separate modeling decisions. MVAR models however make it impossible to “ignore” some of the complexities in the behavior of macroeconomic variables. The great benefit of the approach in the context of for example scenario analysis, is that it ensures a much better awareness of the exact properties of the stochastic behavior of the variables described by the model. It therefore contributes to a “clear set of assumptions” from the scenario definition of Bunn and Salo (1993) from section 1.3.

In short, the Multifactor VAR (MVAR) models described here resolve the shortcomings of current VAR model applications in terms of finding number 3, 4, 5 and 8 from section 19.3. Thereby, these MVAR models

- Allow for a separate modeling of properties in different frequency ranges including an explicit modeling of stochastic trends, i.e. the behavior at the ultra low frequencies. These stochastic trends are the topic of the next section.
- Create a better awareness of properties in different frequency ranges.
- Allow for different samples and observation frequencies for different frequency ranges.
- Produce better estimates of properties in different frequency ranges which are in principle all equally important.
- Allow for the modeling of dependencies between the stochastic properties of the model in different frequency ranges.

#### 20.1.2 Stochastic trends

One of the frequency ranges that needs to be modeled in an MVAR model consists of the ultra low frequencies, with a period length between infinity (zero frequency) and say 100 years (frequency 0.01). Within the conventional horizons, these type of fluctuations do not show up as fluctuations at all. For trending, index type, variables such as the (natural) logarithm of a National Product index, the low frequencies determine the trend around which the scenarios fluctuate. For non-index type



variables such as interest rates or unemployment rates, the low frequencies determine the average value around which the scenarios fluctuate. For two reasons these low frequencies require a different treatment in an MVAR model than the other (higher) frequency ranges. The first is that when time series are filtered for the low frequency ranges, the result will typically be a straight horizontal or trending line. This type of time series obviously does not lend itself very well to be modeled by a VAR model. The second reason is that information on the uncertainty of the long term trends in economic variables will typically have to come from other sources than one single filtered time series because this is in fact only one observation of what can happen in the ultra long run. One possibility to model the low frequency range is to simply use a fixed trend or average in scenario simulations. For short horizons this can be an adequate approach. However, for long horizons we now that there is in fact also uncertainty about the long run trends or averages of economic variables. In fact, for long term strategic decision making this may very well be the most fundamental source of uncertainty. In this section, we therefore describe two models that can be used to model the ultra low frequencies in an MVAR model stochastically. The first is a stochastic linear trend for the (natural) logarithm transform of index variables. The second is a stochastic average for level variables. The models can be seen as the counterpart of the conventional stochastic trend models in which the first order differences of economic time series are modeled and simulated. The major difference is however that in the context of the MVAR models, the properties of the stochastic trends, such as volatilities and correlations, can be modeled explicitly and separately from the other frequency ranges. In conventional first order differencing models, the stochastic trend behavior is often an implicit consequence of the model that is chosen. We start with a more in depth discussion, in a bulleted format, of trend modeling issue in the context of MVAR models. Next, we will present the two trend models. Finally, we will give several possibilities for determining the parameters of the trend models in the light of the fundamental data problem involved.

### *Discussion*

With respect to the trend modeling in the context of MVAR models, the following remarks are important.

- In terms of the frequency domain, trend and averages are situated at the ultra low frequencies. A frequency of  $\omega=0$  represents a periodic component with an infinite period length which is the same as a constant average term. A frequency  $\omega=\varepsilon$  with  $\varepsilon$  very small represents a periodic component with an ultra long, but not infinite, period length. This is the same as a linear trend for which the direction is determined by the exact frequency and phase of that periodic component. Because a spectral density describes the decompositions of the *variance* of a stochastic process over a continuum of frequencies, the average value at frequency  $\omega=0$  is not included in a spectral density and should therefore be specified separately. In a VAR model this is done by a combination of the constant vector  $v$  and the parameter matrices  $A_1, \dots, A_p$ .
- There is a fundamental difference between processes that are integrated of order zero,  $I(0)$  processes, and processes that are integrated of order one,  $I(1)$  processes. In the first, the level of variables is modeled stochastically while in the second, the first order differences of variables are modeled stochastically. These two types of



stochastic processes produce very different confidence intervals. The simplest example of an I(0) process is

$$x_t = \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim N(0, \sigma^2) \quad (20.1.1)$$

The conditional variance for every horizon  $h$  is

$$V(x_{t+h}) = E(x_{t+h}^2) = \sigma^2 \quad (20.1.2)$$

The simplest example of an I(1) process is the random walk model

$$x_t - x_{t-1} = \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim N(0, \sigma^2) \quad (20.1.3)$$

which by recursive substitution results in

$$x_{t+h} = x_t + \sum_{i=1}^h \varepsilon_{t+i} \quad (20.1.4)$$

Assuming a fixed value for  $x_t$  results in a conditional variance of

$$V(x_{t+h}) = E(x_{t+h}^2) = \sum_{i=1}^h \sigma^2 = h\sigma^2 \quad (20.1.5)$$

The conditional variances (20.1.2) and (20.1.5) show the fundamental difference between an I(0) and an I(1) process with respect to the confidence intervals. For an I(0) process the variance is constant for all horizons while for an I(1) process the variance increases linear with the horizon  $h$  resulting in the famous “square root of  $T$  rule” for the standard deviation of a random walk process because the standard deviation from (20.1.5) is  $\sqrt{h}\sigma$ . Because of the constant widening of the confidence intervals for longer horizons, I(1) processes are said to have a *stochastic trend*.

- For some reason we are educated with the idea that the uncertainty in index variables must increase with the horizon. However, a trend stationary model for an index variable is fundamentally no different than a stationary model for an interest rate variable around some fixed expected value which is in fact nothing more than a flat trend. This type of interest rate model is well accepted because we believe interest rates are pulled back by economic forces towards some mean level to prevent interest rates from attaining unrealistically high or unrealistically low (negative) levels. From this perspective, it is strange that the trend stationary model for an index variable is often found to be unrealistic.
- From a frequency domain perspective, a single historical time series gives us only one “observation” for the long term average or trend value of the time series as determined by the ultra low frequencies. This, contrary to higher frequency fluctuations which can complete several cycles within the sample period and therefore provide us with several observations. Despite the single observation, the historical average or trend value of a time series might very well have been

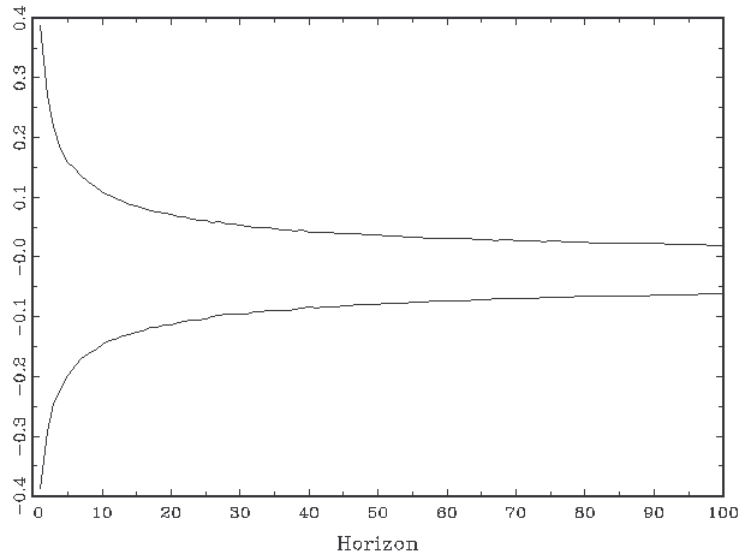
different and is therefore of a stochastic nature, just as the higher frequency fluctuations. In stochastic trend models this uncertainty needs to be modeled, in the end also by choosing appropriate parameters of the models. At the end of this section we will give several possibilities for choosing these parameters.

- One reason for using the following stochastic trend models in the context of MVAR models instead of conventional stochastic trend models in terms of modeling the first order differences of time series, is that the latter models do not allow a explicit modeling of the stochastic trends, in terms of volatilities and correlations. In the first order differencing models, the stochastic trend behavior is often an implicit consequence of the model that is chosen and the implications for the long term uncertainty are not always well known. This can be illustrated by the following simple example. We simulated 10.000 scenarios of annual returns from a Normal distribution with an expected value of 0% and a standard deviation of 20%. For horizons up to 100 years we then calculated the geometrical average return during each of the scenarios over the relevant horizon. Finally, we calculated the 2.5% and 97.5% percentile for each horizon, and hence the 95% confidence interval, for the 10.000 resulting geometrical average returns. For the 1 year horizon, the average return is equal to the first year return and therefore the 95% confidence interval is enclosed by the familiar plus and minus  $1.96 \times \sigma$  value which in this case is 39.2%. The results for other horizons are shown in Table 20.1 and Figure 20.2. The long term geometrical average return from a  $N(\mu, \sigma^2)$  distributions is  $\mu - 0.5 \times \sigma^2$  and in this case equal to  $-2.0\%$ . As the horizon increases, the distribution indeed converges to this number. However, even for an extremely long horizon of 100 years, the 95% confidence interval still ranges from a low average return of  $-6.2\%$  until a high average return of  $+2.0\%$ . For a more realistic horizon in scenario applications of say 10 years, this range is even  $-14.6\%$  until  $10.7\%$ . This shows that what at first sight may seem a pretty clear modeling assumption in terms of stochastic trends, an expected return of 0% with a standard deviation of 20%, in fact still leaves a considerable amount of uncertainty about the actual low frequency trending behavior in the scenarios of which one is often not well aware and which cannot be controlled explicitly.

Table 20.1 Symmetrical 95% confidence interval of geometrical average returns on an annual basis of a random walk model with  $N(0, 20\%)$  distributed annual returns for various horizons, based on 10.000 simulations.

<i>Horizon</i>	<i>2.5% percentile</i>	<i>97.5% percentile</i>
5	-19.9%	15.8%
10	-14.6%	10.7%
20	-11.3%	7.1%
50	-7.9%	3.7%
100	-6.2%	2.0%

Figure 20.2 Symmetrical 95% confidence interval of geometrical average returns on an annual basis of a random walk model with  $N(0,20\%)$  distributed annual returns for horizons up to 100 years, based on 10.000 simulations.



#### *Stochastic logarithmic linear trend*

The previous discussion on the one hands tells us that using a trend stationary MVAR model might not be that strange at all. On the other hand we see that changing the trend stationarity assumption into a stochastic trend can have an enormous impact on the wideness of future confidence intervals. Furthermore, a linear trend is of course also uncertain in itself. Therefore an MVAR model with a stochastic but linear trend is presented here. This model places us for the unpleasant but necessary task to specify the uncertainty in the long term trend. The basic idea is to replace the linear trend with a fixed annual growth by a linear trend for which the growth is stochastic in terms of the scenarios but within a scenario identical for each time period. For scenario  $i$ , starting time period  $t$  and horizon  $h$ , such a linear trend for the logarithm of an index variable can be specified as

$$x_{i,t+h} = x_t + \varepsilon_i \cdot h \quad \text{with} \quad \varepsilon_i \sim N(\mu, \sigma^2) \quad (20.1.6)$$

where  $\varepsilon_i$  is the growth in scenario  $i$  which has expected value  $\mu$  and variance  $\sigma^2$ . Assuming a fixed starting value  $x_t$ , the conditional expected value of this trend for each horizon is

$$E(x_{i,t+h}) = x_t + \mu \cdot h \quad (20.1.7)$$

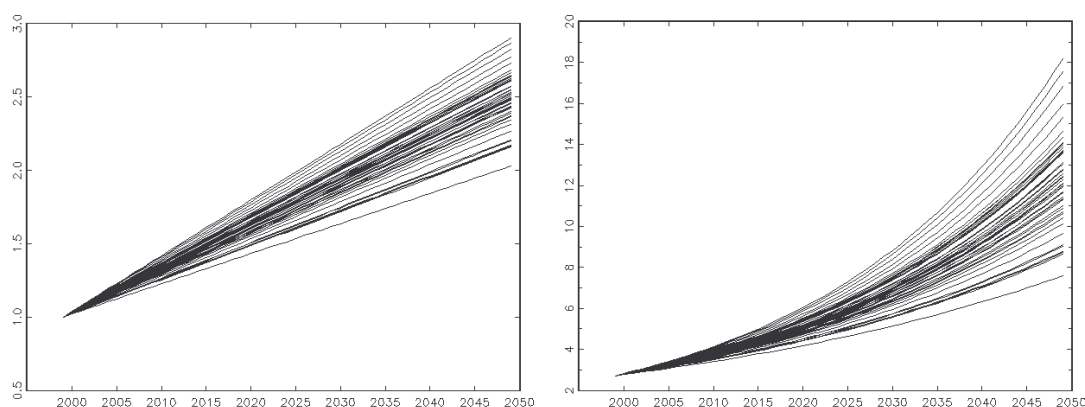
with a conditional variance that increases quadratic with the horizon as

$$V(x_{i,t+h}) = h^2 V(\varepsilon_i) = h^2 \sigma^2 \quad (20.1.8)$$

Hence, the standard deviation increases linear with the horizon as  $h\sigma$ . If it is assumed that this stochastic trend is uncorrelated with the other frequency ranges, it can be simulated as just an extra component of an MVAR model for the ultra low

frequencies. The expressions for the expected value (20.1.7) and variance (20.1.8) can be applied in a straightforward fashion in the analytical construction of the confidence intervals of MVAR models as described in section 20.1.3. Finally, as an example, consider this stochastic trend model with  $\mu=2.7\%$  and  $\sigma=0.4\%$ . This means that 95% of the long term growth rates is approximately between 1.9% and 3.5%. Figure 20.3 illustrates trend scenarios from this model, both for the logarithm of the index and for the original level of the index. The latter has an exponential trend.

Figure 20.3 Example simulations of a logarithmic linear trend model (20.1.6) with an average growth of 2.7% and a standard deviation of 0.4% per annum. Left in terms of the logarithm of the index and right in terms of the original levels of the index.



### Stochastic AR(1) trend

As the non-index variable counterpart of the stochastic logarithmic linear trend for index variables, consider the following simple stochastic AR(1) trend model. For scenario  $i$ , starting time period  $t$  and horizon  $h$ , such an AR(1) trend for a non-index variable can be specified as

$$x_{i,t+h} = \varepsilon_i + \beta x_{i,t+h-1} \quad \text{with} \quad \mu_i = \frac{\varepsilon_i}{1-\beta} \sim N(\mu, \sigma^2) \quad (20.1.9)$$

Under the condition  $|\beta| < 1$ , the long term value of  $x_{i,t+h}$  in scenario  $i$  is  $\mu_i$ . Because here the constant term of the AR(1) model is stochastic, this long term value is stochastic and is therefore different for every scenario but fixed during a specific scenario. Assuming a fixed starting value  $x_t$ , recursive substitution yields

$$\begin{aligned} x_{i,t+1} &= (1-\beta)\mu_i + \beta x_{i,t} \\ x_{i,t+2} &= (1-\beta)\mu_i + \beta(1-\beta)\mu_i + \beta^2 x_{i,t} \\ x_{i,t+3} &= (1-\beta)\mu_i + \beta(1-\beta)\mu_i + \beta^2(1-\beta)\mu_i + \beta^3 x_{i,t} \\ &\vdots \\ x_{i,t+h} &= \left( \sum_{j=0}^{h-1} \beta^j \right) (1-\beta)\mu_i + \beta^h x_{i,t} \end{aligned} \quad (20.1.10)$$

The conditional expected value of this trend for each horizon is therefore

$$\begin{aligned} E(x_{i,t+h}) &= \left( \sum_{j=0}^{h-1} \beta^j \right) (1 - \beta) E(\mu_i) + \beta^t x_{i,t} \\ &= \left( \sum_{j=0}^{h-1} \beta^j \right) (1 - \beta) \mu + \beta^t x_{i,t} \end{aligned} \quad (20.1.11)$$

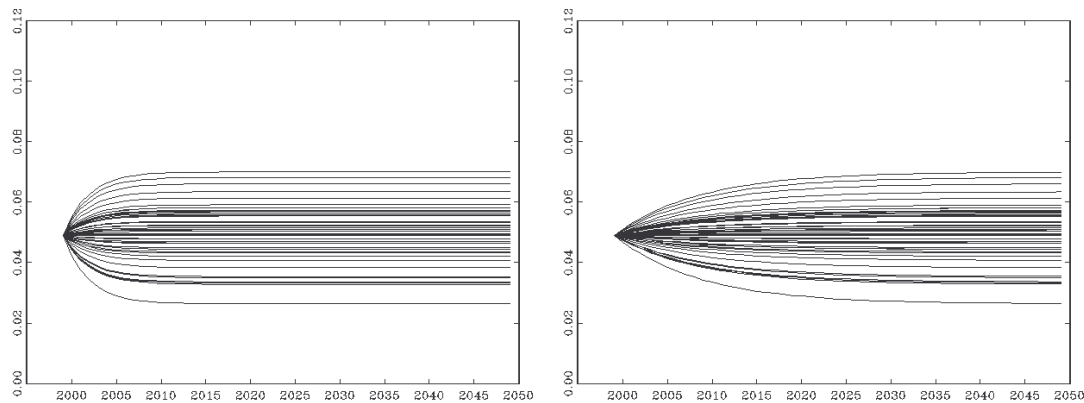
with a conditional variance of

$$\begin{aligned} V(x_{i,t+h}) &= \left( \left( \sum_{j=0}^{h-1} \beta^j \right) (1 - \beta) \right)^2 V(\mu_i) \\ &= \left( \left( \sum_{j=0}^{h-1} \beta^j \right) (1 - \beta) \right)^2 \sigma^2 \end{aligned} \quad (20.1.12)$$

Under the condition  $|\beta| < 1$ , the factor preceding  $\sigma^2$  in (20.1.12) converges to a value of one as the horizon  $h$  increases. Hence, as the horizon increases, the conditional variance first increases and then converges towards the long term variance  $\sigma^2$ . If it is assumed that this stochastic trend is uncorrelated with the other frequency ranges, it can be simulated as just an extra component of an MVAR model for the ultra low frequencies. The expressions for the expected value (20.1.11) and variance (20.1.12) can be applied in a straightforward fashion in the analytical construction of the confidence intervals of MVAR models as described in section 20.1.3. With respect to this stochastic AR(1) trend model, the following remarks are in order.

- It is very different from a conventional AR(1) model in which the values of  $x_{i,t+h}$  for consecutive values of  $t+h$  are stochastic during a scenario, while here the expected value is stochastic in terms of the scenarios  $i$  but determines the entire smooth evolution of  $x_{i,t+h}$  during a single scenario. Furthermore, a conventional AR model describes fluctuations in the entire frequency range  $[0, 0.5]$  while this stochastic AR(1) model only describes ultra low frequency fluctuations.
- The AR structure is only used to produce a smooth transition from the final value of the non-modeled components towards some new long term expected value. A simple alternative would be to assume the same average value for each horizon instead of using such a transition. However, because of the potentially large jumps from the last value from the sample towards the first value from the scenarios, such a model can produce unrealistic scenarios.
- The AR parameter  $\beta$  can be used to change the speed of the transition from the last values from the sample towards the long term averages in the scenarios. Thereby it also influences the shape of the confidence intervals derived from the conditional variance (20.1.12). This can be seen from the sample trend paths in Figure 20.4 which have a horizon of 50 years. The overall long term expected value is  $\mu=5.0\%$ . The uncertainty around this value is  $\sigma=1.0\%$  which means that 95% of the expected values is approximately between 3% and 7%. The left hand side shows a rather fast transition in about 10 years by using  $\beta=0.7$  while the right hand side shows a slow transition in about 30 years by using  $\beta=0.9$ .

Figure 20.4 Sample paths with a horizon of 50 years from the stochastic AR(1) trend model (20.1.9) with  $\mu=5\%$ ,  $\sigma=1.0\%$  and  $\beta=0.7$  (left) and  $\beta=0.9$  (right).



Finally note that, multivariate extensions of both the stochastic logarithmic linear trend and AR(1) trend model are even combinations are straightforward. Besides the expected value and variance for the trend in each variable, also the correlations between the various trend values need to be specified, based on for example cross section information of long term averages and trend values. Once the expected values and this covariance matrix are known, the stochastic trends can be simulated from a multivariate Normal distribution having these first two moments.

#### *Determining trend parameters*

Once a model for the lowest frequencies has been chosen, also values for the parameters of the model that determine its stochastic behavior need to be chosen. In a univariate setting this concerns the overall expected value  $\mu$  and the trend volatility  $\sigma$ . In a multivariate setting also the correlations between the long term trend values need to be chosen. For the long term overall expected value we mention two possibilities:

1. Assume the long term historical averages of representative time series also to hold for the future.
2. Assume forward looking averages, based on for example consensus forecasts, taking into account information about the future that is assumed not to be contained in the historical data.

For the volatility of a stochastic trend model we mention five possibilities. Note that these methods can equally well be applied for determining the correlations between stochastic trends in a multivariate setting.

1. Combine the time series data with cross sectional data (i.e. use panel data) by using long term averages and trend values for the relevant variables from different though comparable countries. The sample standard deviation of such long term averages can be used as an estimate of the uncertainty in the long term expected values. Such cross sectional information can for example be obtained from long term historical time series in Mitchell (1992, 1993, 1995), Maddison (1995), Taylor (2000) and Dimson et al. (2001). Missing data for recent years can be completed from for instance the various publications of the OECD.



2. Use the uncertainty calculated from forecasts given by different forecasting experts which can be obtained from survey results.
3. Use such a volatility that the confidence intervals produced by the conditional distributions of the total MVAR model are consistent with the uncertainty in historical time series. This approach can especially be applied to compensate for the variance of frequency components that are left out of the model and can be implemented by the backtesting procedure described in section 20.1.3.
4. Calculate the volatility that is needed such that the unconditional variance of the total MVAR model is identical to the historical variance of the time series. Just as the third method, this approach can especially be applied to compensate for the variance of frequency components that are left out of the model.
5. Calculate the standard deviation of various long term averages taken from the same sample of data. One possibility is to use averages of sample periods that are as long as the period length of the longest fluctuations that are included in the other component models of the MVAR model. In order to have a sufficient number of observations, the samples need to be overlapping. The possible autocorrelation that is introduced in the consecutive long term averages can be corrected for by using appropriate estimators of the volatility based on unsmoothing techniques.

### 20.1.3 Confidence intervals and backtesting

Besides the use of spectral densities to assess the dynamic properties of the component models of an MVAR model, within the new scenario framework several other tools are available for testing and calibrating the (MVAR) models. The first two of these tools are the familiar confidence intervals and a new backtesting procedure, specifically suited for testing in sample model behavior instead of the out of sample predictive capacities of a model. These are described in this section while two other tools are described in section 20.1.4 (adjusting conditional variances) and section 20.1.7 (adjusting dynamics).

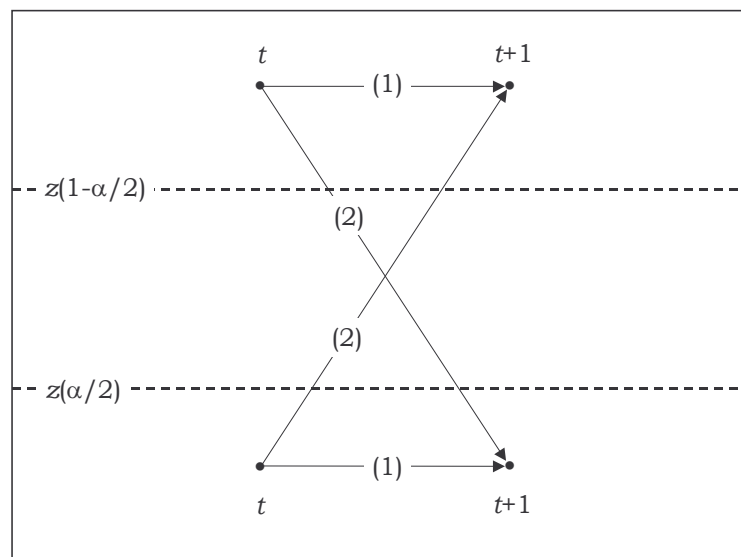
#### *Introduction*

Each of the component models of an MVAR model describes the dynamics in a separate frequency range. These dynamics are represented and controlled by the normalized auto-spectra of the models. A more familiar aspect of model behavior is the total variance for each simulation year. In a sense, the normalized auto-spectra describe how a model dynamically behaves *within* this total variance. In conventional modeling exercises, one is often primarily interested in these so called conditional variances of a model. The following example illustrates that confidence intervals in fact tell very little about the dynamic behavior of a model because they do not measure *how* a variable wanders from period to period through the relevant confidence intervals. For multi-period scenario applications these dynamics are of considerable importance. For an illustrative example in the context of ALM we refer to section 1.2. Figure 20.5 shows that processes with very different dynamics can actually have identical confidence intervals. The dotted lines indicate the boundaries of a  $(1-\alpha)\%$  confidence interval that divide the conditional distributions into three regions. The four points represent two subsequent values for two scenarios. Now suppose that in a first model, the values change during the scenarios according to the paths indicate by (1). Because each value stays with the same region of the



distribution this could be a model with a high positive first order autocorrelation. For a second model, the values change during the scenarios according to the paths indicated by (2). Because each value switches from the one outer region of the distribution to the other outer region this could be a model with a strong negative first order autocorrelation. Although both models clearly describe a very different dynamic behavior, the number of observations in the outer regions of the distributions at the two moments  $t$  and  $t+1$  is the same and hence the confidence intervals of the models will also be identical.

Figure 20.5 Example to illustrate that processes with very different dynamics can have identical confidence intervals.



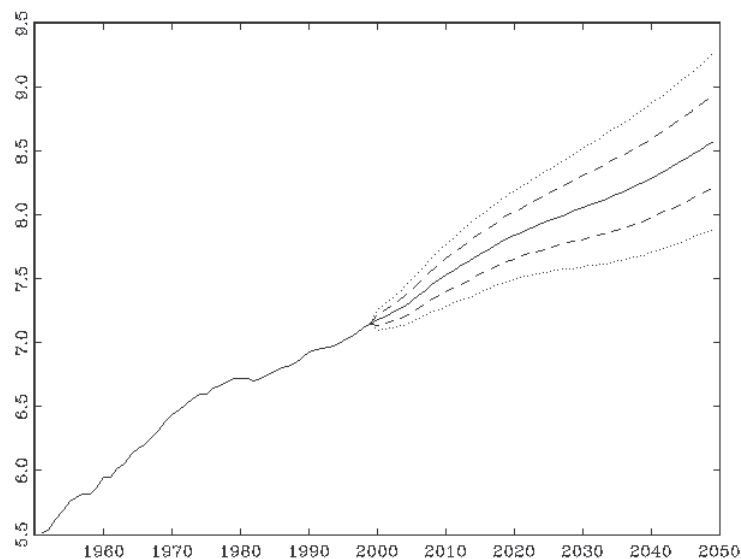
In the remainder of this section we will first show how confidence intervals can be calculated for an MVAR model. Next, we will show how these confidence intervals can be used in a backtesting procedure based on the historical time series data used for estimating a model.

#### *Confidence intervals*

Confidence intervals are mostly defined in terms of percentiles of the conditional variances of a model, that is the variance of simulated scenarios for each future point in time, starting from the end of the available sample data. The conditional variances are a combination of the long term variance of the model and its dynamics in terms of its autocorrelation structure. There are two ways of determining the conditional variances of a model. The first possibility is to calculate the conditional variances directly from the model parameters. If such analytical expressions of the conditional variances are not available due to the complexity of the model, a second possibility is to simulate a large number of scenarios, say 10.000, from the model and calculate a sample variance for each simulated year from these scenarios. Once the conditional variances are known together with the assumed distribution function, the required confidence intervals can easily be calculated.

The confidence intervals of a conventional VAR model can be calculated from the MA representation of the model as described in section 3.6. There, it is also shown how to calculate confidence intervals of transformations of the variables described by the model, such as linear interpolation, integration and differencing. For MVAR models consisting of several “stacked” conventional VAR models for various non-overlapping frequency ranges, only a slight modification of these calculations is required. First, the conditional expectations and conditional variances of the separate models have to be calculated in the usual way. Because of the zero correlation between the models, the individual conditional expectations and variances can then be added to obtain the total conditional expectations and variances for each simulation year. If a Normal distribution is assumed for each component model, then the sum of the models will also have a Normal distribution with the conditional expectations and variances as indicated. Together with percentiles from the standard Normal distribution, these expectation and variances can be used to calculate the desired confidence intervals from (3.6.12). Figure 20.6 shows the 0.1%, 5%, 50% (median), 95% and 99.9% percentile of the conditional distributions of an MVAR model for the logarithm of the National Product Index for the Netherlands with an horizon of 50 years (see section 20.2.4). The percentiles are shown together with a part of the sample time series. The 0.1% and 99.9% percentiles represent the minimum and maximum values of the distributions. Because of the thin tails of the Normal distribution, depending on the number of scenarios, the actual use of the model may result in a wider interval. In case of a 1000 scenarios, the minimum and maximum are approximately the chosen 0.1% and 99.9% percentiles. In some applications, it is not possible to calculate the confidence intervals analytically. When calculating the percentiles numerically on a large number of scenarios in such cases, the 0.1% and 99.9% percentiles can therefore be replaced by the minimum and maximum of the simulated values.

Figure 20.6 Example confidence intervals for conditional annual distributions of an MVAR model of logarithm of the National Product Index for the Netherlands. Shown are a part of the sample time series together with the 0.1%, 5%, 50%, 95% and 99.9% percentiles of scenarios with an horizon of 50 years. That is, the median together with the 90% and 99.8% confidence intervals.



*Backtesting*

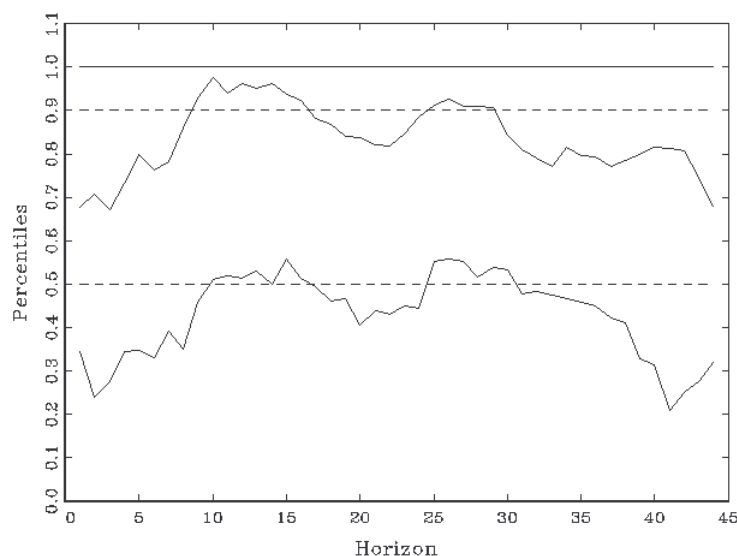
In case both the dynamics (autocorrelations) and the unconditional variance of an economic variable are modeled correctly, the conditional variances and hence the confidence intervals will automatically be consistent with the observed historical behavior of the variable. For two reasons it can be useful to have a procedure to formally test whether the confidence intervals of a model are indeed consistent with the observed historical uncertainty in the time series used for the estimation of the model. The first is when estimation procedures are used that do not automatically guarantee consistency in terms of variance and autocorrelations. This will hold for OLS type of procedures but to a lesser extent also for Yule-Walker estimation methods in case an inappropriate model order is used. The second reason why such a testing procedure can be useful is if one for some reason chooses not to model all frequencies of a time series in an MVAR model but does want to maintain the same conditional variances. The idea is in that case to replace the part of the variance caused by the omitted frequency ranges by extra variance at the frequency ranges that are contained in the model. This will widen the confidence intervals of the model to compensate for the variance caused by the non-modeled (typically low frequency) fluctuations. What is needed in both situations is a tool to determine to what extent the conditional variance of an MVAR model gives an adequate description of the historical stochastic behavior. The following backtesting procedure is in fact a rather severe test for a scenario model in terms of model specification and estimation that can be applied for a much broader class of models than MVAR models alone. What matters is that some model is meant to show stochastic behavior consistent with some historical time series. It is important to note that, as illustrated by the example in Figure 20.5, conditional variances and hence the following backtesting procedure do *not* give all the information about the dynamics of a process through time. Hence, the backtesting procedure should always be accompanied by an assessment of the dynamics of the model, preferably in terms of its (multivariate) spectral densities. The proposed backtesting consists of the following steps.

1. Start by estimating the model. For each component model of an MVAR model this is a one time estimation on the entire associated sample. The estimates are not of a rolling window type because the objective is to have a single scenario model that describes the dynamics of the variables consistent with their empirical behavior during the sample. Rolling window estimates, without using future information, should be used when the objective is to assess the quality of predictions from the model. Here, it is fair to use information from the entire sample because of the scenario application of the model.
2. Based on the estimated model, then construct chosen percentiles of the conditional distributions as those in Figure 20.6, with a chosen horizon, starting from every point in the sample. If the objective is not to compensate variance in omitted frequency ranges, frequency components of the time series that are not explicitly modeled should be taken as given to ensure a fair evaluation of the model. In that case, models should not be judged on information that was not used for estimating them. For example use the 5%, 25%, 75% and 95% percentiles, thereby constituting symmetrical 50% and 90% confidence intervals, and a horizon of 50 years. Asymmetrical confidence interval can be used as well.

3. Running over all starting points and horizons, then determine the overall percentage of the realizations of the time series that falls in the different confidence intervals. Because there is only a single realized path of the time series from each point in the sample, an absolute representation of the true historical conditional distributions is not available. By using the relative percentile approach, results from all points in the sample can be combined. When determining the percentages, skip all cases in which a missing value is encountered. Furthermore, to ensure stable estimates of the percentages, apply a minimum number of total observations for each horizon. Use for example a minimum number of 50 observations.
4. Plot the realized percentages of observations for each horizon in the different confidence intervals together with the theoretical percentages that are to be expected based on the construction of the confidence intervals.

Figure 20.7 shows an example of backtesting results obtained in this way for an MVAR model of the logarithm of the National Product Index for the Netherlands when tested on the 1870-1999 period in terms of the symmetrical 50% and 90% confidence intervals. From these results we can for example see that both for a horizon up to 10 years and a horizon longer than 30 years, the model describes a too small conditional variance. In these cases, for example only 70% of the realizations fall within what should be the 90% confidence intervals according to the model. Not 10%, but 30% of the realizations fall outside these confidence intervals, thereby indicating that the true conditional distributions are wider than the ones from the model.

Figure 20.7 Example backtesting results in terms of realized and expected percentages of observations in symmetrical 50% and 90% confidence intervals of conditional distributions with various horizons for an MVAR model of the logarithm of the National Product Index for the Netherlands tested on the 1870-1999 period.



#### 20.1.4 Adjusting conditional variances

Following the confidence intervals and backtesting procedure from the previous section, a third tool available within the new scenario framework for testing and calibrating the (MVAR) models is the following. The backtesting procedure might indicate too low or too high conditional variances for some horizons. It might also be the case that, based on forward looking information, one wants to work with different conditional variances than the historical ones. A unique feature of MVAR models that can be used in those cases is that because it is possible to change the (relative) variance for separate frequency ranges, it is also possible to change the conditional variances for different horizons while leaving those for other horizons largely intact. Changing the variance of one of the component VAR models without changing its dynamic properties can be done by means of the procedure described in section 3.7.5 which is based on the Yule-Walker equations. Note that, for conventional (VAR) models that describe the entire frequency range by a single stochastic factor, changing the unconditional variance will typically change the conditional variances for all horizons. As a simple example consider the conditional variances (20.1.5) of the random walk model. If the  $\sigma$  parameter is changed in this model, the conditional variances for all horizons will change by the same factor. By having several stochastic factors available, an MVAR model simply offers more flexibility for changing different types of variances.

#### 20.1.5 Truncated VAR (TVAR) models

For the component models for the different frequency ranges of an MVAR model, in principle any type of model can be used as long as the models describe and are estimated on the filtered time series for the relevant frequency range. In the context of the present research, these component models will in principle be of the conventional VAR model type estimated by conventional estimation procedures. However, within the new scenario framework also two other new types of VAR models can be used in order to solve some of the specific shortcomings found in the previous chapter and summarized in section 19.3. In many scenario applications it is desirable to prevent variables, or transformation of variables, from attaining values above or below some specific level. Think of interest rates or the relative changes of a consumer price index (i.e. inflation rates) which should or could be wanted to stay strictly positive. One of the shortcoming is that with conventional VAR models this is difficult to achieve, especially in case of low expected values and high volatilities. The underlying problem are the thin tails of the often assumed Normal distribution. One simple way to prevent scenarios from a VAR model to fall below some critical level is to first simulate all values and then simply replace all simulated values that have fallen below the critical level by the critical level itself. This approach is followed in some of the conventional applications of VAR models for generating scenarios as described in section 18.2.1. From section 3.8 we know however that this approach can lead to unrealistic probability distributions and that it changes the moments of the original distribution such as expected values, standard deviations and correlations. An alternative is to use *Truncated VAR (TVAR) models* as described in detail in section 3.8.3. The distribution of these models is truncate at some  $\alpha\%$

percentile of the original distribution without changing the first two moments (expected value and unconditional variance) of the model. Both the truncating and the fixing of the moments can also be done in terms of transformations of the original model, for example for the first order differences.

### *Effect on dynamics*

A rightful question is to what extent the dynamic behavior of the model is changed by using a TVAR instead of a symmetrical VAR model? Because in a TVAR model the scenarios are simulated by resampling from the original model, one would expect the effects to be only minor. That this is indeed the case, especially for “mild” truncations, can be seen from the following univariate example. An easy way to check the effect on the dynamics is by examining the effect on the auto-spectrum of a TVAR model. Figure 20.8 shows the normalized auto-spectrum of a conventional univariate AR(7) model estimated on the 1949-1999 sample for the filtered  $[1/14, 0.5]$  (business cycle) component of the natural logarithm of the Consumer Price Index for the Netherlands with a conventional one year observation frequency together with the auto-spectrum of its truncated version. The truncation has been done on the 5% percentile of the original model, in such a way that the first two moments (expected value and unconditional variance) of the model have not changed. How this can be accomplished is described in detail in section 3.8.3. Because both models have the same unconditional standard deviation it makes no difference whether we compare normalized or non-normalized auto-spectra. The spectra have been estimated by generating a single 1000 year simulation for each model and then (re-)estimating the AR(7) model by means of the Yule-Walker procedure. The properties of the spectra are shown in Table 20.2.

Figure 20.8 Example auto-spectra of original AR(7) model and Truncated AR(7) model of the  $[1/14, 0.5]$  component the logarithm of the Consumer Price Index for the Netherlands when truncating at the 5% percentile of the original distribution without changing the first two moments of the model.

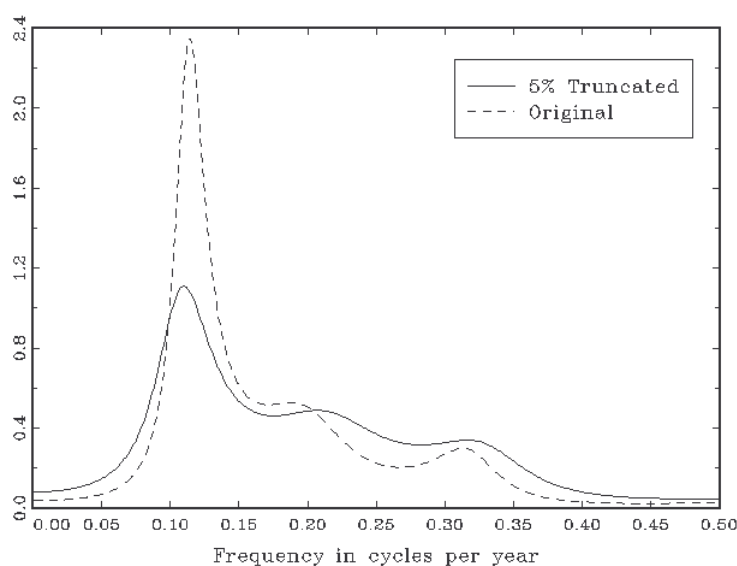




Table 20.2 Statistics of the auto-spectra of original AR(7) model (left) and Truncated AR(7) model (right) of the [1/14, 0.5] component the logarithm of the Consumer Price Index for the Netherlands when truncating at the 5% percentile of the original distribution without changing the first two moments of the model.

<i>Original AR(7) model</i>					<i>Truncated AR(7) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>		<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1000 obs.	0.11	8.88	0.34	0.92	1000 obs.	0.11	9.40	0.33	0.86
	0.20	4.99	0.24	0.78		0.21	4.66	0.30	0.72
	0.32	3.13	0.12	0.83		0.33	3.01	0.17	0.75

From these results the following can be observed:

- The peak frequencies and peak power of the auto-spectra remain practically unchanged by truncating the distribution of the model.
- However, the modulus of the complex roots are significantly reduced, causing a more smooth auto-spectrum. The pseudo-periodic behavior of the original model is therefore somewhat dampened by truncating the distribution of the model.
- In general however, it is fair to say that the dynamic behavior of the model is only mildly changed by using the truncated version. In general, the effects will be smaller as smaller values of  $\alpha$  are used for determining the truncation levels.

#### *Truncated MVAR models*

With respect to the application of TVAR models in an MVAR model the following remarks are in order.

- If the objective is to truncate the total distribution of the model, it is most logical to truncate all component models. In case all models are truncated at say the 5% percentile of the original models, the effect on the total distribution will be a truncation at a lower percentile, say at the 2% percentile. The reason for this at first sight perhaps unexpected effect is that truncating the separate distributions of the component models is in fact less strict than truncating the distribution of the sum of the models. Because of the zero correlation between the models, the distribution of the sum has a lower standard deviation than the sum of the standard deviations of the distributions of the separate models.
- A first possibility to remedy this “diversification” effect in truncated MVAR models is to truncate the individual models at a higher percentile, for example at the  $\alpha=10\%$  percentile, in case a final truncation at the  $\alpha=5\%$  percentile is the objective.
- A second possibility is to directly truncate the required distribution at some desired level instead of at some percentile of the original distributions of the separate models. For an MVAR model this requires a choice about which models to resample from in case the truncation level is violated. The most consistent seems to be to resample from all component models at the same time. The required  $\beta_\mu$  and  $\beta_\sigma$  adjustments from section 3.8 that are needed to maintain the required first two moments can be determined by trial and error or some automated search procedure.
- If stochastic trends are modeled, as described in section 20.1.2, the associated distributions need to be truncated as well. For this it suffices to truncate the underlying distributions as described in section 3.8.1 and 3.8.2.



In short, the Truncated VAR (TVAR) models resolve shortcoming number 6 of current VAR model applications as described in section 19.3. Thereby, these TVAR models

- Allow for the modeling of truncated distributions, thereby preventing the variables described by the models, or transformations of these variables, from attaining values below or above some predefined levels.

#### 20.1.6 EigenValue Restricted VAR (EVR VAR) models

Besides the MVAR and TVAR models from the previous sections, a third important new type of VAR model that is part of the new scenario framework is the class of *Eigenvalue Restricted VAR* (EVR VAR) models as described in detail in section 3.7. These models can be used as one of the component models of an MVAR model for describing the stochastic behavior in a specific frequency range. An important result from the testing of existing VAR models in the previous chapter as summarized in section 19.3 is that multivariate models often give a poor description of the univariate dynamics of the separate variables from the model. In case of a low model order, and hence a low number of eigenvalues, unconstrained models prefer to use the available eigenvalues for “averaging” across the univariate spectral densities by using spectral peaks at intermediate frequencies with a low modulus. In case of a high model order, and hence a high number of eigenvalues, unconstrained models have (more than) enough flexibility to fit each individual univariate spectral density. However, partially because of feed-across effects as described at the end of section 4.7.7, the auto-spectra from high order multivariate models are too erratic and thereby blur the “true” underlying auto-spectra. EVR VAR models can be used to restrict the eigenvalues of a VAR model to specific values. These restrictions can either force the dynamics of a low order VAR model in a desired direction or prevent a high order VAR model from having too many different eigenvalues for each variable while maintaining the required flexibility in terms of the eigenvectors.

In short, the EigenValue Restricted VAR (EVR VAR) models resolve the shortcomings of current VAR model applications in terms of finding number 7, 10, 11, 12 and 13 from section 19.3. By using EVR VAR models, we are able to

- Fix the eigenvalues of a VAR model at some pre-specified values and estimate the optimal corresponding eigenvectors. This can be done either to enforce a certain type of dynamics on a low order VAR model or to reduce the dynamics of a high order VAR model.
- Apply restrictions on the (eigenvalue and eigenvector) parameters of the model that directly determine the model dynamics instead of on the conventional parameters.

### 20.1.7 Adjusting dynamics

Following the confidence intervals and backtesting procedure from section 20.1.3 and the adjustment of conditional variances from section 20.1.4, a fourth and final tool available within the new scenario framework for testing and calibrating the (MVAR) models is the following. In section 20.1.3, by means of the example in Figure 20.5, it was illustrated and stressed that conditional variances do *not* give all the information about the dynamics of a process. They do not take into account how the scenarios from a model behave in a dynamic sense within the conditional variances. This section therefore describes how the dynamics of any type of VAR model, also the component models of an MVAR model, can be controlled without changing the (unconditional) volatilities. This mechanism can be used for example to fine-tune the dynamic model behavior or to deliberately let this behavior be different from the observed historical behavior in order to incorporate forward looking information or to perform a sensitivity analysis. The theory and examples in section 3.3.2 and section 4.5.3 showed how the eigenvalues and eigenvectors of a VAR model are linked to its fundamental dynamic behavior and hence its spectral densities. These relations also form the foundation for the EVR VAR models from the previous section. Furthermore however, they offer the possibility to directly influenced the dynamic behavior of the model by changing the fundamental eigenvalue and eigenvector parameters of the model. It is the most intuitively appealing if the eigenvalues and eigenvectors are defined in a polar instead of a cartesian format. In that case, the procedure for changing the dynamics of a VAR model works as follows.

The eigenvalues of a general  $n$  dimensional VAR( $p$ ) model in terms of polar coordinates are defined by the modulus  $R_i$  and frequency  $\omega_i$  (cycles per year) for  $i=1, \dots, np$ . For real eigenvalues  $\omega_i=0$ . The eigenvectors are defined by the modulus  $\Delta_{j,i}$  and phase  $\phi_{j,i}$  (lead in years) for  $j=1, \dots, np$  and  $i=1, \dots, np$ . Once the relevant parameters are changed according to the desired specifications, based on (A.1.8) and (3.3.11), the eigenvalues  $\lambda_i$  and eigenvectors  $v_i$  can be calculated as

$$\lambda_i = R_i \exp(i\omega_i 2\pi) = R_i (\cos(\omega_i 2\pi) + i \sin(\omega_i 2\pi)) \quad (20.1.13)$$

and

$$v_i = \begin{bmatrix} \Delta_{1,i} \exp(i\phi_{1,i}\omega_i 2\pi) \\ \vdots \\ \Delta_{np,i} \exp(i\phi_{np,i}\omega_i 2\pi) \end{bmatrix} = \begin{bmatrix} \Delta_{1,i} (\cos(\phi_{1,i}\omega_i 2\pi) + i \sin(\phi_{1,i}\omega_i 2\pi)) \\ \vdots \\ \Delta_{np,i} (\cos(\phi_{np,i}\omega_i 2\pi) + i \sin(\phi_{np,i}\omega_i 2\pi)) \end{bmatrix} \quad (20.1.14)$$

When constructing these eigenvalues and eigenvectors, the normalization of the eigenvectors from (3.7.29) and section 3.3.2 should be taken into account. The special structure (3.3.12) of the eigenvectors for the VAR(1) representation of a VAR( $p$ ) model should be taken into account as well. Together these two special relations make that each  $np$  dimensional eigenvector has in fact only  $n-1$  true parameters. From the eigenvalues and eigenvectors, the  $A_v$  and  $P$  matrix from (3.3.15) can be calculated. Next, (3.3.15) gives the conventional parameters matrices of the VAR( $p$ ) model in terms of the VAR(1) representation defined by (3.2.35). With

respect to this procedure for changing the eigenvalues and eigenvectors of a VAR model, the following remarks are in order.

- Each of the resulting *eigenvectors* cannot be identical to one of the other eigenvectors or some other linear combination of the other eigenvectors. In this case the parameter matrices cannot be calculated because the  $P$  matrix in (3.3.15) has a rank smaller than  $np$  and hence cannot be inverted.
- In principle all the resulting *eigenvalues* can be identical. However, in most cases this will be undesirable because then, as described in section 3.7.5, all the parameter matrices become diagonal matrices and hence the model cannot describe any cross correlations. Choosing a different value for only one of the eigenvalues is already sufficient to remove this restriction.
- From practical experiments we have learned that in case one wants to be able to recalculate the eigenvalues and eigenvectors that were originally set by hand, from the new parameter matrices, one should never use fully identical eigenvalues, even though this is in principle not necessary. This can be implemented by multiplying the modulus  $R_i$  of each next identical eigenvalue by a factor 0.9999. Without this minor adjustment, the normalized eigenvectors that are (re-)calculated from the modified model can be different from the eigenvectors used for constructing the model. Both sets of eigenvectors however result in exactly the same parameter matrices, so the model itself in fact has not changed. Equation (3.3.11) shows that in case of completely identical eigenvalues, the dynamics of a VAR model are described by a sum of identical cosines that are only different in terms of phase and amplitude shifts as defined by the eigenvectors. The  $\Delta_{j,i}$  and  $\phi_{j,i}$  values can then no longer be uniquely attributed to the individual eigenvalues. However, the total dynamics remain unchanged.
- The most special property of the procedure is that can be used to change the dynamics of an existing estimated VAR model by changing the parameters that *directly* determine its dynamics. The  $\omega_i$  parameters determine the period length of the pseudo-periodic fluctuations. The  $R_i$  parameters determine the rate at which these fluctuations are dampened and hence the extent to which these fluctuations show periodic behavior. Finally, the  $\Delta_{j,i}$  and  $\phi_{j,i}$  parameters respectively determine the extent to which the fluctuations of the eigenvalues are present in the individual variables and with what phase shifts. Seen in this way, these parameters are in fact the foundations of the auto-spectra, coherence spectra and phase spectra of the model and hence its dynamics. As stated before, this can be a very useful way of changing the stochastic properties of scenarios generated from a VAR model, for example for performing a sensitivity analysis or to incorporate forward looking information in the scenarios that is not contained in the historic sample period that was used for the estimation of the model. Think for example of examining the effects of a lengthening or shortening of the business cycle or of the business cycle becoming more or less pronounced (periodic). The relations can even be used to model very special types of factor dependencies in an MVAR model as described in section 20.1.1. The business cycle properties could for example become dependent on the level of the underlying long wave fluctuations.
- Indeed it is also possible to change the dynamic behavior of a VAR model by changing the autocovariance matrices of the process by hand and then to apply the Yule-Walker equations given by (3.2.30) and (3.2.31) to find the parameters of

the corresponding VAR model. There are two reasons why applying the approach in terms of the eigenvalues and eigenvectors as described here is to be preferred. The first is that defining the required dynamics in terms of the eigenvalues and eigenvectors is much easier and more transparent than in terms of the autocovariance matrices. The second reason is that, especially in case of high dimensional or high order models, changing the autocovariance matrices of a VAR model often leads to matrices (3.2.28) and (3.4.45) that are not positive definite and thereby imply an invalid stochastic process with a negative variances for some linear combinations of the variables. This results in a failure of the Yule-Walker equations to produce the parameters of the corresponding VAR model. In order to maintain a positive definite process, exactly the right combination of parameters from the autocovariance matrices should be changed at the same time, instead of changing one single parameter at a time. Without the “intervention” of the structure in terms of the eigenvalues and eigenvectors, it is virtually impossible to find the right combinations.

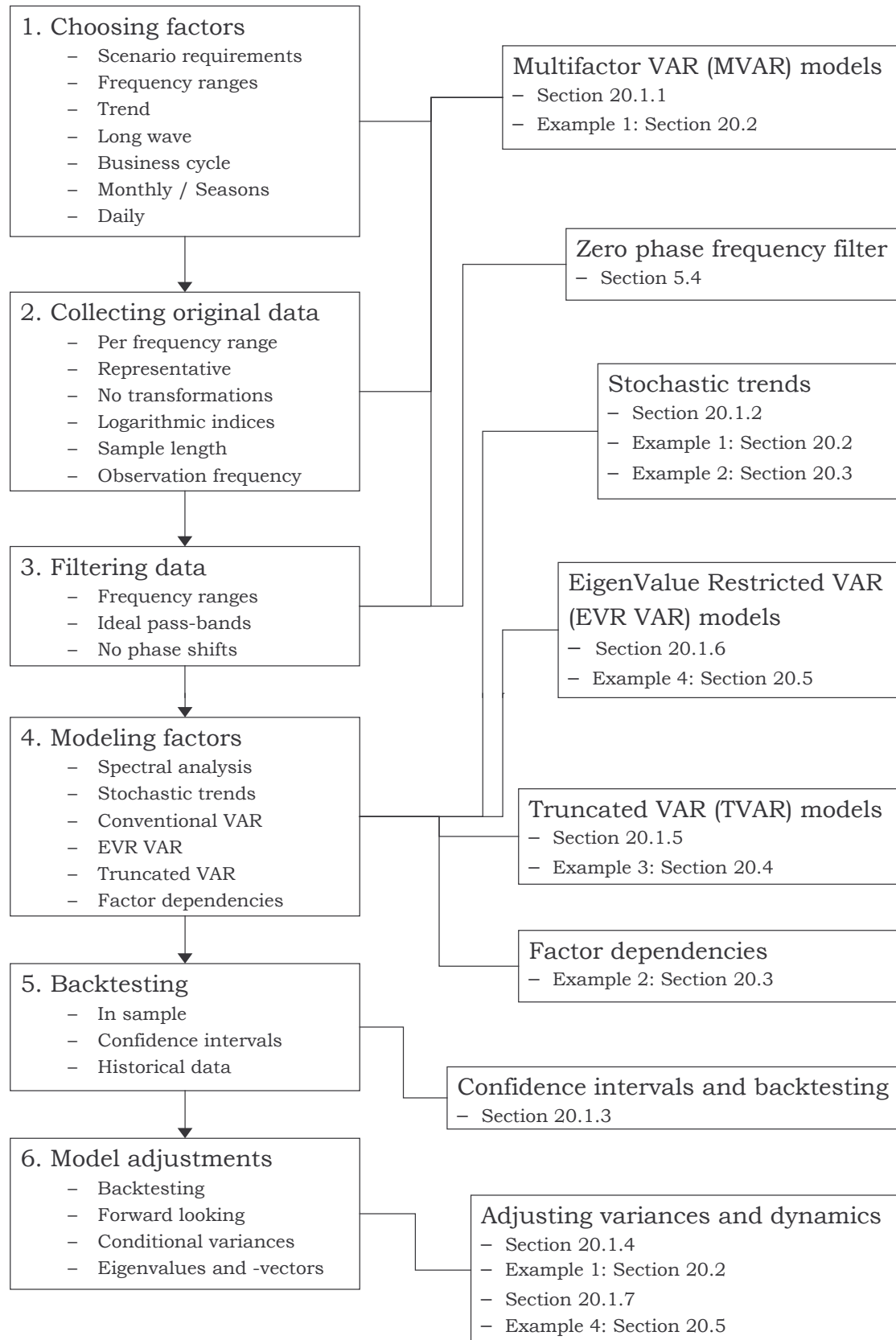
### 20.1.8 Overview

The previous sections introduced the various components of the new frequency domain VAR scenario framework. To close the formal presentation of this new framework, this section provides an overview of the consecutive steps that need to be taken to actually construct a scenario set according to this framework. Furthermore, it shows how these steps are linked to the specific components of the framework and in which of the examples in the subsequent sections the workings of these components are illustrated. Thereby, this overview can serve as a reference for studying the workings of the new framework. The discussion will be based on Figure 20.9. The left hand side of this figure shows the consecutive steps while the right hand side shows the link to the previous components and upcoming examples. The various steps are the following.

#### 1. *Choosing factors*

The first step in constructing the to the framework fundamental MVAR models is to state for which parts of the total frequency range  $[0, 0.5]$  a separate model needs to be made. The required decomposition will to a large extent be determined by the requirements that follow from the intended use of the scenarios. If for example monthly scenarios are needed with an horizon of one year, it will suffice to model the business cycle fluctuations and the combined seasonal and monthly fluctuations as separate stochastic factors. If for example annual scenarios are needed with an horizon of thirty years, the trend, long wave and business cycle fluctuations need to modeled as separate stochastic factors. The selected frequency ranges do not necessarily need to cover the entire frequency range  $[0, 0.5]$ . It can be a deliberate choice not to model some specific frequency ranges. It is important however that the selected frequency ranges are non-overlapping to ensure a zero correlation between the filtered time series following from the third step. Also, modeling some frequency range more than once does not make much sense.

Figure 20.9 Overview of steps to be taken in the new scenario framework together with the relations to the various components and examples.





## 2. *Collecting original data*

Once the stochastic factors that need to be modeled are determined, the second step in constructing an MVAR model is to collect the time series data for each of the factors on which the component models will be estimated. A key feature of MVAR models is that for each of the component models, samples of different length and with different observation frequencies can be used such that the data can be specifically adjusted to give the best and most representative information on the frequency range that needs to be described by the component model. In this way for example a sample of two centuries with a 5-year observation frequency can be used for the long wave type fluctuations, a postwar annual sample can be used for business cycle fluctuations, a five year monthly sample can be used for seasonal and monthly fluctuations and so on. Ideally the underlying total time series should be one and the same. Unfortunately, in case of widely varying frequency ranges in practice this will in general not be possible. For example, a two century long daily time series of equity returns will typically not be available which forces us to use different types of time series data for the different frequency ranges. Fortunately, from a conceptual frequency domain perspective this does not significantly alter the validity of the MVAR approach. Furthermore, all data should be obtained as much as possible in its original format without any transformation or filtering applied to it. This means that for example the first order differencing operator that is often applied to convert trending time series into stationary time series, should *not* be applied. The only exception is that for index type variables such as stock prices or volumes of the national product it is best to apply a (natural) logarithmic transformation to convert exponential trending behavior into linear trending behavior because exponential trends cannot be handled well in the frequency domain filtering procedure in the next step. For this also see the experiments in section 5.5.6.

## 3. *Filtering data*

The third step is to dissect the collected time series as accurately as possible into the fluctuations in the selected frequency ranges such that the resulting uncorrelated filtered time series can be used for the estimation of the different component models. This filtering needs to be based on perfect, non-overlapping, pass-bands and without inducing any phase shifts, that is with as least distortion of the original time series behavior as possible. For this the *zero phase frequency filter* described in section 5.4 with the required pass-bands can be used. Because of the additive character of this decomposition and the zero correlation property, once models have been constructed for the different frequency ranges, the resulting simulations can be simply add up to construct one total simulation as illustrated in Figure 20.1.

## 4. *Modeling factors*

The fourth step is to use the filtered time series for the estimation of the different component models of the total MVAR model. Besides conventional VAR models and estimation procedures, for reasons explained in sections 20.1.5 and 20.1.6, these can also be based on the new types of Truncated VAR (TVAR) or EigenValue Restricted VAR (EVR VAR) models. The dynamic properties of these component models are analyzed and set as much as possible in the frequency domain in terms of their spectral densities (instead of in terms of the conventional autocorrelation matrices) to ensure a clear understanding and better awareness of the final model

behavior. Furthermore, separate models need to be made for the ultra low frequencies that determine the averages and trending behavior of the variables in the model. A final important remark with respect to this step is that despite the at first sight perhaps restricting zero correlation property between the models for the different frequency ranges, with MVAR models it is actually possible to model the more complex relations of macroeconomic behavior between the different frequency ranges. Examples of these factor dependencies are given in section 20.1.1.

### 5. *Backtesting*

Once a basic MVAR model has been constructed in the previous steps, its conditional variances or confidence intervals can be tested on the behavior of the historical time series that have been used for the estimation using the backtesting procedure described in section 20.1.3. It is important to note that, as illustrated by the example in Figure 20.5, conditional variances and hence the backtesting procedure do *not* give all the information about the dynamic behavior of a process within the conditional variances. Hence, the backtesting procedure should always be accompanied by an assessment of the dynamics of the model, preferably in terms of its (multivariate) spectral densities, especially since this dynamic behavior is of fundamental importance in multi-period scenario applications. For this, also see the example in section 1.2.

### 6. *Model adjustments*

As a final step, both the conditional variances and the dynamics of the MVAR model can be adjusted using the procedures described in sections 20.1.4 and 20.1.7. This can be done to refine the model properties or to deliberately deviate from the observed historical behavior of the variables, either to incorporate forward looking information in the model or to perform a sensitivity analysis. Note that in the conventional scenario applications of VAR models, as described in sections 18.2.1 and 18.2.2, such adjustments often only concerned the unconditional expected values, the unconditional variances and at the most some contemporaneous correlation number. The new scenario framework offers much richer and more explicit model structures with element that are directly related to different aspects of macroeconomic (scenario) behavior. Therefore, it is now possible to separately change for example long and short term volatilities, low and high frequency fluctuations, the dynamics in different frequency ranges, state dependent volatilities, etc.

### *Examples*

Now that in the preceding sections the new scenario framework has been described in a formal sense, in the following sections four examples of this new framework will be described. Most of these examples are strongly related to the experiments performed to actually develop the framework and are meant to gain experience with the new framework. Each example illustrates one or more of the specific components of the new framework in isolation, as indicated in Figure 20.9. An example that combines all components is not included. However, combining the various components is straightforward. The four examples are the following.

1. Section 20.2 illustrates the basic workings of the fundamental MVAR models by constructing a univariate (AR) version of such a model for the National Product Index for the Netherlands. The example includes a logarithmic linear stochastic



trend, long wave and business cycle model as well as the first application of the backtesting procedure. Furthermore, the effects of changing the conditional variances for different horizons will be illustrated by applying the procedure outlined in section 20.1.4. Finally, the resulting MVAR model will be compared to conventional modeling approaches in terms of AR models for the first order differences, hence the annual growth rate, of the National Product Index.

2. Section 20.3 illustrates how complex factor dependencies as referred to in section 20.1.1 can be modeled in the new framework by constructing a univariate MVAR model for the Long Term Interest Rate for the Netherlands including the level effect that is observed in both short and long term interest rates and described in respectively section 12.2.1 and 13.2.1. The model also illustrates the workings of the stochastic AR(1) trend model described in section 20.1.2. In general, the example shows how MVAR models can be applied to model non-index type variables such as Interest Rate variables instead of index type variables such as the National Product Index as in the first example.
3. Section 20.4 illustrates the workings of the Truncated VAR (TVAR) models by constructing a univariate model for the Consumer Price Index of the Netherlands with the restriction that the implied annual inflation rates need to be strictly positive. Along the way another empirical level effect is revealed and modeled, this time in the consumer price inflation.
4. Finally, section 20.5 illustrates the working of the EigenValue Restricted VAR (EVR VAR) models by constructing a six dimensional VAR model for the business cycle fluctuations in the Dutch economy with restrictions on the types of business cycle fluctuations present in the model. The objective is to let these be confined to the well known Juglar and Kitchin type business cycle fluctuations as found in Part III. The EVR VAR model will be compared to unrestricted VAR models of different orders in terms of the multivariate spectral densities. Also, the effects of changing the dynamic properties of the model, in terms of its eigenvalues and eigenvectors, on the spectral densities will be explored by using the procedure outlined in section 20.1.7.

To end the description of the components of the new frequency domain VAR scenario framework in this section, we would like to point in an informal sense to the similarity of the combination of the MVAR models together with the stochastic trend modeling and EVR VAR models on the one hand and the existing *common cycle* and *co-integration* models on the other hand. For example Breitung and Candelon (2000 and 2001) state that if there is a common cycle in a time series process at some frequency, then there exists a linear combination which eliminates the cyclical behavior at that frequency (compare a component model of an MVAR model for some specific frequency range or an EVR VAR model with restrictions to achieve identical complex eigenvalues of some frequency). This resembles the well-known co-integration approach introduced by Engle and Granger (1987). The co-integration hypothesis can even be seen as a test for a common “cycle” at the zero frequency (compare the stochastic trend or other low frequency component model of an MVAR model). Also see section 3.7.6 for a formal description of a co-integration model.

## 20.2 Example 1: Multifactor VAR models

This section presents the first of four examples of the new frequency domain VAR scenario framework as described in section 20.1. The overview in Figure 20.9 shows how this example is related to the different components of the new framework. This example illustrates the basic workings of the fundamental MVAR models (section 20.1.1) by constructing a univariate (AR) version of such a model for the National Product Index for the Netherlands. The example includes a logarithmic linear stochastic trend (section 20.1.2), long wave and business cycle model as well as the first application of the backtesting procedure (section 20.1.3). Furthermore, the effects of changing the conditional variances for different horizons will be illustrated (section 20.1.4). Finally, the resulting MVAR model will be compared to conventional modeling approaches in terms of AR models for the first order differences, hence the annual growth rate, of the National Product Index.

### 20.2.1 National Product Index

Consider the case in which we want to simulate univariate scenarios for the real (instead of nominal) National Product Index for the Netherlands. The scenarios should have an horizon of fifty years and an annual observation frequency. They need to describe both long wave type fluctuations and Juglar and Kitchin type business cycle fluctuations as found as important stylized facts in Part III. The scenario properties need to be consistent with and based on the annual time series for the Netherlands for the 1870-1999 sample period with interpolation across the first and second World War as also used for the analysis in Chapter 7. Section 6.1 gives a description of the data while the time series themselves and a description of their sources can be found in Appendix F. All filtered components of the time series used here are identical to those from Chapter 7 as obtained from filtering with the zero phase frequency filter with the appropriate pass-bands.

#### *Estimation and model dynamics*

All estimation is done by the Yule-Walker method described in section 3.4.4. The results in section 4.7 and Chapter 19 indicate that the Yule-Walker method produces the best estimates. Where relevant, this also automatically results in the same spectral densities as those from the Maximum Entropy procedure as obtained in Part III. The dynamic properties of the models are analyzed by means of the (complex) roots of the models and the properties of the spectral densities as described in section 6.4.1. This resulted in the following two model components for a univariate MVAR model of which the basics are described in section 20.1.1.

#### *Long wave*

The properties of the long wave type fluctuations are based on the  $[1/70, 1/30]$  component of the natural logarithm of the index which has been linearly interpolated for the 1916-1919 and 1939-1949 periods to reduce the influence of the two World Wars. Because of the long period length of around 50 years it is logical to use the longest available sample period, 1870-1999. Because of the very smooth nature of the filtered time series it suffices to use a five year observation frequency. That is, one value for every five year period ending with the value for 1999. In this way, lower

order models can be used to describe the long wave without a loss of information. Using only an AR(2) model is sufficient for an adequate description of the long wave dynamics. The volatility of the series is 6.3% as can be seen from Table 7.2. The left hand side of Figure 20.10 shows the auto-spectrum of the long wave component for this AR(2) model. Note that the horizontal axis shows only the  $[0, 0.10]$  frequency range and that the frequencies in terms of a five year observation frequency have been transformed into the conventional annual observation frequency. Table 20.3 shows the properties of the spectral peak. As expected it describes very regular periodic behavior (modulus 0.96) with a period length of around 45 years. Finally note that in case scenarios are required for a higher observation frequency (for example one year) than the one that the model is estimated on (here five years), these can be obtained by means of linear interpolation between adjacent observations.

Figure 20.10 Auto-spectra of AR models for long wave (left) and business cycle component (right) of logarithm of National Product Index for the Netherlands.

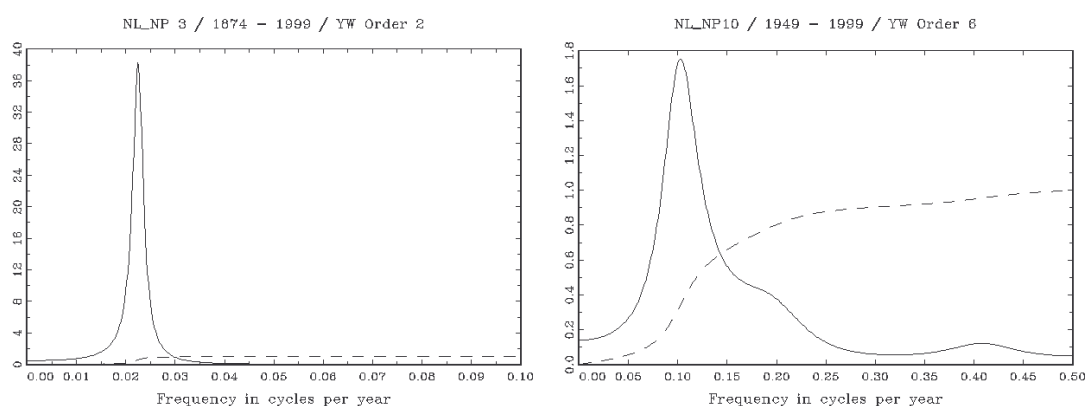


Table 20.3 Statistics of auto-spectra of AR models for long wave (left) and business cycle component (right) of logarithm of National Product Index for the Netherlands calculated from the roots of the models.

<i>NL Long wave AR(2) model</i>					<i>NL Business cycle AR(6) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>		<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1870-1999	0.02	44.4	0.51	0.96	1949-1999	0.10	9.9	0.42	0.87
						0.20	4.9	0.20	0.74
						0.41	2.5	0.07	0.75

### *Business cycle*

The business cycle fluctuations in the scenarios are based on the information in the  $[1/15, 0.5]$  component of the natural logarithm of the index. Because of the quality of the data and the changes in business cycle properties that have taken place, here the 1949-1999 period is considered the most representative. Because of the higher frequency of the fluctuations, a shorter sample can be used but at the same time the data needs to have a higher observation frequency. Here we use a conventional one year observation frequency. The volatility of the series is 2.0% as can be seen from Table 7.2. Using the same AR(7) model as in Chapter 7 results in the auto-spectrum on the right hand side of Figure 20.10 with the spectral properties from the right hand side of Table 20.3. The spectrum shows that the model describes the desired

Juglar and Kitchin type of pseudo-periodic behavior with period lengths of around respectively 10 and 4.5 years. A third spectral peak describes a small amount of variance at the very high frequencies.

#### *Trend and total model*

The AR models separately describe the long wave and business cycle fluctuations of the scenarios. Because of the non-overlapping frequency ranges, the underlying time series used have correlation zero and can indeed be modeled separately and added afterwards without loss of information. What is still missing is an extremely low frequency path around which these fluctuations evolve in the scenarios generated. The lowest frequency fluctuations ( $\omega=0$ ) are in fact no fluctuations at all. They describe the average values or trending paths of the scenarios. We first use the simple assumption that in all scenarios this is a logarithmic linear trend with a average growth rate of 2.7% per annum. This is the average annual growth observed for the Netherlands during the 1870-1999 period as indicated in Table 7.1.

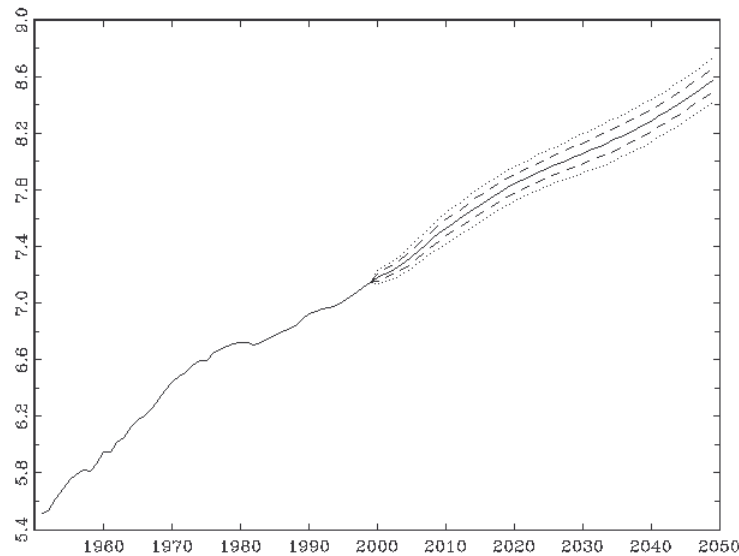
### 20.2.2 Confidence intervals and backtesting

Next, we calculated the confidence intervals of the univariate MVAR model described in the previous section and performed the backtesting procedure as described in section 20.1.3.

#### *Confidence intervals*

Figure 20.11 shows the 0.1%, 5%, 50% (median), 95% and 99.9% percentile of the conditional distributions of the MVAR model for the logarithm of the National Product Index as described in the previous section with an horizon of 50 years. In order to be consistent with the value for the last year of the sample period (1999), the conditional distributions, and hence also scenarios generated from the model, are shifted up or down to correct for the value of the frequency components that are not modeled. In this case this concerns the components from the  $[0, 1/70]$  and  $[1/30, 1/15]$  frequency ranges. Implicitly the assumption is here that all missing components remain at their values at the end of the sample period. The 0.1% and 99.9% percentiles represent the minimum and maximum values of the distributions.

Figure 20.11 Confidence interval for the conditional annual distributions of MVAR model of logarithm of National Product Index for the Netherlands with a linear trend with a 2.7% growth rate and two separate AR models for the long wave and business cycle frequency range. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



From these confidence intervals the following can be observed:

- The expected evolution of the index is dominated by the linear trend and the predictions for the long wave component.
- The conditional variance in the long run stabilizes at the stationary unconditional variance of the sum of the two component AR models. This shows that implicitly this MVAR model is a so called *trend stationary* model. In a sense one could say that the unconditional variance is a mixture of conditional variances calculated from different starting points.
- Compared to the results from a traditional model in terms of the annual growth of the index describing all frequencies at once, the confidence intervals shown here are rather small. To see this, compare for example the results in section 20.2.6.

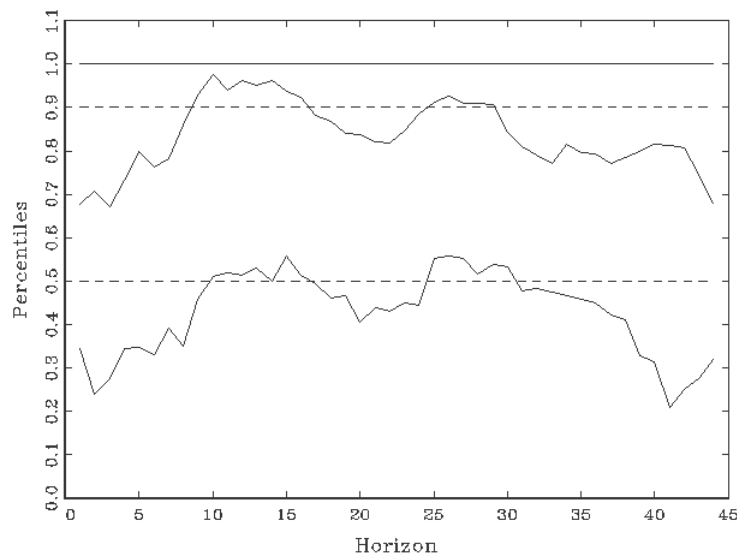
The final observation is a very important one because of the great influence of the (scenario) variance in general on the results from (ALM) models that are based on these scenarios. In the remainder of this section and in subsequent sections we will discuss various tools for both analyzing and modifying the variance of MVAR models.

### *Backtesting*

One stand with respect to the issue of the relatively low conditional variances could be that we know that both the unconditional variance and the dynamics of the two AR models are consistent with the historical time series and hence the conditional variances must also be correct. Here we know this is the case because of the use of the Yule-Walker estimation method (exact replication of first  $p$  autocovariances in case of an  $AR(p)$  model) and the thorough analysis of the models in terms of their spectral properties relative to the empirical knowledge from Part III. If this was not the case, another tool for determining to what extent the conditional variance of an MVAR model gives an adequate description of the stochastic behavior is the

backtesting procedure described in section 20.1.3. The results for the univariate MVAR model described in the previous section are shown in Figure 20.12. Here it is assumed that the frequency components that are not modeled by the MVAR model, the  $[1/T, 1/70]$  and  $[1/30, 1/15]$  frequency ranges, are given when backtesting the model.

Figure 20.12 Backtesting results in terms of realized and expected percentages of observations in confidence intervals of conditional distributions with various horizons for MVAR model of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period.



From these backtesting results the following can be observed:

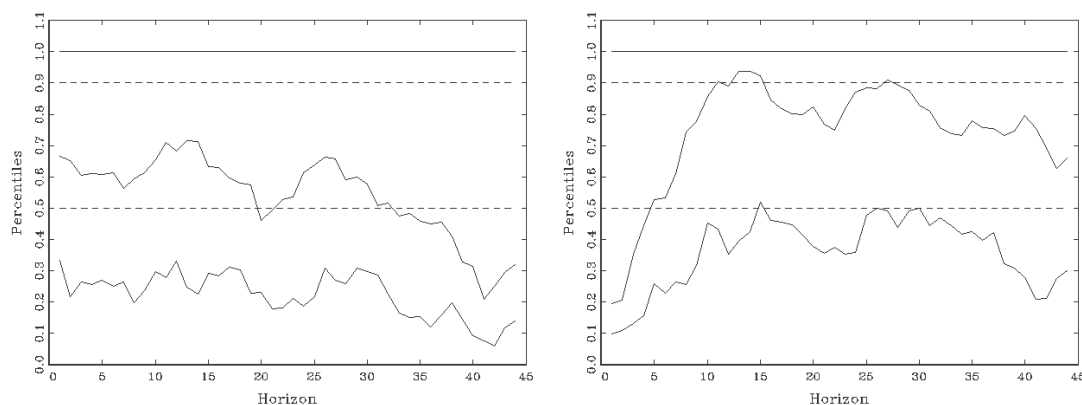
- Both for an horizon up to 10 years and an horizon longer than 30 years, the MVAR model describes a smaller conditional variance when compared to the historical conditional variance. In these cases for example only 70% of the realizations fall within the model's 90% confidence intervals. Not 10% but 30% of the realizations fall outside these confidence intervals, thereby indicating that the historical conditional distributions are wider than the ones from the model.
- For the horizon up to 10 years, the low variance is caused by the fact the MVAR model uses a business cycle model based on the variance of the  $[1/15, 0.5]$  frequency component for the 1949-1999 sample while the backtesting is done on the entire 1870-1999 sample. Table 7.2 shows that the business cycle volatility is 2.0% during the postwar 1949-1999 period while it is 3.3% during the prewar 1870-1913 period. Using this latter volatility for the business cycle AR model results in much better backtesting results for the short term horizon.
- For the 10 to 30 year horizon, the conditional variance described by the MVAR model is indeed consistent with the historical behavior of the time series as was to be expected from the construction of the model.



### 20.2.3 Adjusting conditional variances

The backtesting procedure, such as applied in the previous section, might indicate too low or too high conditional variances for some horizons. As described in section 20.1.4, a unique feature of MVAR models is that because it is possible to change the (relative) variance for separate frequency ranges, it is also possible to change the conditional variances for different horizons. To see why this is so, consider the following experiment based on the MVAR model for the logarithm of the National Product Index described in the previous sections. First, we set the variance of the long wave component model to a very small value while leaving that of the business cycle component model unchanged. Second, the other way around, we set the variance of the business cycle component model to a very small value while leaving that of the long wave component model unchanged. The backtesting results from these two modifications in the MVAR model are shown in Figure 20.13.

Figure 20.13 Backtesting results of MVAR model of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period when reducing the variance of the long wave model (left) or reducing the variance of the business cycle model (right).



When compared to Figure 20.12, from these results the following can be observed:

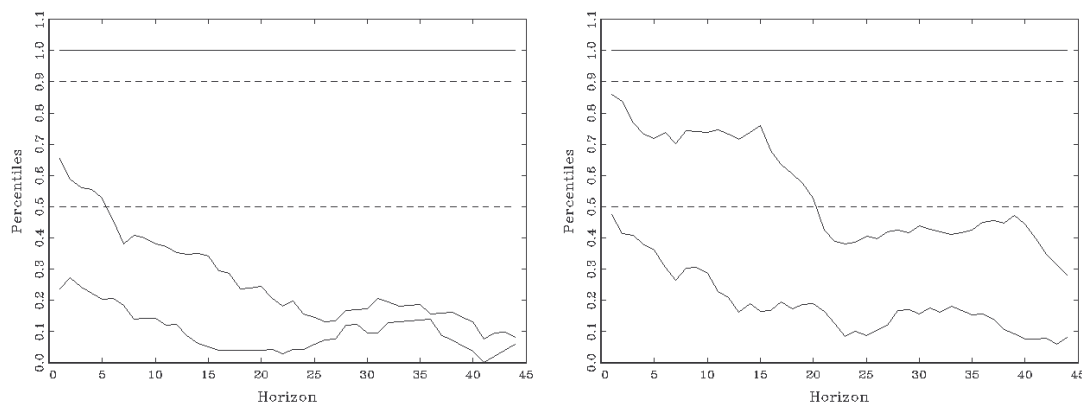
- Lowering the unconditional variance of either of the components reduces the conditional variance for all horizons. This can be seen from the fact that all percentages of observations in the various confidence intervals have decreased and can be understood by noting that the total variance is the sum of the variances from the two component models.
- However, lowering the variance of the long wave component has a greater impact on the conditional variances for long horizons than for short horizons. For example the percentages for a one and two year horizon have hardly changed compared to the original results while for the longer horizons they are reduced very much. This can be understood from the fact that low frequencies correspond to high level autocorrelations which in turn cause a slowly increasing conditional variance. The impact on the conditional variance for short horizons will therefore in that case be relatively small.



- The other way around, lowering the variance of the business cycle component has a greater impact on the conditional variances for short horizons than for long horizons. High frequencies correspond to low level autocorrelations which in turn cause a fast increasing conditional variance. The impact on the conditional variance for short horizons will therefore in this case be relatively big.

Another application of changing the variances of the component models is the following. Suppose now we assume that the frequency components that are not modeled by the MVAR model, the  $[1/T, 1/70]$  and  $[1/30, 1/15]$  frequency ranges, are *not* known when backtesting the model as was assumed in section 20.2.2. From the left hand side of Figure 20.14 it can be seen that the model has far too narrow confidence intervals to model the uncertainty in the non-modeled components correctly. This shows that it is not fair to judge models on information that was not used for estimating them. An obvious remedy is to increase the variance of the components that *are* modeled, the  $[1/70, 1/30]$  and  $[1/15, 0.5]$  frequency ranges, to compensate for the extra variance caused by the components that are *not* modeled. So, the variance in the non-modeled components is in fact substituted by the variance of components that are modeled. This has been done by increasing the variance of the long wave component to the historical variance of the entire  $[1/T, 1/15]$  frequency range from Table 7.2 (standard deviation 12.4% instead of 6.3%). Furthermore, for the business cycle component the variance for the 1870-1913 period has been used (standard deviation 3.3% instead of 2.0%), to prevent the disturbing, but correct, effect described following Figure 20.12. The right hand side of Figure 20.14 shows the backtesting results for this adjusted model. The results have clearly improved because of the increased conditional variances but are not satisfactory. Especially for the longer horizons the confidence intervals are still too narrow to model the uncertainty in the non-modeled components correctly.

Figure 20.14 Backtesting results of MVAR model of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period when assuming the non-modeled frequency components unknown (left) and when substituting the variance of the non-modeled frequency components by additional variance of the modeled components (right).



#### 20.2.4 Modeling a logarithmic linear stochastic trend

Until now we used a linear trend for the National Product Index scenarios with a constant annual growth of 2.7%. The fluctuations around this linear trend are modeled by means of two stationary AR models for respectively the long wave and the business cycle fluctuations. Our MVAR model is therefore in fact a trend stationary model. Although the linear trend hypothesis is supported by the references mentioned at the end of section 5.6 and the filtering results in section 7.2, one might wonder whether this is indeed an appropriate modeling assumption. One reason for asking this question is that the backtesting results in both Figure 20.12 and 20.14 indicate too narrow confidence intervals for (very) long horizons. If the linear trend is not an appropriate assumption, what are then the alternatives within the context of MVAR models and what are their effects on for instance the backtesting results?

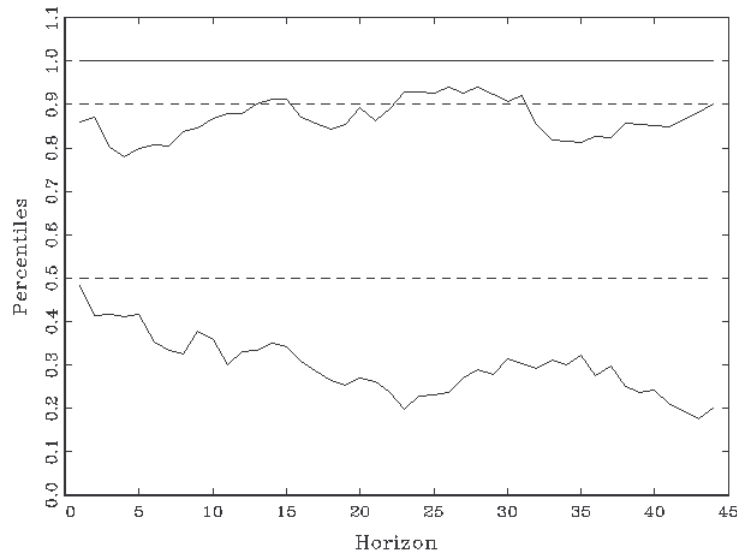
To answer this question we now apply the stochastic logarithmic linear trend model (20.1.6) described in section 20.1.2 with  $\mu=2.7\%$  and  $\sigma=0.4\%$  on the final model of the previous section in which the variance in the non-modeled components was substituted by the variance of components that are modeled. The 0.4% standard deviation is determined by trail and error to arrive at the appropriate 90% confidence intervals in the backtesting procedure<sup>100</sup>. The results from the backtesting procedure for the resulting model are shown in Figure 20.15. When compared to the right hand side of Figure 20.14, from these results the following can be observed:

- The 90% confidence intervals of this MVAR model with a stochastic trend are consistent with the uncertainty in the historical time series for all horizons considered. Note that the linear trend with its linear increasing standard deviation works well here because the extent to which the confidence intervals in Figure 20.14 are too narrow also increases approximately linear with the horizon.
- Although the results for the 50% confidence interval have also improved, they still point at too narrow confidence intervals. Because the broader 90% confidence intervals *are* consistent we know this is not the case. Apparently, the shapes of the conditional historical and model distributions are different. This should come as no surprise because in this model the variance of some frequency ranges has been replaced by variance in other frequency ranges, thereby changing the dynamics of the model. From the example in Figure 20.5 we know that variance is not the whole story, it is also important what type of fluctuations constitute this variance.

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<sup>100</sup> A remarkable, probably lucky, coincidence is that the long term average growth rates from a cross section for the Netherlands, United Kingdom and United States from Table 7.1 are respectively 2.7%, 2.0% and 3.3% and have a standard deviation of 0.5% which is close to the 0.4% found here. Of course the 0.5% is based on only three observations.

Figure 20.15 Backtesting results of MVAR model of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period including a linear stochastic trend together with substituting the variance of the non-modeled frequency components by variance of the modeled components.

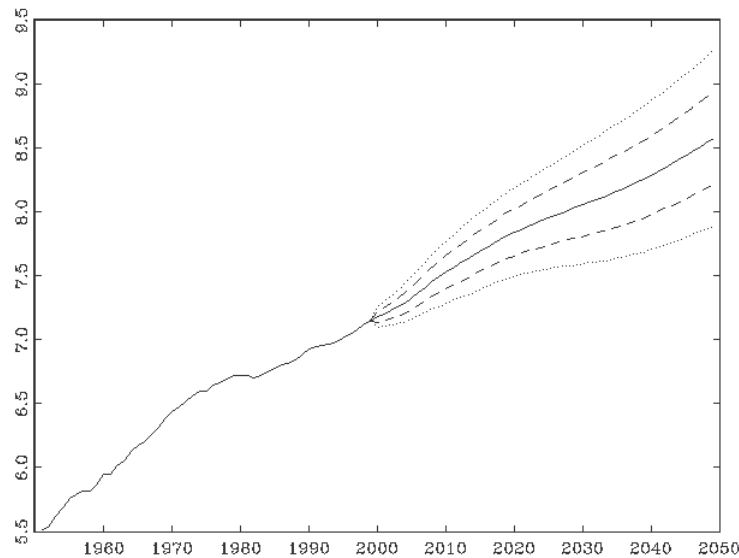


This example in fact demonstrates two applications of a stochastic trend in an MVAR model. The first is to widen the confidence intervals of the model to compensate for the variance from non-modeled (low frequency) fluctuations. The second is to allow for an explicit modeling of the uncertainty in the ultra low frequencies. In case the horizon is long enough for the models on the other frequencies to level out on their average values of zero, the trend directly determines the uncertainty in the long term average growth rate of the index in the scenarios. The result is an MVAR model with a separate and therefore both flexible and transparent modeling of

- a) Long wave and business cycle dynamics, consistent with those empirically observed,
- b) Uncertainty in the long term average growth rate of the National Product Index that is directly controllable and
- c) Confidence intervals that are consistent with historical uncertainty.

Figure 20.16 shows the confidence intervals for the MVAR model with the stochastic trend. Compared to the confidence intervals of the original model shown in Figure 20.11, these intervals are much wider as was to be expected.

Figure 20.16 Confidence intervals for conditional annual distributions of MVAR model of logarithm of National Product Index for the Netherlands with a stochastic linear trend with an expected growth rate of 2.7% and standard deviation of 0.4% per annum and two separate AR models for the long wave and business cycle frequency range. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



### 20.2.5 First order differencing

The principles of an MVAR model with separate models for different frequency ranges can just as well be applied on the first order differences of an index variable which is consistent with the data representations used in conventional applications of VAR models (see section 18.2.1). Consider for example the typical first data representation from Table 18.1. From section 20.1.2 we also know that confidence intervals from stochastic models in terms of levels,  $I(0)$  models, are fundamentally different from those in terms of first order differences,  $I(1)$  models. Therefore in this section the effects are examined of specifying an MVAR model in terms of the frequency components of the first order differences of the logarithm of the National Product Index instead of its levels.

#### *Filtering*

To obtain the  $[1/70, 1/30]$  and  $[1/15, 0.5]$  frequency components of the growth rates, in fact two filters have to be applied on the logarithm of the index. The first is the first order differencing filter and the second is an ideal zero phase frequency filter with the appropriate pass-band. Because of the linearity of these filters it does not matter in what order they are applied on the time series, as long as they are both applied. For this also see the final paragraph of section 5.1. In theory, it should therefore make no difference whether we first calculate the first order differences and then apply the band-pass filters to the resulting series or, the other way around, if we apply first order differencing on the filtered components of the levels of the time series. Because the frequency components from the logarithm of the index are already used in the previous example models, the second approach was used here.

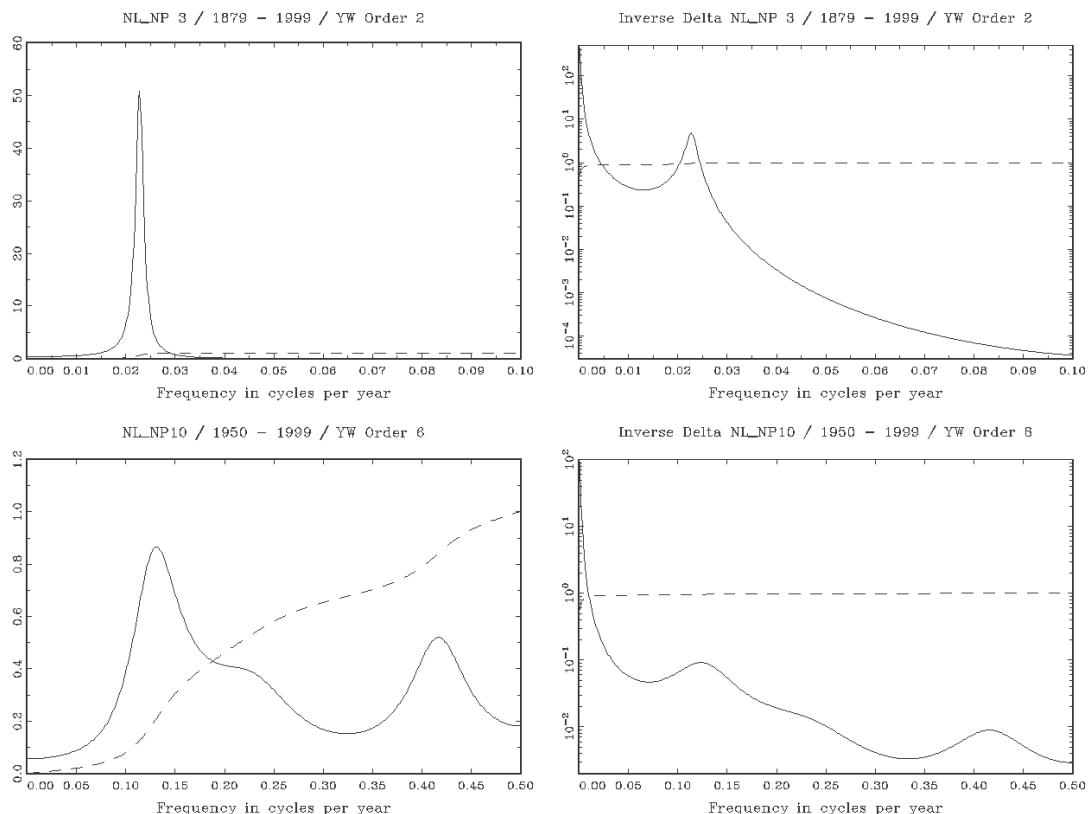
### Model estimation and dynamics

To ensure a fair comparison of the two types of MVAR models, for both the long wave and business cycle model, identical model orders and sample periods are used as for the MVAR model from section 20.2.1 in terms of the levels of the logarithm of the index. The models are again estimated by the Yule-Walker procedure. As usual the dynamics of the models are considered in terms of their auto-spectra. Because the models are specified here in terms of the first order differences of the logarithm of the index, these auto-spectra in principle describe the dynamics of these first order differences. In order to assess the corresponding dynamics of the levels of the logarithm of the index, which can be compared to the auto-spectra from the earlier models, the Power Transfer Function (PTF) of the inverse first order differencing operator from section 5.2.5 is applied on the non-normalized auto-spectra.

Table 20.4 Statistics of auto-spectra of AR models for long wave (left) and business cycle component (right) of first order differences of logarithm of National Product Index for the Netherlands calculated from the roots of the models.

<i>NL Long wave AR(2) model</i>					<i>NL Business cycle AR(6) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>		<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1871-1999	0.02	43.9	0.50	0.97	1950-1999	0.13	8.0	0.32	0.83
						0.23	4.3	0.24	0.71
						0.42	2.4	0.21	0.80

Figure 20.17 Auto-spectra of AR models for long wave (top) and business cycle component (bottom) of first order differences (left) of logarithm of National Product Index for the Netherlands and implied auto-spectra of levels (right) of logarithm of National Product Index on a logarithmic scale.



From comparing the results in Table 20.4 and Figure 20.17 to those in Table 20.3 and Figure 20.10, the following can be observed:

- The auto-spectra for the first order differences of the component models are very similar to those implied by the models estimated in terms of the levels. The business cycle spectrum in terms of the growth of the National Product from Figure 20.17 can for example be compared to that from Figure 17.2. Furthermore, the frequencies of the spectral peaks are very similar. One exception is that the Juglar peak now has a higher frequency of 0.13 (period length 7.95) instead of 0.10 (period length 9.92). This disturbance is probably caused by the fact that the first order differencing operator amplifies the high frequency fluctuations.
- While the auto-spectra of the first order differences as implied by the models estimated in terms of the levels are very similar to the auto-spectra of the models directly estimated in terms of the first order differences, the other way around is certainly not the case. The auto-spectra for the levels implied by the estimates in terms of the first order differences, as shown on the right hand side of Figure 20.17, are dominated by the ultra low frequency fluctuations which is clearly not the case for the auto-spectra shown in Figure 20.10. Note that the integrated spectra indicate that almost 100% of the variance is described by these ultra low frequencies. These extreme and incorrect auto-spectra are caused by a combination of two facts. The first is that the spectra on the left hand side show a small amount of variance at the low frequencies that we know is not really there because of the perfect pass-bands of the applied zero phase frequency filtering. The spectra should have zero variance below frequencies of respectively  $\omega=1/70$  and  $\omega=1/15$ . The second fact is that these spurious small variances are amplified in the spectra on the right hand side by multiplying them by the large values of the PTF of the inverse first order differencing operator at these frequencies, as shown in Figure 5.6.

Figure 20.18 shows the confidence intervals from the MVAR model in terms of the first order differences assuming a constant average growth rate of 2.7%. Figure 20.19 shows the backtesting results for this model both with and without using the higher 3.3% standard deviation for the level of the business cycle component from the 1870-1913 period instead of the original 2.0% from the 1949-1999 period. From these results the following can be observed:

- The expected values for the various horizons, as represented by the 50% percentile in Figure 20.18, are very similar when modeling components of the first order differences or of the levels of the index (compare Figure 20.11).
- The confidence intervals of the levels of the index are wider when the first order differences are modeled than when the levels are modeled. This is consistent with the auto-spectra shown in Figure 20.17 and the discussion on the fundamental difference between  $I(0)$  and  $I(1)$  models in section 20.1.2. As we shall see in the next section (see Figure 20.21) they are not as wide as the situation in which the first order differences for *all* frequencies are modeled. This makes sense because here only parts of the frequencies are modeled, namely the  $[1/70, 1/30]$  and  $[1/15, 0.5]$  frequency ranges. For similar reasons, a comparison with the confidence intervals for the stochastic trend model, as shown in Figure 20.16, is not fair because there also the ultra low frequencies are modeled stochastically.

- That the confidence intervals of the model are too wide is confirmed by the backtesting results. For horizons of more than 10 years for example, 70% of the realizations fall within the 50% confidence intervals. Not 50% but 30% of the realizations are outside these confidence intervals, thereby indicating that the true conditional distributions are less wide than the ones from the model.
- The reason that the model produces confidence intervals that are too wide is probably that by applying the filtering procedure, time series have already been obtained that show sufficiently stationary behavior. Hence, these filtered series require no further manipulation such as the conventional first order differencing operator to render stationary time series. Doing this anyway introduces additional variance in the model.
- When correcting for the low variance on the short horizons by using the higher standard deviation for the level of the business cycle component, as explained in section 20.2.2, the widening impact on the confidence interval is larger for the longer horizons than in the case of modeling the levels of the index. This is again caused by the  $I(1)$  property of the model in which extra variance in the first order differences cumulates into the variance of the levels as the horizon increases. Using the higher variance therefore causes more consistency for the short horizons but actually worsens the backtesting results for the longer horizons. The property of MVAR models illustrated in section 20.2.3 of separately changing the conditional variances for different horizons by changing the (relative) variance of separate frequency ranges is therefore lost when modeling in terms of first order differences.

Figure 20.18 Confidence interval for the conditional annual distributions of MVAR model of first order differences of logarithm of National Product Index for the Netherlands with an average annual growth of 2.7% and two separate AR models for the long wave and business cycle frequency range. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.

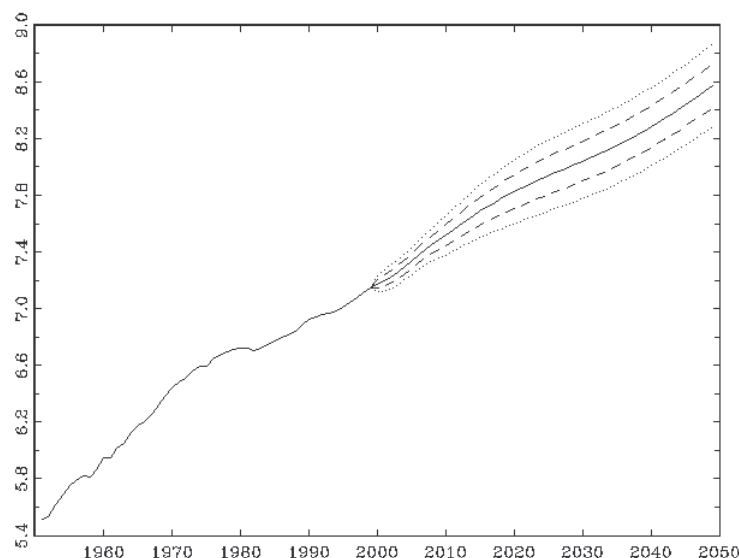
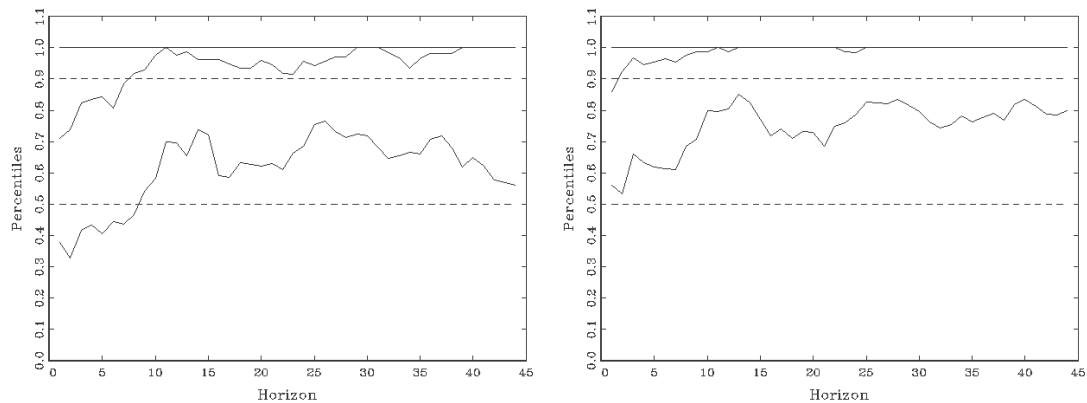




Figure 20.19 Backtesting results for MVAR model of first order differences of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period both with (right) and without (left) using the higher 3.3% standard deviation for the level of the business cycle component instead of the original 2.0%.



To summarize, also for MVAR models it holds that models in terms of the first order differences of a variable describe a very different type of stochastic behavior than models in terms of the levels of the variable. More specifically, they produce wider confidence intervals. Here, MVAR models in terms of the first order differences show a poor consistency with the empirical behavior of the time series in terms of incorrect implied spectral densities for the levels of the variable and confidence intervals that are too wide. Furthermore, a useful property of MVAR models in terms of allowing for a flexible calibration of the confidence intervals, is lost.

### 20.2.6 Conventional approach

Typical conventional applications of VAR models for generating scenarios are elaborately tested in Chapter 19. Nevertheless to close the first example of the new scenario framework as described in section 20.1, in this section we analyze yet another conventional application, in this case a univariate AR model for the National Product Index of the Netherlands. There are two reasons for doing this. The first is that the new scenario framework contains new tools in terms of the confidence intervals and the backtesting procedure for analyzing model behavior which might provide new information on the performance of conventional models in terms of their consistency with the empirical knowledge. The second reason is to enable a direct comparison between the new MVAR models and conventional VAR models by applying them on the same problem of modeling the National Product Index.

#### *Model specification*

To ensure a fair comparison between a conventional model and the MVAR models analyzed in the previous sections, the relevant models need to be based on an approach that is identical where possible and only differs in the most distinguishing aspects of the models. The model here is therefore based on the growth of the National Product Index for the Netherlands for the 1870-1999 sample period, calculated as the first order differences of the logarithm of the series. The logarithm of the index has first been linearly interpolated for the 1916-1919 and 1939-1949

World War periods. Because no filtering is applied, the entire frequency range is modeled by a single model. An AR(10) model was estimated using the Yule-Walker procedure. The order  $p=10$  was chosen to provide the model with a similar time window of information as the MVAR models. The long wave components of the latter models use an order of  $p=2$  with a five year observation frequency which means a total lag of  $2 \times 5 = 10$  years. The business cycle component uses an order  $p=6$  model with an annual observation frequency. Figure 20.20 shows the auto-spectrum of the resulting AR(10) model, both in terms of the growth rates on which the model was estimated and in terms of the implied logarithm of the level of the index. The latter is obtained by applying the PTF of the inverse first order differencing filter as described in section 5.2.5. Table 20.5 shows the properties of the spectral peaks in the auto-spectrum of the first order differences.

Figure 20.20 Auto-spectrum of AR model for first order differences (left) of logarithm of National Product Index for the Netherlands and implied auto-spectrum of levels (right) of logarithm of National Product Index on a logarithmic scale.

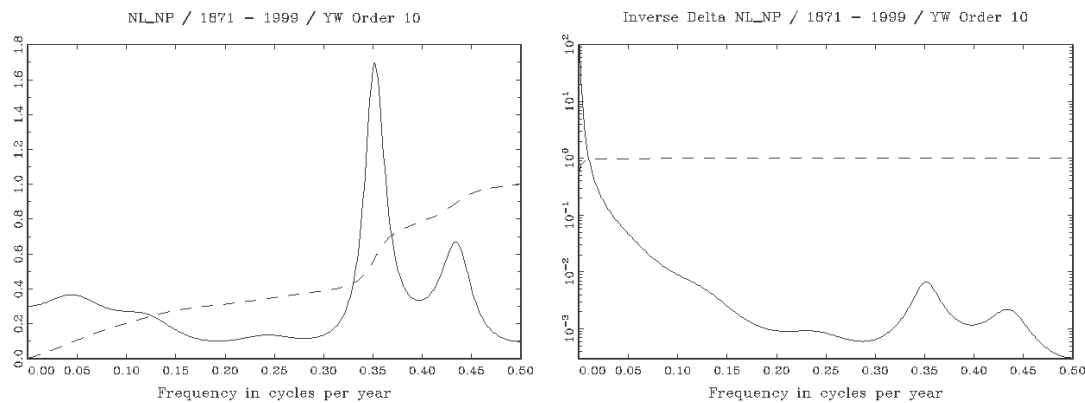


Table 20.5 Statistics of auto-spectrum of AR model for first order differences of logarithm of National Product Index for the Netherlands calculated from its roots.

<i>NL Growth AR(10) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1870-1999	0.05	22.2	0.18	0.77
	0.13	7.9	0.11	0.77
	0.24	4.1	0.08	0.77
	0.35	2.9	0.22	0.92
	0.44	2.3	0.18	0.89

From these results the following can be observed:

- The auto-spectra are far from consistent with the empirical knowledge about long wave and business cycle fluctuations. The only positive thing about these spectra is that the model does show roots in the range of the Juglar and Kitchin type fluctuations. Because of their low Peak Power and modulus these are however hardly visible in the actual spectra. The long wave type fluctuations are missing entirely.

- Instead of showing long wave and business cycle behavior, the auto-spectrum of the growth rates is dominated by high frequency fluctuations that describe more than 50% of the total variance. This is very probably caused by the very irregular fluctuations in the growth rates during the 1870-1913 period as a result of a poor quality of the data. For this, also see the filtered stochastic component for this period in Figure 7.2 and the corresponding Maximum Entropy auto-spectrum in Figure 7.3.
- Just as for the MVAR model in terms of the first order differences from the previous section, the auto-spectrum of the level of the index is dominated by the ultra low frequency fluctuations. From this we see that the model describes an  $I(1)$  process which has a stochastic trend. Again this is caused by the possibly incorrect variance at the ultra low frequencies in the left hand side auto-spectrum that is amplified in the right hand side auto-spectrum by the PTF of the inverse first order differencing operator.

In short, these results confirm the finding from Chapter 19 that with one overall model it is difficult to estimate the dynamic behavior of a variable on all frequency ranges correctly because some frequencies may dominate the variance of the data representation (here the high frequency fluctuations in the growth rates) that is by chance used for estimating the model, while in another data representation fluctuations in other frequency ranges may actually be more important (here the long wave and business cycle fluctuations in the levels of the index). Figure 20.21 shows the confidence intervals of this traditional model while Figure 20.22 shows its backtesting results on the 1870-1999 period.

Figure 20.21 Confidence interval for the conditional annual distributions of AR model of first order differences of logarithm of National Product Index for the Netherlands. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.

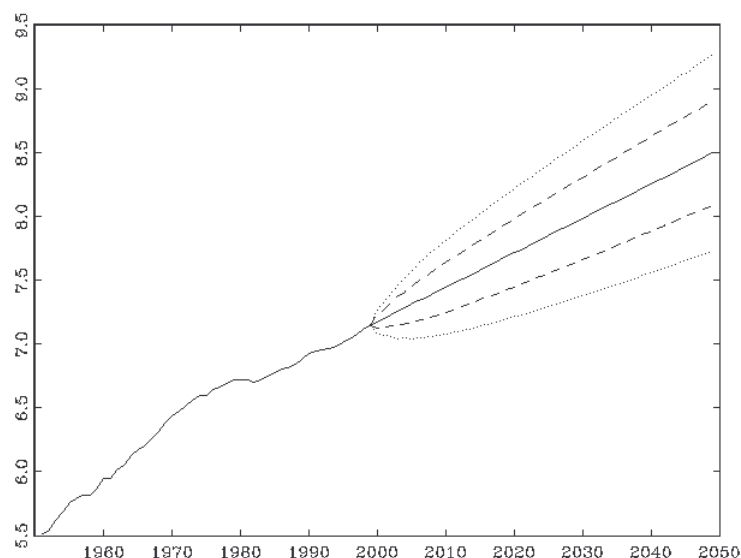
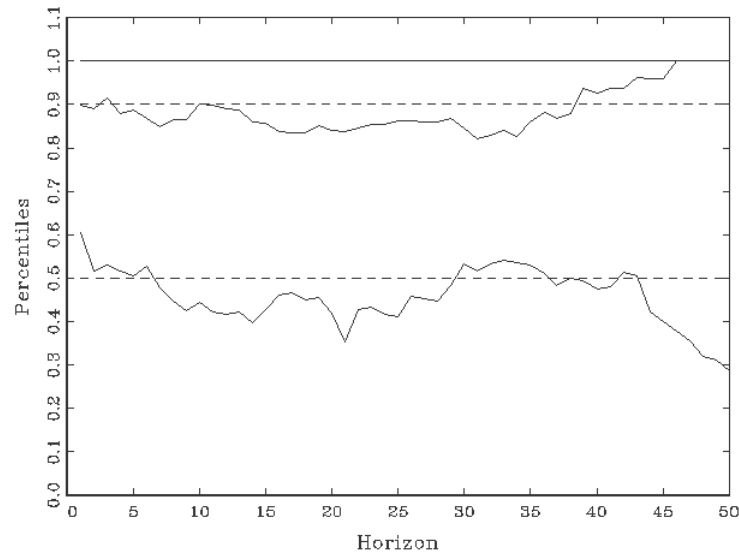


Figure 20.22 Backtesting results for AR model of first order differences of logarithm of National Product Index for the Netherlands tested on the 1870-1999 period.



From these results the following can be observed:

- The confidence intervals are wider than those of the MVAR model in terms of the first order differences, as shown in Figure 20.18. This is logical because now the first order differences for *all* frequencies are modeled instead of only specific parts of the frequencies, namely the  $[1/70, 1/30]$  and  $[1/15, 0.5]$  frequency ranges.
- The 50% percentiles of the confidence intervals indicate a practically constant expected growth rate without a prediction of for example the long wave fluctuations that are clearly visible in the confidence intervals of all MVAR models, even in those that are modeled in terms of the first order differences. This is consistent with the absence of the corresponding peaks in the auto-spectra shown in Figure 20.20. The dynamics and hence also the model predictions are dominated by high frequency fluctuations. The “flat” and hence not very informative predictions of the model compared to the more logical extension of the long wave fluctuations in the MVAR models suggest that the latter class of models might actually result in better long term predictions.
- Judging from the backtesting results, the confidence intervals are however not too wide. Up to an horizon of 40 years the model matches both the 50% and 90% confidence interval well. Because we already know that the dynamics of the model are incorrect, this again illustrates the important point that the confidence intervals should never be the only criterion for judging the performance of a scenario model. Another illustration that confidence intervals don’t tell the whole story is the fact that in this case estimating an AR(1) instead of an AR(10) model gives similar confidence intervals and performance in terms of the backtesting procedure while by definition, because of its low order, this model cannot describe any pseudo-periodic behavior at all and therefore must give a poor description of the dynamics of the variable. These are practical illustrations of the stylized example shown in Figure 20.5.

*Comparison with MVAR models*

The results for the conventional model can now be compared to those of the various MVAR models from the previous sections. Some important differences between the models are discussed next.

- Because one of the model objectives is to give an adequate description of the uncertainty in long wave type of fluctuation, using the long sample 1870-1999 is in a sense a necessity. Because in the conventional model it is not possible to use different samples for different frequency ranges, the dynamics of the model are unavoidably influenced by the poor quality and non-representative high frequency fluctuations of the 1870-1913 period. Note that the *low* frequency information in this sample may very well be useful. An MVAR model can use the low frequency information from this period, while at the same time discarding the high frequency information.
- Even when using the long sample and a high model order of  $p=10$ , the conventional model does not pick up the dynamics of the long wave type fluctuations which as we know are present in the data. One consequence is that this type of fluctuations is also not present in the long term predictions generated by the model (50% percentile). With one overall model it is difficult in general to estimate the dynamic behavior of variables on all frequency ranges correctly.
- Because of its one factor nature, for the conventional model it is only possible to change the wideness of the confidence intervals for all horizons at the same time.
- A related problem is that in the conventional model, dependencies between the model behavior in different frequency ranges cannot be modeled. An example of such dependencies is the level effect in interest rate variables which describes a dependency between the variance of the business cycle fluctuations and the underlying low frequency level of the interest rates. The next section presents MVAR models that can model this type of factor dependencies.
- The use of spectral densities improves the awareness of the dynamic properties of the models and forces one to think about and take a stand on the model behavior at the various frequencies. An important example is the modeling of the trends, that is the behavior of the model at the important ultra low frequencies. The conventional model has a stochastic trend. The exact stochastic behavior of the average growth rate per scenario is implicitly hidden in the estimated model and cannot be modeled explicitly. In the terminology of the Bunn and Salo (1993) scenario definition from section 1.3 this can impossibly be called a very “clear set of assumptions”. One might argue that the average growth rate in scenarios generated by the model is not unclear at all. After all, isn’t it equal to the 2.7% average growth during the sample period? The answer is a firm no! Even for long horizons there can be a considerable, implicit, amount of uncertainty about the long term average growth rate in the model. This is illustrated by the example in Figure 20.2.

## 20.3 Example 2: Factor dependencies

This section presents the second of four examples of the new frequency domain VAR scenario framework as described in section 20.1. The overview in Figure 20.9 shows how this example is related to the different components of the new framework. This example illustrates how complex factor dependencies (section 20.1.1) can be modeled within in the new framework by constructing a univariate MVAR model for the Long Term Interest Rate for the Netherlands including the level effect that is observed in both short and long term interest rates and described in respectively section 12.2.1 and 13.2.1. The model also illustrates the workings of the stochastic AR(1) trend model (section 20.1.2). Finally, the example shows how MVAR models can be applied to model non-index type variables such as Interest Rate variables instead of index type variables such as the National Product Index.

### 20.3.1 Long Term Interest Rate

Consider a scenario modeling case with very similar objectives as those described at the start of section 20.2.1, only now for the Long Term Interest Rate for the Netherlands instead of the National Product Index. Besides the obvious differences such as different available sample periods, the important difference between these examples is that interest rates are non-index type variables which may require a different modeling approach. The basic models for the two components of the MVAR model, the long wave and business cycle fluctuations, are described next.

#### *Long wave*

The properties of the long wave type fluctuations are based on the  $[1/70, 1/30]$  component of the Long Term Interest Rate for the 1814-1999 sample period with a five year observation frequency. Estimating an AR(3) model with the Yule-Walker method leads to the results shown on the left hand side of Figure 20.23 and Table 20.6. From this we see that the model describes a rather regular pseudo-period behavior with a period length of around 50 years as was to be expected. The volatility of the series, and hence the model, is 0.9% as can be seen from Table 13.2.

Figure 20.23 Auto-spectra from AR models for long wave (left) and business cycle component (right) of Long Term Interest Rate for the Netherlands.

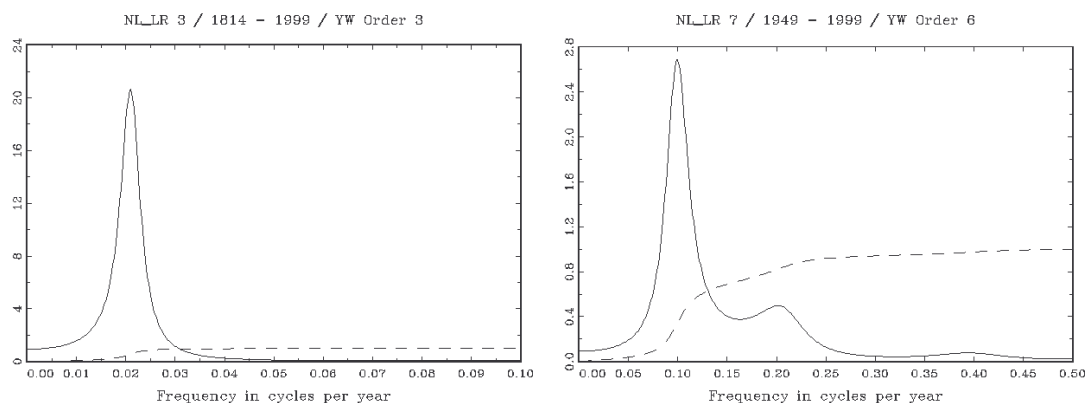


Table 20.6 Statistics of auto-spectra of AR models for long wave (left) and business cycle component (right) of Long Term Interest Rate for the Netherlands calculated from the roots of the models.

<i>NL Long wave AR(3) model</i>					<i>NL Business cycle AR(6) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>		<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1814-1999	0.02	47.6	0.51	0.93	1949-1999	0.10	10.1	0.38	0.92
						0.21	4.8	0.20	0.82
						0.40	2.5	0.05	0.76

### *Business cycle*

The right hand side of Figure 20.23 and Table 20.6 show the dynamic behavior of an AR(6) model estimated on the 1949-1999 sample for the  $[1/15, 0.5]$  component with a conventional one year observation frequency. This model, which is the same as the Maximum Entropy estimate from Chapter 13, describes the familiar Juglar and Kitchin type of pseudo-periodic behavior with period lengths of around respectively 10 and 4.5 years. The volatility of the series, and hence the model, is 0.8% as can be seen from Table 13.2.

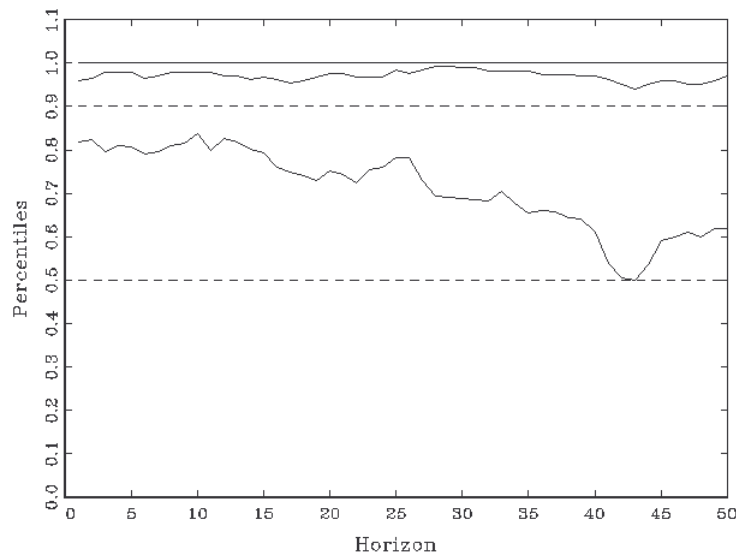
### *Backtesting*

The backtesting procedure described in section 20.1.3 can be applied to the MVAR model for the Long Term Interest Rate consisting of the long wave and business cycle AR models. The procedure assumes all non-modeled frequency components, those for the  $[0, 1/70]$  and  $[1/30, 1/15]$  frequency ranges, as given. The results are shown in Figure 20.24 for the 50% and 90% confidence intervals. From these results the following can be observed:

- The confidence intervals of the model are too wide compared to uncertainty in the historical time series. This can especially be seen from the 50% confidence intervals. For horizons up to 15 years, 80% of the realizations fall within regions that according to the model should describe only 50% of the realizations.
- From analyzing the confidence intervals of the model starting from every point in the sample (not shown here), it becomes clear that when interest rates are low, the model can describe confidence intervals that include negative interest rates. This confirms that in some cases the confidence intervals produced by the model are too wide.



Figure 20.24 Backtesting results in terms of realized and expected percentages of observations in confidence intervals of conditional distributions with various horizons for MVAR model of Long Term Interest Rate for the Netherlands tested on the 1814-1999 period.

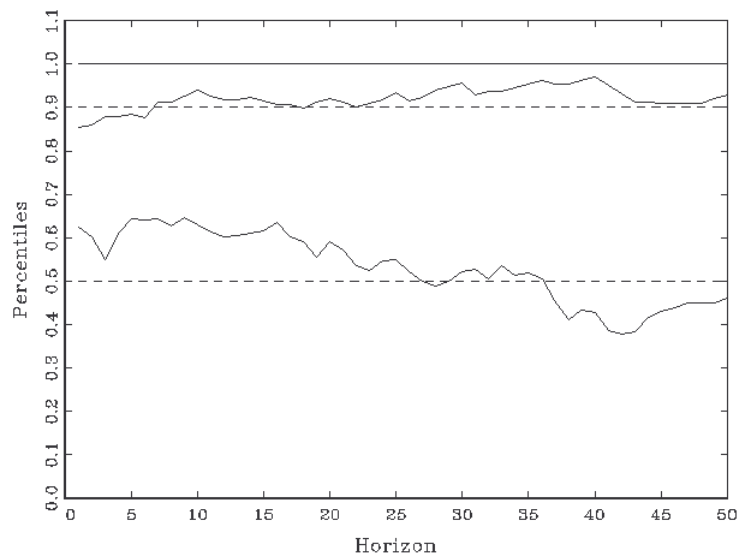


The reason for the wide confidence intervals of the MVAR model, including negative interest rates, is as follows. The volatility of the business cycle model is 0.8% which is based on the 1949-1999 sample period during which interest rates were the highest in the entire history, as can be seen in Figure 13.1. During the prewar and interwar periods however, interest rates were lower while the volatility of the business cycle fluctuations was only 0.2% then, as can be seen from Table 13.2. Because of the constant volatility of the model, the 0.8% volatility is just as well applied during the low interest rate periods causing confidence intervals that include negative interest rates. Because interest rate in the past have never been below zero, this means that confidence intervals from the model are too wide in the case of low interest rates.

The phenomenon described here is actually the *level effect* that is observed in both short and long term interest rates and described in respectively section 12.2.1 and 13.2.1. The best solution for the wide confidence intervals would therefore obviously be to explicitly model this effect that describes a positive relation between the volatility at the business cycle frequencies and the underlying level of the low frequency fluctuations. Section 20.3.3 describes an MVAR model that actually has this property. Here we perform a simple test to confirm our hypothesis by setting the volatility of the business cycle model at a constant 0.4% which is a time weighted average of the low 0.2% for the prewar and interwar periods and the high 0.8% for the postwar period. The backtesting results from Figure 20.25 indeed show that:

- *On average* the confidence intervals of the model are now much more consistent with the historical uncertainty in the time series. However, considering high and low interest rate situations separately, still indicates confidence intervals that are respectively too narrow and too wide.
- Confidence intervals that include negative interest rates no longer occur when starting simulations from historical time periods with low interest rates.

Figure 20.25 Backtesting results in terms of realized and expected percentages of observations in confidence intervals of conditional distributions with various horizons for MVAR model of Long Term Interest Rate for the Netherlands with a constant business cycle volatility of 0.4% instead of 0.8%, tested on 1814-1999.



### 20.3.2 Modeling a stochastic AR(1) trend

Before we move on to the modeling of the level effect in interest rates, the modeling of the ultra low frequencies in non-index type variables, the average or expected value in this case, is considered. Just as was the case in section 20.2.4, a stochastic modeling of the trend or average can be used to:

- Compensate for the variance in the confidence intervals of the model that is missing because some frequency ranges are not modeled.
- Model the uncertainty in the long term trend or level of a variable.

Now suppose that our example MVAR model, still with the fixed 0.4% standard deviation for the business cycle model, is extended by the stochastic AR(1) trend model (20.1.9) as described in section 20.1.2 with

- An expected value that is equal to the final value from the sample,  $\mu=5.0\%$ .
- A volatility of the trend of  $\sigma=1.0\%$ .
- An autoregressive parameter of  $\beta=0.7$ .

The stochastic AR(1) trend model can start from the last values from the sample for the non-modeled components, in this case the value of the Long Term Interest Rate minus the values of its long wave and business cycle components. The non-modeled components will probably often be those from low frequency ranges which renders it consistent to model them by means of the stochastic AR(1) trend model.

The choice for the volatility in the trend model of  $\sigma=1.0\%$  is the average of the outcomes of the final two procedures, number 4 and 5, to determine the stochastic trend parameters as described at the end of section 20.1.2.

4. Here the  $[1/70, 1/30]$  component has a standard deviation of 0.9% while the time weighted average standard deviation of the  $[1/15, 0.5]$  component is 0.4%. In order to achieve the total historical standard deviation of 1.7% from Table 13.1, the standard deviation of the trend component should be 1.4%. This is so because  $\sqrt{0.9\%^2 + 0.4\%^2 + 1.4\%^2} = 1.7\%$  which uses the fact that there is zero correlation between the three component models.
5. Without using unsmoothing techniques, the 1814-1999 sample for the Long Term Interest Rate produces 117 long term average estimates with a sample period of 70 years. These averages have a standard deviation of 0.6%.

Furthermore, together with the 0.7 autoregressive parameter, the  $\sigma=1.0\%$  value produces consistent backtesting results as described by procedure number 3 at the end of section 20.1.2. Figure 20.26 shows these backtesting results when assuming the non-modeled components are not known. Hence, the variance of the non-modeled frequency components is substituted by additional variance for the components that are modeled.

Figure 20.26 Backtesting results of MVAR model of Long Term Interest Rates for the Netherlands tested on the 1814-1999 period including an AR(1) stochastic trend together with substituting the variance of the non-modeled frequency components by additional variance of the modeled components.

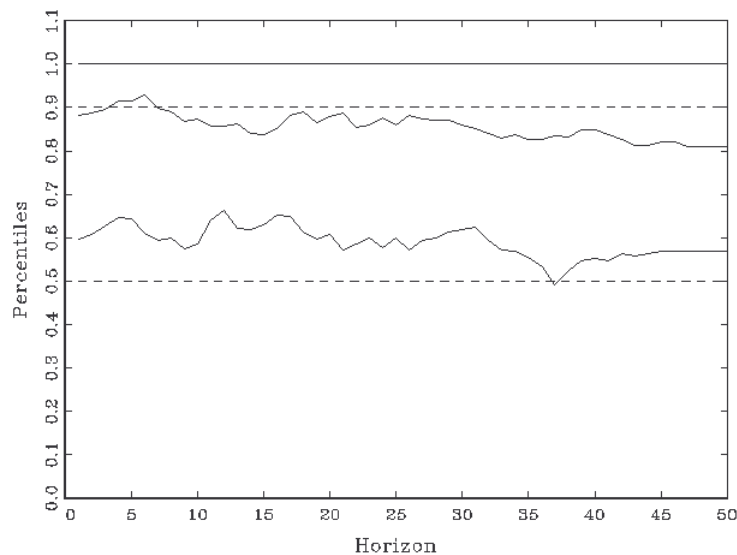


Figure 20.27 Confidence intervals for conditional annual distributions of MVAR model of Long Term Interest Rate for the Netherlands with a stochastic AR(1) trend with an expected value of 5.0%, a standard deviation of 1.0%, an autoregressive parameter of 0.7 and two separate AR models for the long wave and business cycle frequency range. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.

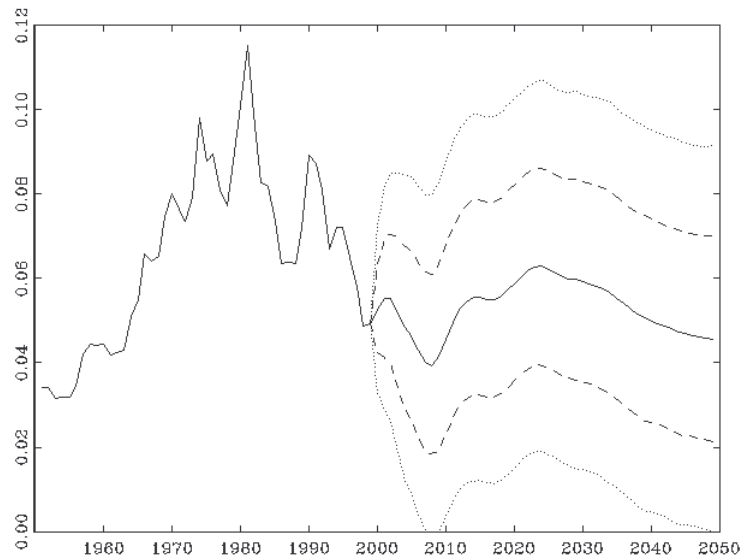


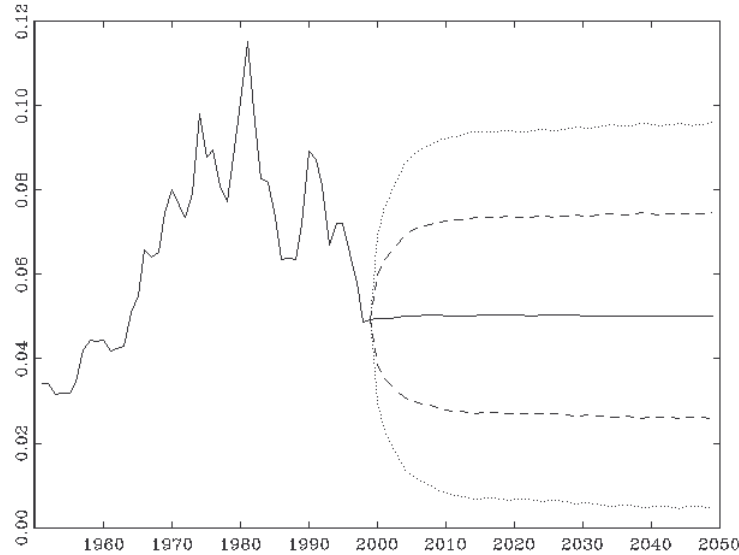
Figure 20.27 shows the confidence intervals produced by the same model starting from the 1999 values. From these results the following can be observed:

- Because of the additional volatility introduced by the stochastic trend, the model now again shows (small) probabilities of negative interest rates. This will be even more so in case of lower initial interest rate levels.
- The confidence intervals produced by the model are clearly symmetrical around the conditional expectations. This is not consistent with the strongly skewed distribution shown in Figure 13.3.
- The conditional expectations wildly fluctuate according to the combination of the predictions from the long wave and business cycle AR models. This in itself is nothing strange. However, for a scenario analysis application, one might wonder whether such a dominating conditional expectation pattern is desirable. After all, scenario analysis is exactly supposed *not* to be predictive. Of course, constant expected values are still a prediction as well.

As we shall see in the next section the problem of the negative interest rates can be reduced by adequately modeling the level effect. This will also produce more skewed distributions. In case the distributions are still not skewed enough one might consider using a skewed distribution for the stochastic trend such as a lognormal distribution or a truncated Normal distribution as described in section 20.1.5. For producing more “neutral” conditional expectations one can use *stationary starting values*. This means that the simulations from the two AR models are started from initial values that are equal to the long term averages of the model, all zero in this case, instead of starting from the actual values at the end of the sample period. The conditional expected values of the models will then no longer show the damped fluctuations towards the long term averages because they are already at those

averages at the start of the simulations. An example of the confidence intervals from such simulations is shown in Figure 20.28.

Figure 20.28 Confidence intervals for conditional annual distributions of MVAR model of Long Term Interest Rate for the Netherlands with a stochastic AR(1) trend and two separate AR models for the long wave and business cycle frequency range, both applied with stationary starting values. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



### 20.3.3 Level effect

In the previous sections we mentioned several times that the absence of an explicit modeling of the level effect in interest rates can (partially) cause the following related problems:

- Inconsistent backtesting results for the conditional distributions of the MVAR model with a constant volatility for the business cycle AR model.
- Confidence intervals that include negative interest rates, especially when the simulations start from low levels of interest rates.
- Symmetrical conditional distributions instead of the desired skewed distributions as those empirically observed and shown in Figure 13.3.

The level effect in long term interest rates is described in for example section 13.2.1 and stylized fact *SF 42* in section 17.2.1. The effect describes a positive relation between the volatility of the business cycle fluctuations and the level of the underlying low frequency fluctuations in interest rates. The level effect is an example of dependencies between the properties of the stochastic factors of an MVAR model and can be modeled as follows.

- Use the following estimated linear relation from Figure 13.3 for describing the standard deviation of the  $[1/15, 0.5]$  component at some time  $t$  in scenario  $s$  as a function of the sum of the values of the AR(1) trend component and the  $[1/70, 1/30]$  component at the same time  $t$  in scenario  $s$ .

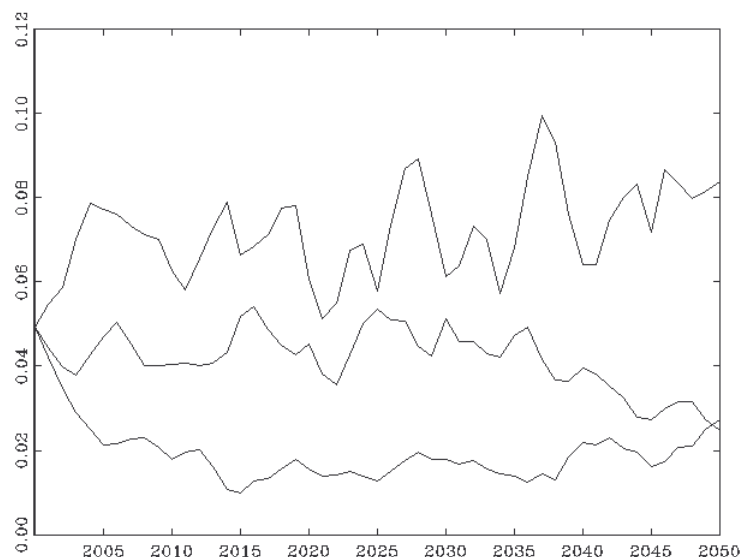
$$\sigma_{[1/15, 0.5], s, t} = -0.38\% + 0.18 \cdot \mu_{Trend + [1/70, 1/30], s, t} \quad (20.3.1)$$

In this way the fixed volatility of the business cycle component is replaced by a state dependent volatility where the state is the underlying level of the interest rate as determined by the other factors modeled. To prevent zero volatilities, a minimum volatility of 0.2% is applied.

- First simulate the trend and long wave components of the MVAR model. Based on their values at time  $t$  in scenario  $s$ , the required standard deviation of the business cycle model can be calculated from (20.3.1). Next, using the procedure described in section 3.7.5 of adjusting the standard deviations of a VAR model, the volatility of the business cycle AR(6) model can be adjusted accordingly without changing the dynamic properties of the model. After having sampled a value from this adjusted model for time  $t$  in scenario  $s$ , the same procedure is repeated for time  $t+1$ ,  $t+2$  and so on. The parameters of the business cycle AR(6) model therefore change during the simulations depending on the simulations from the trend and long wave models.

Figure 20.29 shows three scenarios that are sampled from the MVAR model with the stochastic trend and stationary starting values from the previous section, but now with the level effect modeling as described here. The three scenarios are chosen such that the level of interest stays respectively at a low, medium and high value during the entire horizon. Their averages are approximately 2%, 4% and 7% which according to the (truncated) relation (20.3.1) leads to business cycle standard deviations of respectively 0.2%, 0.3% and 0.9%. The resulting level effect is clearly visible in the scenarios.

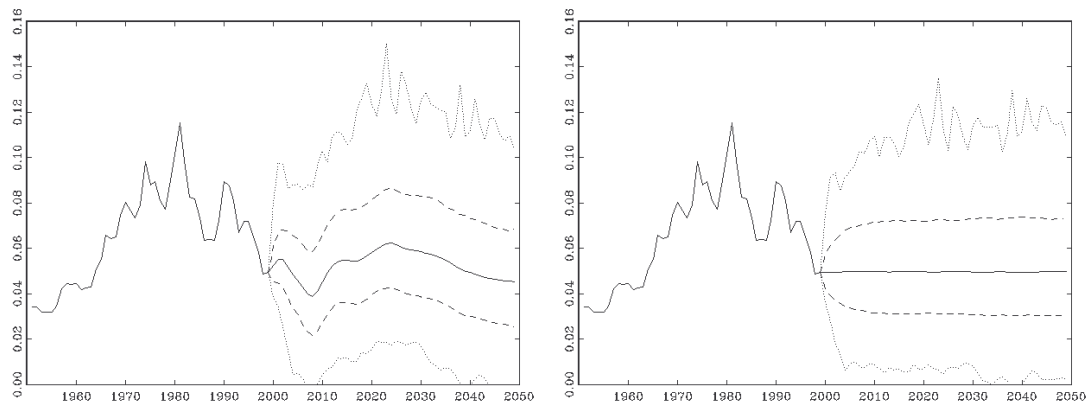
Figure 20.29 Three example interest rate scenarios from MVAR model which includes a level effect for the volatility of the business cycle frequency range.



As indicated in section 20.1.3, the confidence intervals of an MVAR model with factor dependencies such as the level effect in interest rates can no longer be determined analytically but have to be determined by means of simulation. Figure 20.30 shows the confidence intervals of the final MVAR model for the Long Term Interest Rate based on 10.000 scenarios. The results for both starting from the historical values and using stationary starting values are shown. From these results the following can be observed:

- We know that the level effect is by construction included in the model.
- The level effect also introduces a skewness in the distributions of the model.
- Negative interest rates can still occur.

Figure 20.30 Confidence intervals for conditional annual distributions of MVAR model of Long Term Interest Rate for the Netherlands with a stochastic AR(1) trend and two separate AR models for the long wave and business cycle frequency range, both applied with (right) and without (left) stationary starting values. The volatility of the business cycle model is determined by the level effect (20.3.1). Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



The result of this section is an MVAR model with a separate and therefore both flexible and transparent modeling of

- Long wave and business cycle dynamics, consistent with those empirically observed,
- Uncertainty in the long term average Long Term Interest Rate,
- Confidence intervals that are consistent with historical uncertainty and
- The level effect for the volatility of the business cycle fluctuations which also causes skewed overall distributions.

The only remaining problem of this MVAR model is that negative interest rates can still occur, especially when starting from low interest rate levels. In order to prevent this from happening the following adjustments come to mind.

- Use a higher AR parameter in the stochastic trend model thereby causing a slower increase in the conditional variance of the model. This will only reduce the probability of negative interest rates for the relatively short horizons.



- Also model a level effect for the volatility of the long wave fluctuations based on the level of the underlying trend. Although such a level effect might very well be supported by the data, modeling it means that low interest rate scenarios will remain low during the entire simulation horizon (unless another trend model is applied).
- Truncate the distributions for one or more of the three component models at the required (low) levels. This kind of truncating is also relevant for example to prevent having negative price inflations and is the topic of the third example described in the next section.

## 20.4 Example 3: Truncated distributions

This section presents the third of four examples of the new frequency domain VAR scenario framework as described in section 20.1. The overview in Figure 20.9 shows how this example is related to the different components of the new framework. This example illustrates the workings of the Truncated VAR (TVAR) models (section 20.1.5) by constructing a univariate model for the Consumer Price Index of the Netherlands with the restriction that the implied annual inflation rates need to be strictly positive. Along the way another empirical level effect is revealed and modeled, this time in the consumer price inflation. This second example of how complex factor dependencies (section 20.1.1) can be modeled within in the new framework concerns a slightly different type of level effect as the one in the previous section. Here the volatility of a component depends on the *transformation* (first order differences) of another component.

### 20.4.1 Consumer Price Index

Consider a scenario modeling example with very similar objectives as those described at the start of section 20.1.1, only now for the Consumer Price Index for the Netherlands instead of the National Product Index. Besides the obvious differences such as different available sample periods, the important difference between these examples is that, because of for example the inflation limiting policy of the European Central Bank, the inflation rates from this scenario model are required to be strictly positive. As we know from the analysis in Chapter 10, deflation often occurred before the second World War but hardly ever since. The basic models for the two components, the long wave and business cycle fluctuations, are described next.

#### *Long wave*

The properties of the long wave type fluctuations are based on the  $[1/70, 1/40]$  component of the natural logarithm of the Consumer Price Index for the 1813-1999 sample period with a five year observation frequency. Estimating an AR(3) model with the Yule-Walker method leads to the results shown on the left hand side of Figure 20.31 and Table 20.7. From this we see that the model describes a rather regular pseudo-period behavior with a period length of around 50 years as was to be expected. The volatility of the series, and hence the model, is 8.5% as can be seen from Table 10.2.

### Business cycle

The right hand side of Figure 20.26 and Table 20.6 show the dynamic behavior of an AR(7) model estimated on the 1949-1999 sample for the  $[1/15, 0.5]$  component with a conventional one year observation frequency. This model, which is the same as the Maximum Entropy estimate from Chapter 10, describes the familiar Juglar and Kitchin type of pseudo-periodic behavior with period lengths of around respectively 10 and 4.5 years. The volatility of the series, and hence the model, is 1.7% as can be seen from Table 10.2.

Figure 20.31 Auto-spectra from AR models for long wave (left) and business cycle component (right) of Consumer Price Index for the Netherlands.

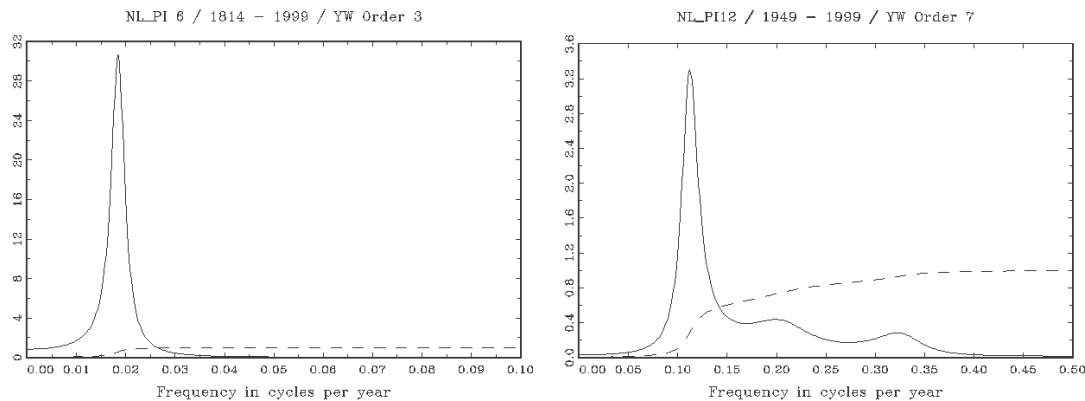


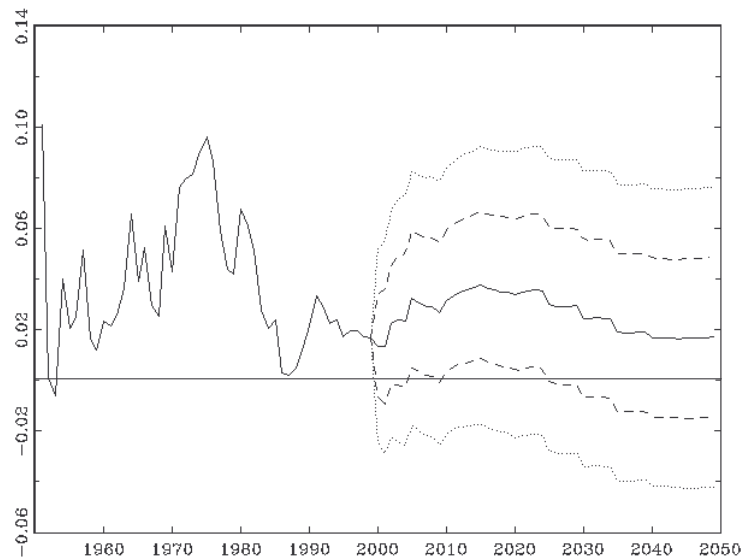
Table 20.7 Statistics of auto-spectra of AR models for long wave (left) and business cycle component (right) of Consumer Price Index for the Netherlands calculated from the roots of the models.

<i>NL Long wave AR(3) model</i>					<i>NL Business cycle AR(7) model</i>				
	<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>		<i>Freq</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>
1813-1999	0.02	44.1	0.50	0.95	1949-1999	0.11	9.0	0.33	0.94
						0.21	4.9	0.21	0.79
						0.33	3.1	0.11	0.83

### Confidence intervals

Backtesting the model on the historical time series of the Consumer Price Index is not very useful because of the major structural breaks in the past behavior of the index. These breaks, as described in Chapter 10, concern the change from an average 0% inflation rate into a positive average inflation rate somewhere around the start of the 20<sup>th</sup> century and the (related) fact that deflation hardly ever occurred anymore after the second World War. Figure 20.32 does show the confidence intervals of the inflation rates implied by the MVAR model in terms of the logarithm of the index. These confidence intervals of the first order differences of the original model are calculated by the procedure described in section 3.6.4. The model assumes a fixed logarithmic linear trend with a average inflation rate of 2.5% per annum, a typical forward looking expected inflation rate in current scenario applications.

Figure 20.32 Confidence interval for the conditional annual distributions annual inflation rates as implied by MVAR model of logarithm of Consumer Index for the Netherlands with a linear trend with a 2.5% inflation rate and two separate AR models for the long wave and business cycle frequency range. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



From these confidence intervals the following can be observed:

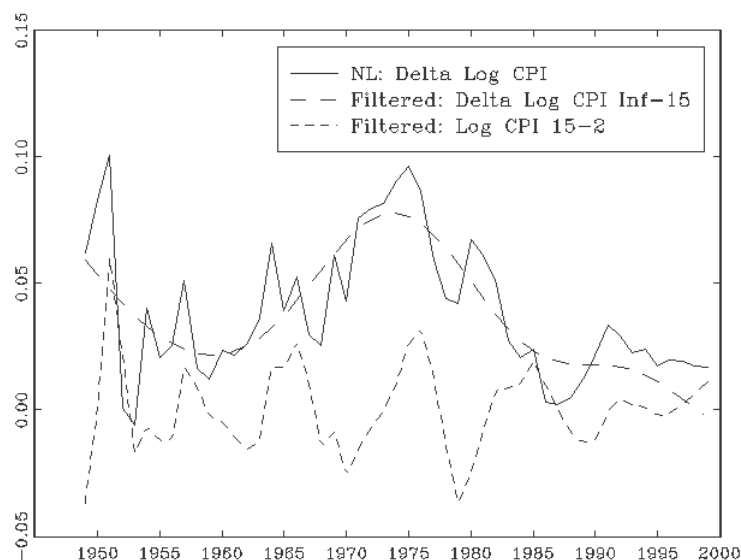
- The stepwise pattern in the percentiles of the confidence intervals, and hence in the scenarios themselves, are caused by the fact that first the level of the long wave component of the logarithm of the index is simulated at a five year observation frequency and then completed by means of linear interpolation. This causes five year periods of identical first order differences of the logarithm of the index, and hence of the inflation rates.
- Inflation rates are not strictly positive at all, contrary to the objective of the model. Instead, there is a substantial probability of negative inflation rates in the confidence intervals of the MVAR model, especially at the end of the horizon. Note that this is even so without substituting the variance of the non-modeled components by additional variance of the long wave and business cycle models and also without a stochastic trend model. Such model adjustments would increase the probability of negative inflation rates even further.

One possible cause of the negative inflation rates might be that the historical average inflation rate for the 1950-1999 sample period is 3.8% while in the MVAR model we assume it to be only 2.5% without explicitly changing the volatility around this lower level of inflation rates. The modeling of some kind of level effect in the Consumer Price Inflation, similar to that for the Long Term Interest Rate in the previous section, is therefore considered in the next section.

### 20.4.2 Another level effect

Although this was not described in Chapter 10 on the empirical behavior of Consumer Price Indices, upon further inspection there indeed seems to be a level effect in the volatility of the Consumer Price Inflation. That this effect remained unnoticed before, is because in Chapter 10 the focus of the analysis in the frequency domain was on the *level* of the logarithm of the indices, instead of on their *first order differences*, although at the end of section 10.1 there is a short note on a dependency between inflation volatility and the level of inflation. The level effect describes a positive relation between the business cycle volatility of the *price index* and the underlying low frequency level of the *inflation rate*. It is observed for the postwar period in the Netherlands, the United Kingdom and the United States. Figure 20.33 illustrates the level effect for the Netherlands by showing the postwar inflation rate together with its low frequency  $[0, 1/15]$  component and the  $[1/15, 0.5]$  component of the logarithm of the index. The  $[0, 1/15]$  component is calculated as the first order differences of the sum of the  $[0, 1/T]$  trend component and the  $[1/T, 1/15]$  periodic component of the logarithm of the index. From Figure 20.33, it is clear that the business cycle volatility in the level of the price index is relatively high when inflation rates are high and vice versa. The difference with the level effect in interest rates is that here the volatility of the business cycle fluctuations in the *level* of the index depends on the low frequency fluctuations in the *first order differences* of the index. Modeling this specific type of factor dependency in terms of transformations of the variables poses no problems in the MVAR model. What matters is that the required volatility of the representation in which the business cycle model is specified can be calculated. Transforming the low frequency components in order to calculate the required volatility should never be a problem.

Figure 20.33 Illustration of the level effect in the Consumer Price Inflation for the Netherlands. Shown are the postwar inflation rates together with the  $[0, 1/15]$  low frequency component of the inflation rates and the level of the  $[1/15, 0.5]$  business cycle component of the original logarithm of the index.



Following a similar approach as in section 12.2.1 and 13.2.1 for the level effect in the Short and Long Term Interest Rates, the level effect in the Consumer Price Inflation for the Netherlands was estimated as the linear relation shown in Figure 20.34. The relation is based on the 1949-1999 sample period and uses three intervals for the level of the inflation rate on the horizontal axis: 0%-2%, 2%-4% and 4%-8% with respectively 13, 18 and 20 observations. The standard deviation of the  $[1/15, 0.5]$  component of the logarithm of the price index is on the vertical axis. Figure 20.35 shows the confidence intervals of the same MVAR model as in section 20.4.1 but now with the level effect of Figure 20.34 incorporated with a minimum standard deviation of 0.2%. The confidence intervals are numerically obtained from 10.000 simulations of the model.

Figure 20.34 Estimated level effect in the Consumer Price Inflation for the Netherlands, linearly linking the level of the inflation rate (horizontal axis), calculated as the first order differences of the  $[0, 1/15]$  component of the logarithm of the index, to the standard deviation of the  $[1/15, 0.5]$  component of the logarithm of the index (vertical axis).

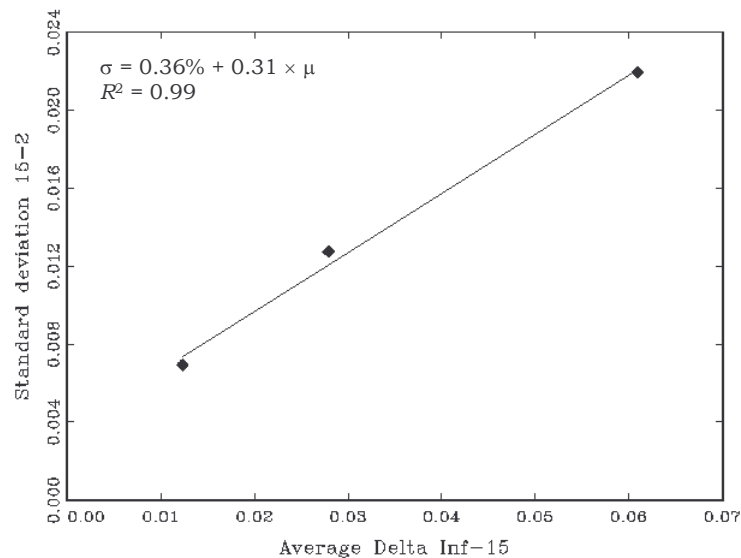
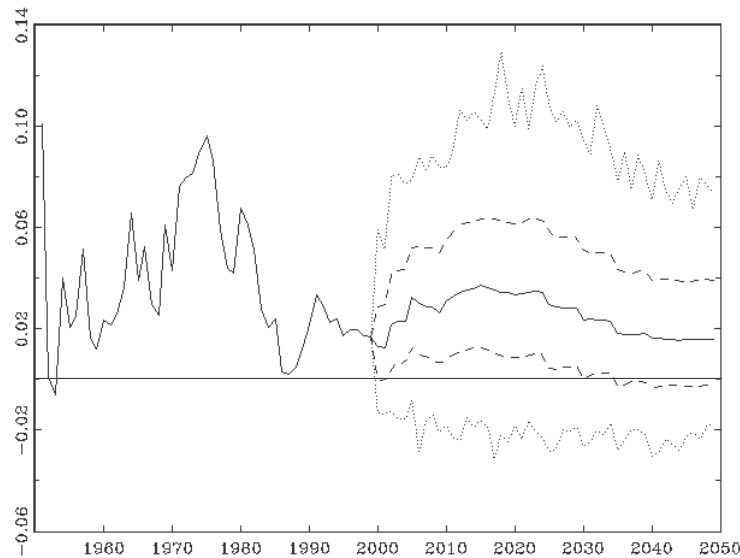


Figure 20.35 Confidence intervals for conditional annual distributions of MVAR model of Consumer Price Index for the Netherlands with a fixed 2.5% inflation rate trend and two separate AR models for the long wave and business cycle frequency range. The volatility of the business cycle model is determined by the level effect from Figure 20.29. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



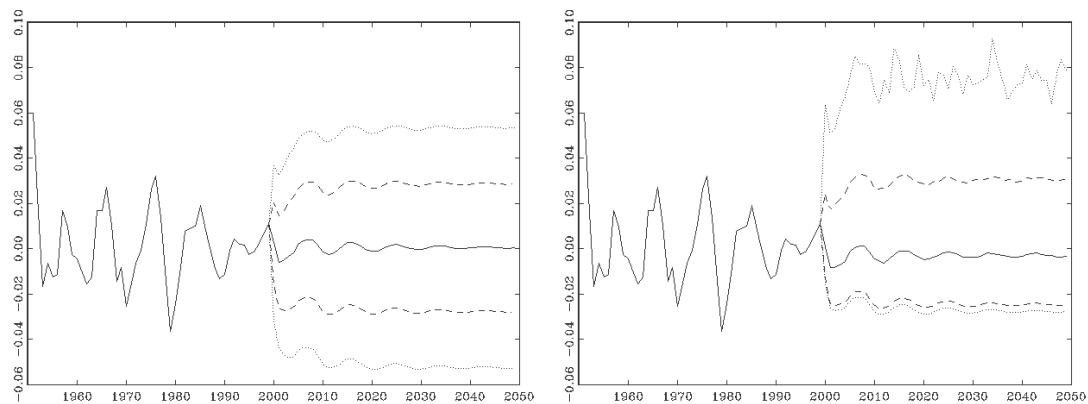
From comparing these confidence intervals with those from Figure 20.32 the following can be observed:

- By introducing the level effect, the distribution of the inflation rates has become skewed.
- By introducing the level effect, the probability of negative inflation rates has indeed decreased. For example the 5% percentile at the end of the horizon is now around a 0% inflation rate while without the level effect it was still around -1.75%. However, negative inflation rates still occur with a probability of more than 5%. The model in the next section uses another approach to limit the probability of negative inflation rates: Truncated VAR (TVAR) models.

### 20.4.3 Positive inflation rates

In the previous section we saw that modeling the level effect in the Consumer Price Inflation rates alone is not enough to prevent the MVAR model from producing confidence intervals with negative inflation rates. An obvious way to prevent negative inflation rates from occurring is to use the Truncated VAR (TVAR) models described in section 20.1.5. As a univariate example of these models, consider the original AR(7) model for the  $[1/14, 0.5]$  component of the logarithm of the Consumer Price Index described in section 20.4.1. The unconditional expected value and standard deviation of this model are respectively 0% and 1.7%. Suppose we want to truncate the distribution of the model at the 5% percentile of the original distribution, without changing the 0% expected value (first moment) and 1.7% standard deviation (second moment). In this case, the required factors in the procedure of section 3.8.3 turn out to be  $\beta_\mu = -0.20$  and  $\beta_\sigma = 1.5$ . Figure 20.36 shows the confidence intervals of the resulting truncated AR model, calculated on 10.000 scenarios.

Figure 20.36 Confidence intervals for conditional annual distributions of AR model of  $[1/14, 0.5]$  component of logarithm of Consumer Price Index for the Netherlands, both with (right) and without (left) truncating at the 5% percentile of the original distribution. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



From these results the following can be observed:

- The unconditional median (50% percentile) of the truncated model is a little below 0% because the distribution has become skewed. The long term average is still 0% by construction.
- The distribution on the right is correctly truncated at the 5% percentile of the original distribution on the left, without changing the expected value and standard deviation of the distribution. Of course higher moments of the distribution, such as the skewness, have changed.
- Another result, which is not shown here, is that the scenarios from the truncated VAR model behave “normally” in the sense that they are not absorbed by the truncation level. Furthermore, the truncation induces a kind of level effect. Low level scenarios are less volatile than high level scenarios.

Note that here the truncation was done such that the first two moments of the distribution of the model itself did not change. As indicated in section 20.1.5 and 3.8.3, this may just as well be the first two moments of the distribution of some transformation of the model, for example the first order differences. Doing this for more than one representation of the model at the same time is not possible because there are only two parameters in the procedure,  $\beta_\mu$  and  $\beta_\sigma$ .

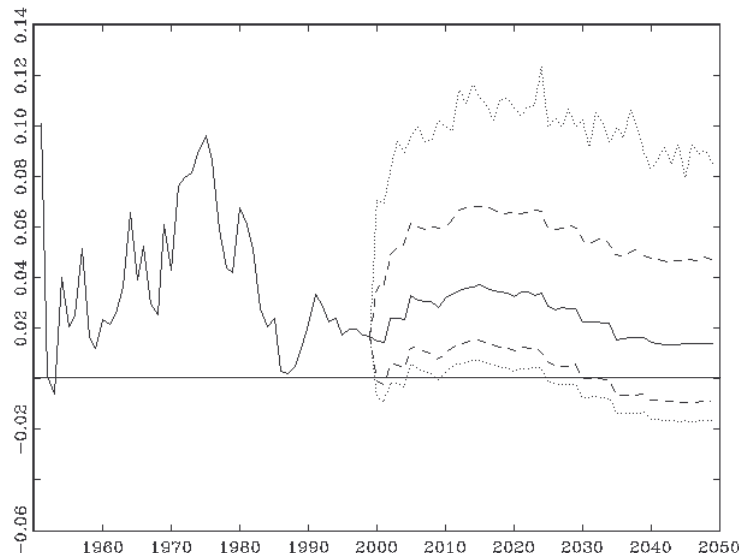
#### *Truncated MVAR model*

Finally, the TVAR models are applied in the MVAR model from section 20.4.1 for the Consumer Price Index of the Netherlands in order to limit the probability of negative inflation rates. There are two differences with the previous example application of truncated AR models. The first is that now the distributions of two component models need to be truncated: the long wave model and the business cycle model. The second difference is that the objective is now not to truncate the distributions of the models themselves, but the distributions of their *first order differences* that make up the inflation rate. Judging from the confidence intervals of the non-truncated model in Figure 20.32 it seems that we may need a truncation level at approximately the  $\alpha=10\%$  percentile of the distribution of the first order differences. In this case, the required factors in the procedure of section 3.8.3 turn out to be  $\beta_\mu=-0.10$  and  $\beta_\sigma=1.08$



for the business cycle model and  $\beta_\mu = -0.60$  and  $\beta_\sigma = 1.85$  for the long wave model. These are chosen such that the first two moments of the distributions of the original models, and not their first order differences, have not changed. As noted before, maintaining the same first two moments for some (single) other representation of the models, for example the first order differences, is also possible. To be able to judge the isolated effect of using the truncated distributions, the level effect described in the previous section is not included here. Figure 20.37 shows the confidence intervals of the first order differences (inflation rates) of this truncated MVAR model, calculated on 10.000 scenarios.

Figure 20.37 Confidence interval for the conditional annual distributions of truncated MVAR model of logarithm of Consumer Index for the Netherlands with a linear trend with a 2.5% inflation rate and two separate AR models for the long wave and business cycle frequency range. The distributions of the first order differences of the two AR models are truncated at their 10% percentiles in such a way that the first two moments of the original models do not change. Shown are the 0.1%, 5%, 50%, 95% and 99.9% percentiles.



Comparing these confidence intervals to those from Figure 20.32 for the corresponding non-truncated MVAR model, the following can be observed:

- The minimum values of the truncated distributions are approximately the 5% percentiles from the non-truncated distributions instead of the intended 10% percentiles. Because of this, the model is less effective in reducing the probability of negative inflation rates than intended. The reason for this unexpected effect is the “diversification” effect described at the end of section 20.1.5.
- The expected value and standard deviation of the original model have not changed, by construction. The median (50% percentile) is a little lower than in the non-truncated model because the distribution has become skewed with especially a wider range for the top 5% probability mass. Furthermore, the truncation induces a kind of level effect. Low level scenarios are less volatile than high level scenarios.

So, truncating the distributions of the models at the 10% percentile of the distribution of the first order differences is effective in reducing the probability of negative inflation rates. However, negative inflation rates can still occur. Possible extensions of the model that can help resolve this problem, but are not further explored here, are the following.

- The remedies for the “diversification” effect in truncated MVAR models as suggested at the end of section 20.1.5.
- Combine the MVAR model with the level effect from the previous section with the truncated distributions. After all, the level effect is an empirical fact and both approaches reduce the probability of negative inflation rates. Hence, combining them will be more effective. In this case, given some value of  $\alpha$ , the required truncation levels need to be determined numerically based on a large number of scenarios as done in Figure 20.35. The diversification effect needs to be taken into account here as well.
- If a stochastic trend is modeled that can generate negative inflation rates, this distribution needs to be truncated as well.

## 20.5 Example 4: EigenValue Restricted VAR models

This section presents the last of four examples of the new frequency domain VAR scenario framework as described in section 20.1. The overview in Figure 20.9 shows how this example is related to the different components of the new framework. All previous sections of this chapter described a different aspect of *univariate* MVAR models. This section describes a *multivariate* application. Because all univariate MVAR principles from the previous sections can be generalized rather straightforward into a multivariate setting, these MVAR principles will not be discussed here. Instead, this example illustrates the working of the EigenValue Restricted VAR (EVR VAR) models (section 20.1.6) by constructing a six dimensional VAR model for the business cycle fluctuations in the Dutch economy with restrictions on the types of business cycle fluctuations present in the model. The objective is to let these be confined to the well known Juglar and Kitchin type business cycle fluctuations as found in Part III. The EVR VAR model will be compared to unrestricted VAR models of different orders in terms of the multivariate spectral densities. Also, the effects of changing the dynamic properties of the model, in terms of its eigenvalues and eigenvectors, on the spectral densities will be explored by using the procedure outlined in section 20.1.7.

### Models

We will estimate various six dimensional VAR models for the business cycle components of the same variables as used in Chapter 19 for the postwar 1949-1999 sample period for the Netherlands. This allows for a direct comparison of the performance of the EVR VAR models, in terms of being consistent with the empirical “benchmark” knowledge from Part III, with the conventional VAR model applications tested in Chapter 19. The variables used are shown in Table 20.8 and are the same as the filtered stochastic components for the postwar period from the relevant chapters of Part III. We focus only on a business cycle model as one of the components of a more general multivariate MVAR model. The principles of this model can also be applied to a long wave model and can also be combined with the factor dependencies and truncated distributions from the previous sections. The EVR VAR model is then used to describe the basic dynamics of the variables while the level effect and truncated distributions change the volatility and shape of the distributions, while maintaining the same dynamics.

Table 20.8 Data representation used for estimating six dimensional business cycle VAR models for the 1949-1999 sample period for the Netherlands.

	<i>Representation</i>
National Product Index	[1/15, 0.5] log(NP)
Consumer Price Index	[1/15, 0.5] log(PI)
Real Industry Wage Index	[1/15, 0.5] log(WI)
Short Term Interest Rates	[1/15, 0.5] SR
Long Term Interest Rates	[1/15, 0.5] LR
Real Equity TRR Index	[1/15, 0.5] log(TR)

*Reference business cycle dynamics*

Figure 20.38 and Table 20.9 give a summary of the business cycle dynamics that the EVR VAR models are meant to describe. Figure 20.38 shows the univariate Maximum Entropy estimates of the auto-spectra of the six variables as taken from the relevant chapters of Part III. These spectra are dominated by the familiar Juglar and Kitchin type of pseudo-periodic behavior with period lengths of around respectively 10 and 4.5 years. From the bivariate Maximum Entropy estimates, also the bivariate properties in terms of the coherence and phase spectra relative to the National Product Index are available. Besides properties of the univariate spectra, Table 20.9 also shows the coherence and phase numbers at the Juglar and Kitchin peak frequencies. These are taken from the summarizing Tables 17.12 and 17.14.

*Judging model performance*

For judging the performance of the models in terms of describing business cycle dynamics that are consistent with the empirical knowledge, a very similar approach is followed as for the testing of existing scenario applications of VAR models in Chapter 19. The dynamics of the models are tested by comparing the spectral densities with those from Figure 20.38. A first difference with the procedure described in section 18.2.4 is that it now concerns the *levels* of the logarithms of the index variables instead of the *first order differences*. A second difference is that now also the complete coherence and phase spectra are compared instead of only the values at the Juglar and Kitchin frequencies in the reported tables. Finally, also the actual eigenvalues and eigenvectors of the models are reported in terms of the polar coordinates as described in section 3.3.2.

What follows are three sections. Section 20.5.1 reports the results for unrestricted VAR models of various orders. This section:

- Confirms the relevant shortcomings of current VAR model applications as found in Chapter 19, but now in terms of the same data used for estimating the alternative EVR VAR models later on which allows for a direct comparison of the model performance.
- Provides new information by also looking at the complete coherence and phase spectra instead of at two frequencies only.
- Provides input as to what the EVR VAR model should look like in terms of the values and multiplicity of the different eigenvalues.

Section 20.5.2 explores the effects on the model dynamics when directly changing the eigenvalues and eigenvectors of a VAR model by using the procedure outlined in section 20.1.7. This section

- Provides input for the estimation of the EVR VAR models.
- Shows how the dynamics of a VAR model can be adjusted by hand.

Section 20.5.3 finally estimates EVR VAR models for various model orders with the appropriate restrictions on the eigenvalues.

Figure 20.38 Reference normalized auto-spectra from the relevant univariate Maximum Entropy estimates from Part III for the variables from Table 20.8. The fine dotted lines show the corresponding 95% white noise confidence intervals.

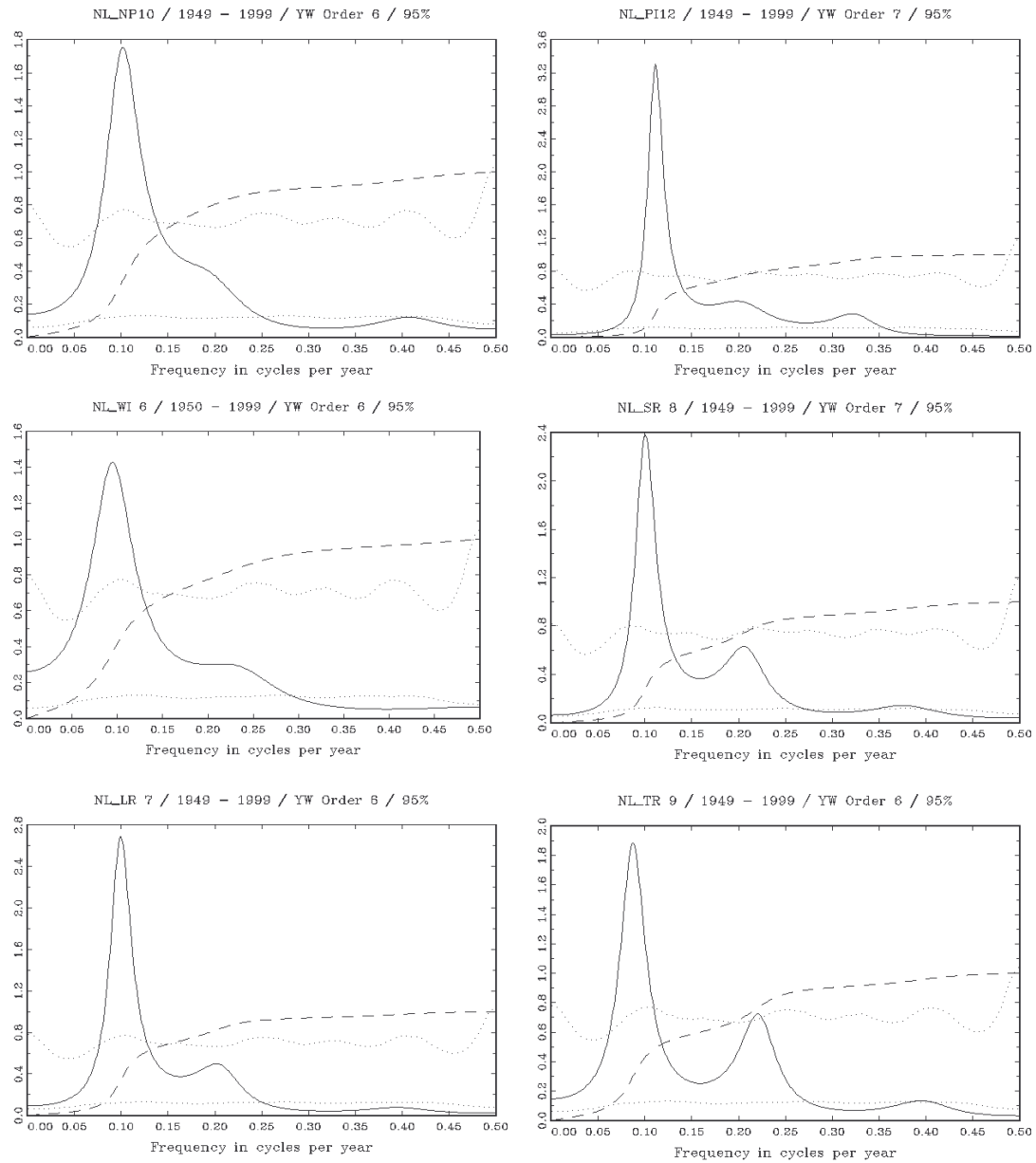


Table 20.9 Properties of Juglar and Kitchin peaks in normalized auto-spectra from the relevant univariate and bivariate Maximum Entropy estimates from Part III for the variables from Table 20.8.

	<i>Juglar</i>					<i>Kitchin</i>				
	<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>	<i>Period</i>	<i>PP</i>	<i>Mod</i>	<i>Coh</i>	<i>Phase</i>
NP	9.9	0.4	0.9	-	-	4.9	0.2	0.7	-	-
PI	9.0	0.3	0.9	0.6	-4.2	4.8	0.2	0.8	0.6	-1.4
WI	10.6	0.4	0.8	0.9	-1.6	4.2	0.2	0.7	0.5	-0.9
SR	10.1	0.3	0.9	0.9	-1.9	4.8	0.2	0.8	0.6	-0.6
LR	10.1	0.4	0.9	0.9	-2.7	4.8	0.2	0.8	0.7	-0.6
TR	11.5	0.3	0.9	0.2	4.7	4.5	0.2	0.9	0.9	0.7
Average	10.2	0.4	0.9	0.7	-	4.7	0.2	0.8	0.6	-

### 20.5.1 Unrestricted six dimensional business cycle VAR models

For reasons described at the end of the introduction of section 20.5, this section reports the dynamic properties of unrestricted VAR models for the business cycle fluctuations in the six variables of Table 20.8. The models are estimated by the MLS procedure described in section 3.7.1. Because there are no restrictions, this is equivalent to OLS estimation for each equation separately. The reason for using the MLS procedure is that it directly provides us with a multivariate residual sum of squares that can be compared to the ones from the estimated EVR VAR models in terms of for example a multivariate  $R^2$ .

#### *Order selection criteria*

The results of the order selection criteria from Table 20.10 indicate a high order model as being appropriate. Only the SC criterion still points to a low order model, although if order  $p=5$  would be included, this criterion points to a high order model as well. These results are identical to those from Table 19.11 for models in terms of the first order differences of the variables from Table 20.8 used here. Note that the best results in section 19.2 were indeed obtained for the high order models. Based on the order selection criteria, unrestricted VAR models for all orders  $p=1, \dots, 4$  were estimated. To illustrate the main findings however, it suffices to report only the results for the extreme orders  $p=1$  and  $p=4$ .

Table 20.10 Ordering of model orders according to the lowest values of the various order selection criteria based on unrestricted MLS estimates.

	$FPE(p)$	$AIC(p)$	$SC(p)$	$CAT(p)$	$HQ(p)$
1	4	4	1	4	4
2	3	3	2	3	3
3	2	2	4	2	2
4	1	1	3	1	1

#### *Standard deviations*

Because the model standard deviations of the variables can be changed with the procedure described in section 18.1.4 without changing the dynamic properties of a model, only the normalized spectra are reported here. The results for the unrestricted VAR(4) model in Table 20.11 indicate that, especially for high order unrestricted VAR models, the model standard deviations can be very different from the sample estimates.

Table 20.11 Model and sample standard deviations for unrestricted VAR(4) model for the variables from Table 20.8. The last column shows the relative sample volatilities compared to the volatility of the filtered National Product Index.

	<i>Model Stdev</i>	<i>Sample Stdev</i>	<i>Relative Sample Stdev</i>
NP	1.7%	2.0%	1.0
PI	1.5%	1.7%	0.9
WI	1.8%	2.1%	0.9
SR	1.8%	1.7%	0.9
LR	0.9%	0.8%	0.4
TR	16.1%	17.9%	9.1

*Unrestricted VAR(1) model*

The following figures and tables describe the dynamic properties of the estimated unrestricted VAR(1) model for the variables from Table 20.8. Figures 20.39, 20.40 and 20.41 respectively show the auto-spectra, coherence spectra and phase spectra for this model. Table 20.12 shows the spectral properties at the Juglar and Kitchin frequencies. Table 20.13 reports the eigenvalues and eigenvectors of this model in terms of the polar coordinates as described in section 3.3.2. In this and all similar subsequent tables the eigenvalues are grouped based on the frequency intervals  $[0, 0.12]$ ,  $[0.12, 0.25]$  and  $[0.25, 0.50]$  for respectively the Juglar, Kitchin and high frequency type fluctuations.

Figure 20.39 Normalized auto-spectra from unrestricted MLS estimates for six dimensional VAR(1) model for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

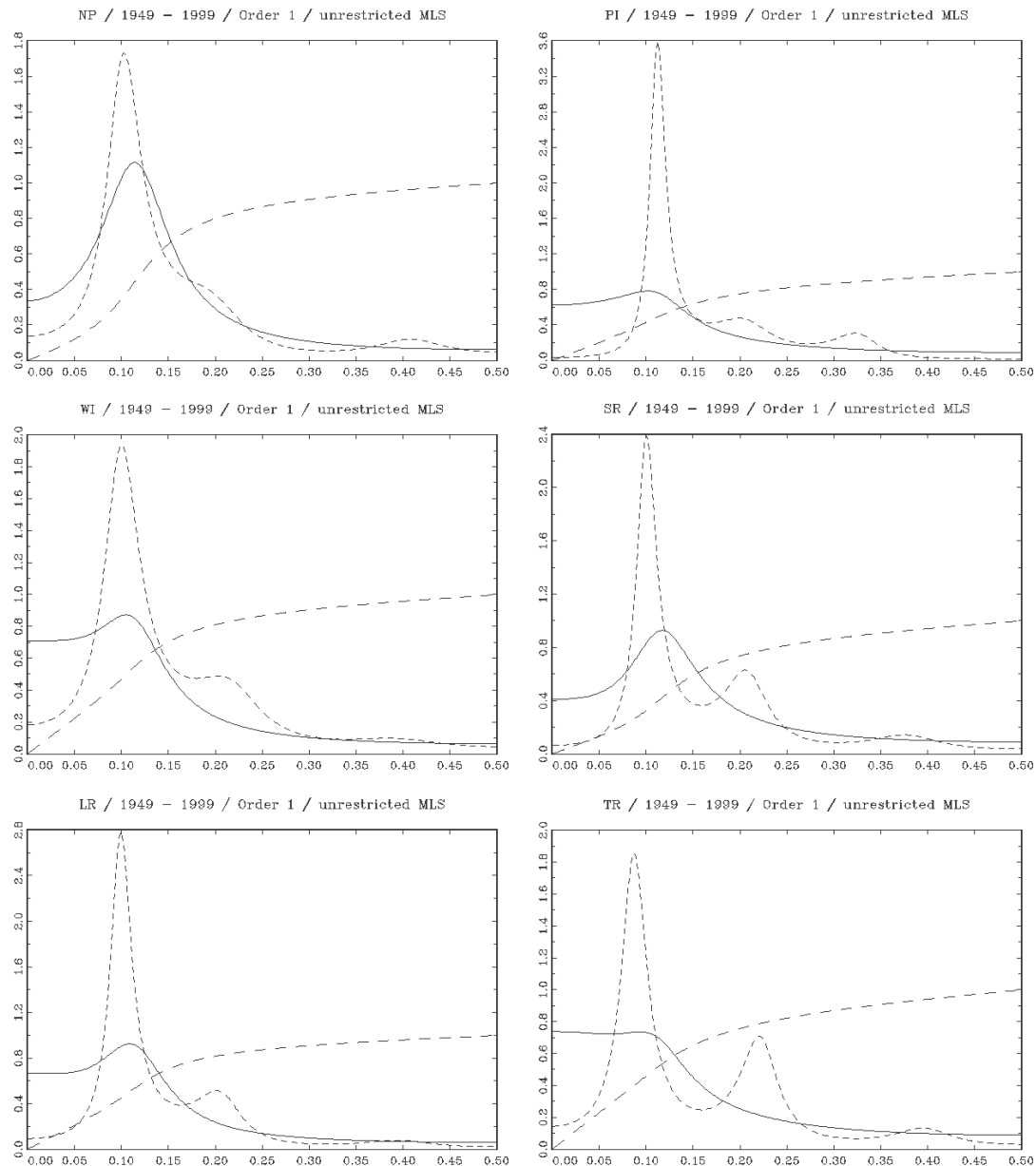




Figure 20.40 Coherence spectra relative to the NP index from unrestricted MLS estimates for six dimensional VAR(1) model for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

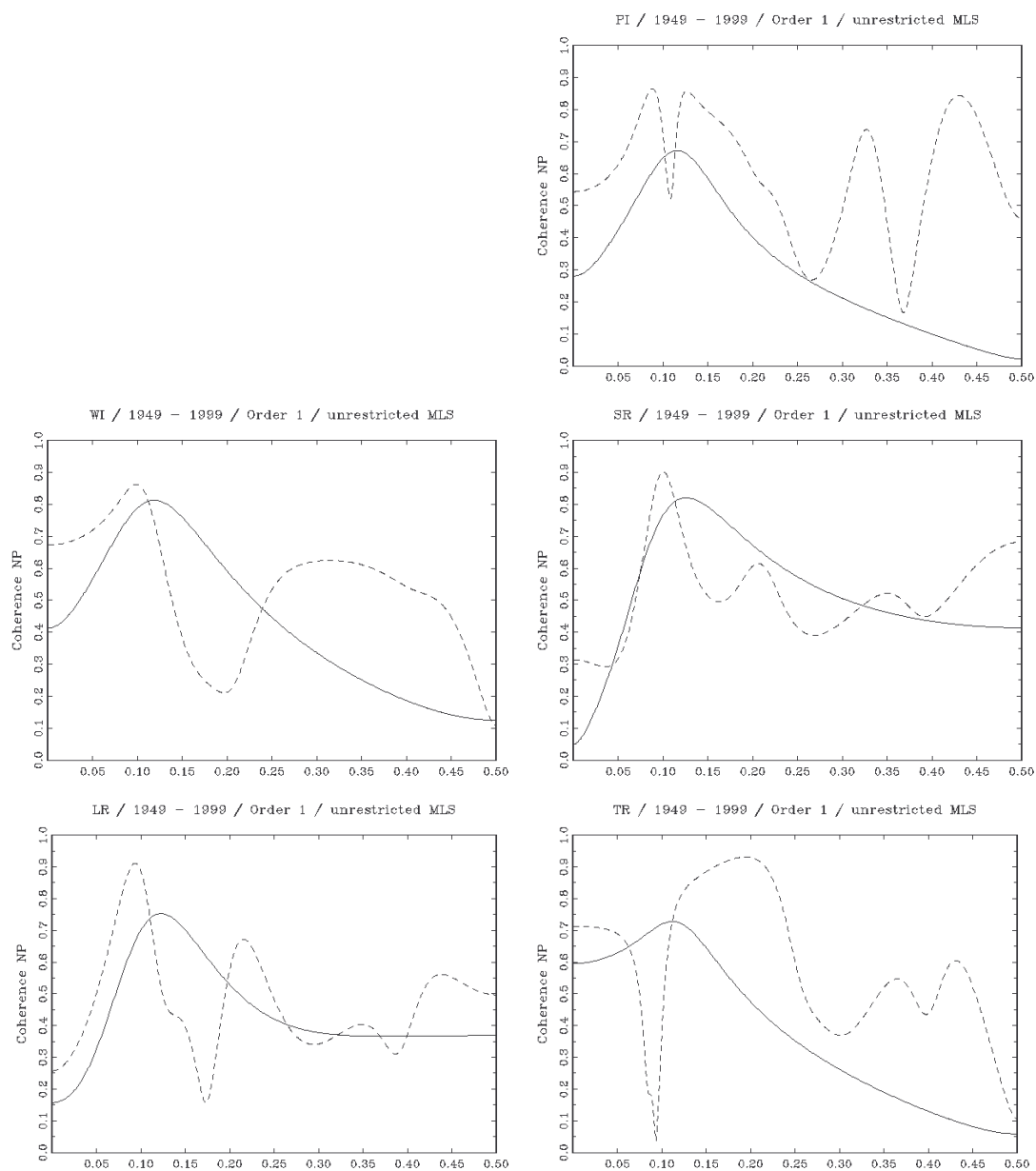


Figure 20.41 Phase spectra in years relative to the NP index from unrestricted MLS estimates for six dimensional VAR(1) model for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

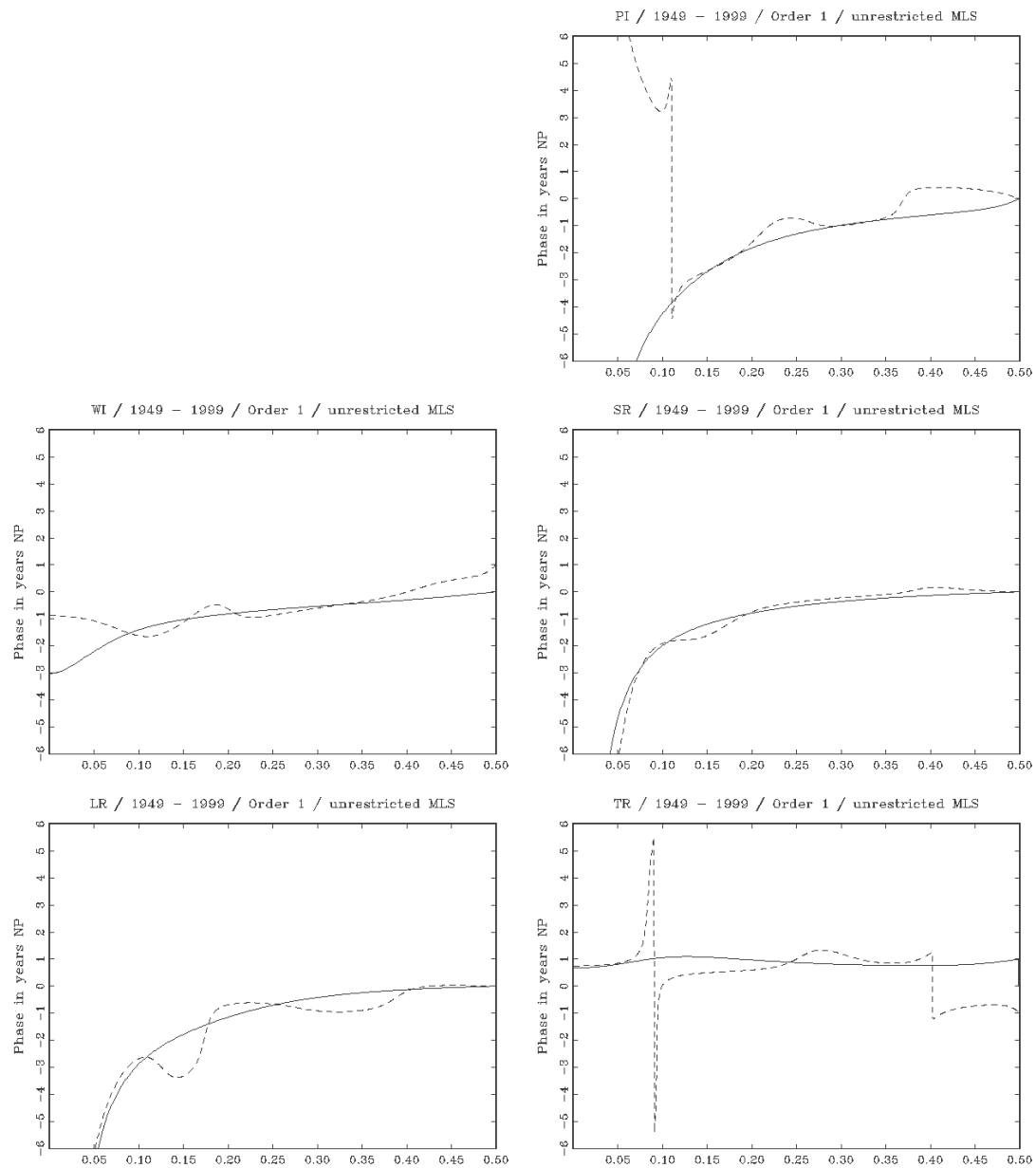


Table 20.12 Peak Power, coherence and phase in years relative to the NP index for Juglar and Kitchin frequencies from unrestricted MLS estimates for six dimensional VAR(1) model for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	PP	Coh	Phase	PP	Coh	Phase
NP	0.6			0.2		
PI	0.7	0.6	-4.3	0.2	0.3	-1.6
WI	0.7	0.8	-1.4	0.2	0.5	-0.7
SR	0.6	0.8	-2.0	0.3	0.6	-0.7
LR	0.7	0.7	-2.9	0.2	0.5	-0.9
TR	0.7	0.7	1.1	0.2	0.4	0.9
Average	0.7	0.7		0.2	0.5	

Table 20.13 Eigenvalues and eigenvectors in polar coordinates of unrestricted MLS estimates for six dimensional VAR(1) model for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>			
	1	2	3	4
R	0.8	0.5	0.4	0.4
$\omega$	0.11	0.06	0.0	0.0
$1/\omega$	8.8	16.6		
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>			
	1	2	3	4
NP	1.0	1.0	1.0	1.0
PI	0.5	1.6	-8.0	0.8
WI	0.8	2.0	6.4	-3.1
SR	0.7	0.8	-6.6	1.6
LR	0.3	1.1	3.4	0.7
TR	3.4	8.1	-85.8	1.5
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>			
	1	2	3	4
NP	0.0	0.0	0.0	0.0
PI	-4.1	-3.8	0.0	0.0
WI	-1.4	-6.1	0.0	0.0
SR	-1.6	2.9	0.0	0.0
LR	-2.6	2.0	0.0	0.0
TR	1.0	-0.8	0.0	0.0

From these results for the unrestricted VAR(1) model the following can be observed:

- The *eigenvalues and eigenvectors* of the unrestricted VAR(1) model from Table 20.13 consist of two pairs of complex conjugate eigenvalues and two positive real eigenvalues. The real eigenvalues describe low frequency fluctuations in the model that, as we know, are by definition not there because of the applied high-pass filter. The frequency of the first complex eigenvalue matches reasonably with Juglar type fluctuations. The modulus of 0.8 is however lower than the 0.9 from Table 20.9. Instead of the Kitchin type fluctuations, the second complex eigenvalue described low frequency fluctuations that are not present in the series. The relative modulus of the eigenvector corresponding to the first eigenvalue are reasonably consistent with the relative volatilities of the series shown in Table 20.11. For example, the corresponding fluctuations have a relatively small amplitude in the Short Term Interest Rate (SR) and a relatively high amplitude in

the real equity Total Return index (TR). Also the phase numbers from this eigenvector seem reasonable when compared to the phase numbers from Table 20.9. All variables are lagging the National Product index (NP) except the real equity Total Return index (TR) that shows a lead of one year.

- The *auto-spectra* of the unrestricted VAR(1) model, as shown in Figure 20.39 are clearly inconsistent with the reference spectra. The low valued real eigenvalues and the low frequency second eigenvalue with a low modulus cause significant variance at the ultra low frequencies that is by definition not present in the series. The Juglar spectral peak is only mildly visible because of the presence of these other eigenvalues. The auto-spectra from the unrestricted VAR(1) model are a kind of smooth average of the univariate reference spectra. This averaging property can also be seen from the relatively high frequency of the Juglar eigenvalue and its lower modulus.
- The *coherence and phase spectra* of the model, as shown in Figure 20.40 and 20.41, are much smoother than the reference spectra. This is caused by the fact that the six dimensional VAR(1) model has less eigenvalues ( $n \times p = 6 \times 1 = 6$ ) than the typical two dimensional VAR(6) models used for the Maximum Entropy estimates ( $n \times p = 2 \times 6 = 12$ ) and therefore also less flexibility. These spectra are smooth averages of the more volatile reference spectra. The phase numbers of the VAR(1) model in Table 20.12 are therefore not very different from those from the reference spectra in Table 20.9. The coherence at the Juglar and Kitchin frequencies are lower because of the smoother pattern in the coherence spectra. The average coherence over all frequencies as reported in Table 20.30 is pretty much the same.

The results from this unrestricted VAR(1) model first of all confirm finding number 7 from section 19.3. A sufficient number of eigenvalues is not enough to guarantee that a multivariate model will actually also describe the expected univariate dynamics. A six dimensional VAR(1) model can have three pairs of complex conjugate roots which should be enough to describe the Juglar, Kitchin and high frequency spectral peaks that are typically present in the univariate reference spectra from Figure 20.38. Apparently the MLS estimator prefers more smooth spectra to minimize the multivariate residual sum of squares. Despite the same number of eigenvalues, a major difference with the univariate models is of course that now the *same* eigenvalues have to be used for each value instead of a (slightly) *different* set for each variable in the separate univariate models. Intuitively more appealing would be that the multivariate model would *combine* the information from the univariate time series to obtain a better estimate of the business cycle dynamics.

The results also confirm finding number 9 from section 19.3. Phase numbers are not very different across models and are robust in that sense. Apparently these are picked up well by the criteria used in the MLS estimation procedure. From Chapter 19 we know that this also holds for other types of estimation procedures such as the Yule-Walker method.

*Unrestricted VAR(4) model*

The results for the unrestricted VAR(2) and VAR(3) models are not reported here because they show the same shortcomings as the VAR(1) model. They do have only complex eigenvalues which can be attributed in three sets to respectively the Juglar, Kitchin and high frequency fluctuations. However, major changes occur for the unrestricted VAR(4) model of which the dynamics properties are shown in Figures 20.42, 20.43 and 20.44 and Tables 20.14 and 20.15.

Figure 20.42 Normalized auto-spectra from unrestricted MLS estimates for six dimensional VAR(4) model for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

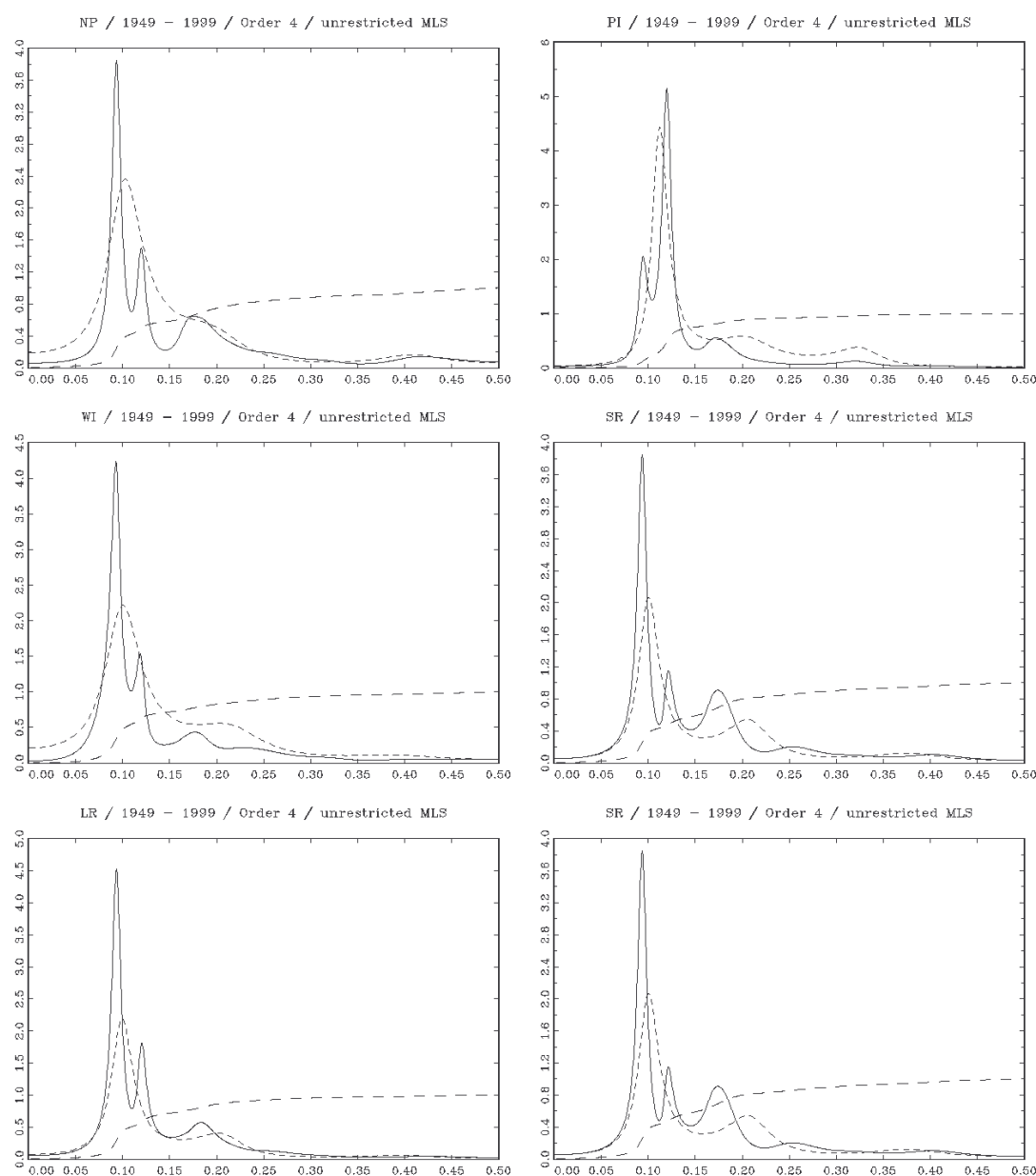


Figure 20.43 Coherence spectra relative to the NP index from unrestricted MLS estimates for six dimensional VAR(4) model for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

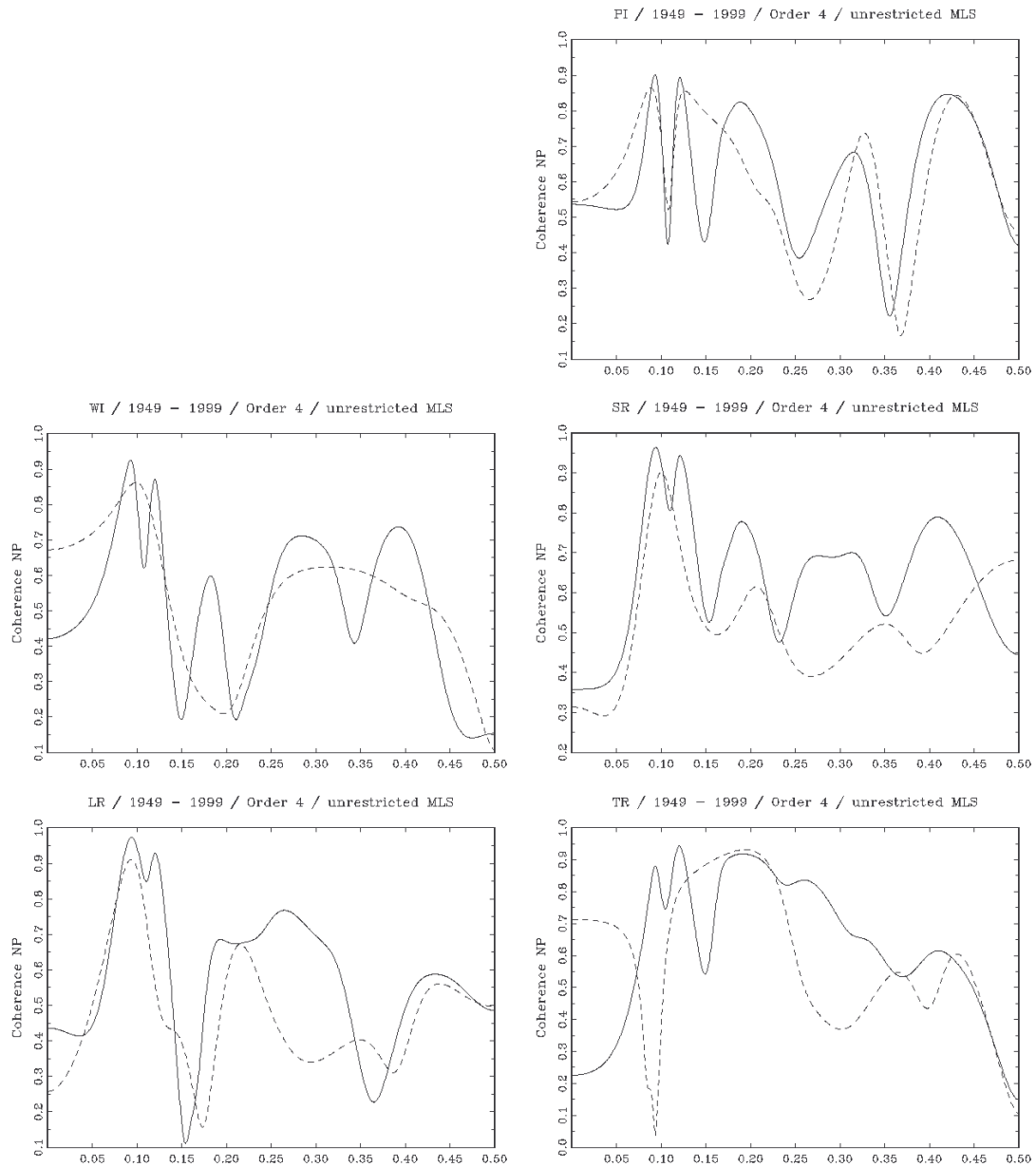


Figure 20.44 Phase spectra in years relative to the NP index from unrestricted MLS estimates for six dimensional VAR(4) model for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

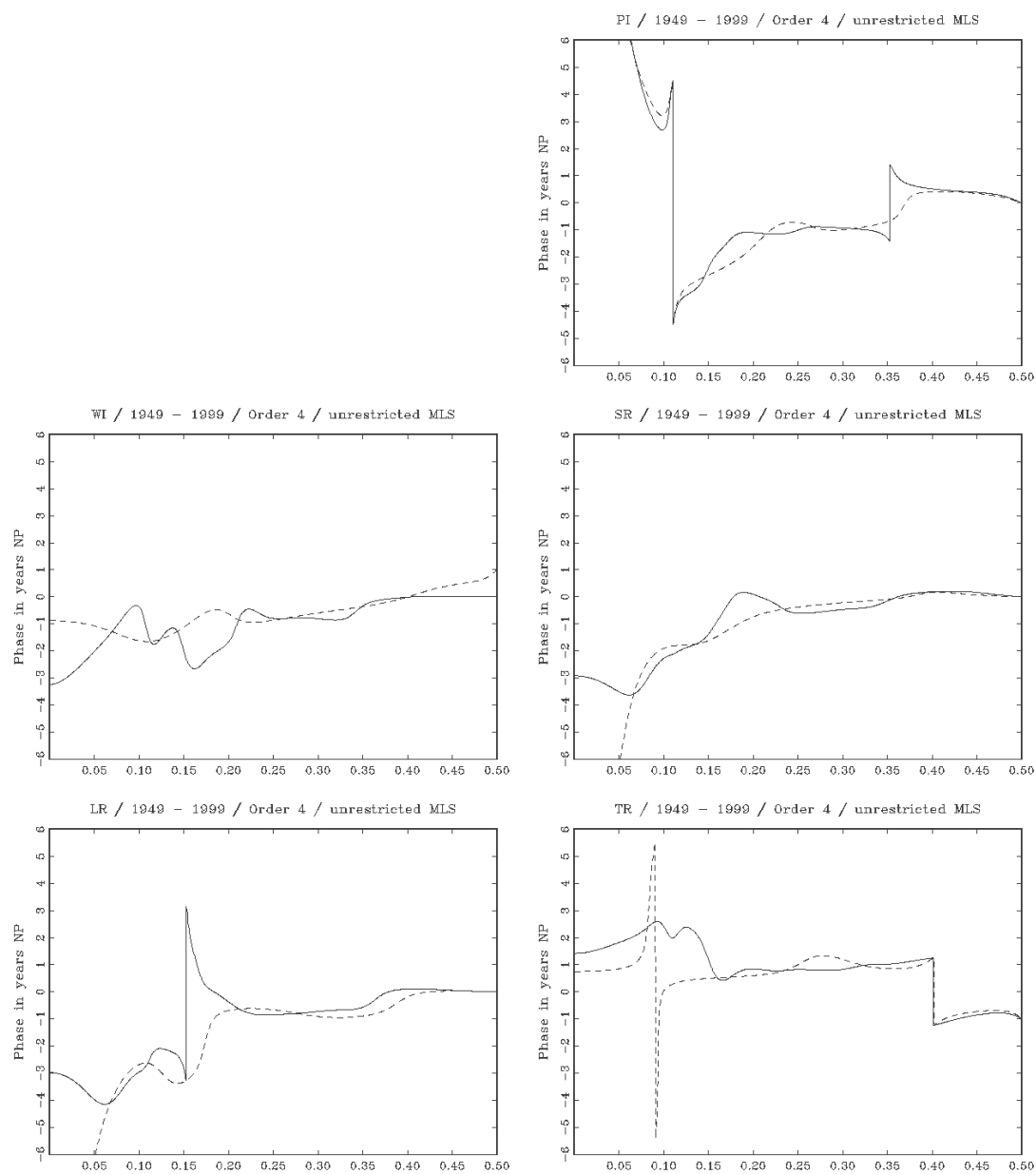




Table 20.14 Peak Power, coherence and phase in years relative to the NP index for Juglar and Kitchin frequencies from unrestricted MLS estimates for six dimensional VAR(4) model for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	PP	Coh	Phase	PP	Coh	Phase
NP	0.3			0.3		
PI	0.7	0.7	2.7	0.2	0.7	-1.2
WI	0.6	0.8	-0.4	0.2	0.3	-0.5
SR	0.3	0.9	-2.3	0.4	0.6	-0.2
LR	0.4	0.9	-3.0	0.2	0.7	-0.8
TR	0.5	0.8	2.4	0.3	0.9	0.8
Average	0.5	0.8		0.2	0.6	

Table 20.15 Eigenvalues and eigenvectors in polar coordinates of unrestricted MLS estimates for six dimensional VAR(4) for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
R	1.0	0.9	1.0	0.9	0.9	0.8	0.2	0.8	0.8	0.6	0.8	0.8
$\omega$	0.09	0.08	0.12	0.17	0.18	0.23	0.15	0.25	0.32	0.43	0.41	0.45
$1/\omega$	10.7	12.2	8.3	6.0	5.5	4.4	6.6	3.9	3.1	2.3	2.5	2.2
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
NP	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
PI	0.6	0.3	1.7	0.6	0.8	0.6	0.6	0.4	1.8	0.9	0.5	0.3
WI	1.0	2.3	0.9	0.3	1.0	1.2	1.4	0.4	1.4	2.7	0.4	1.3
SR	1.0	0.7	0.9	0.6	1.4	0.9	0.5	1.2	1.1	1.3	0.8	1.4
LR	0.6	0.3	0.6	0.3	0.9	0.8	0.3	0.6	0.3	0.8	0.2	0.5
TR	5.5	19.5	7.1	6.1	4.7	17.8	10.4	17.5	5.2	11.8	3.3	4.2
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
NP	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
PI	2.9	-5.8	-3.5	-1.8	-1.1	-0.9	-0.3	0.1	-0.8	0.1	0.4	0.4
WI	-0.4	-2.7	-2.1	2.7	-2.4	0.9	-0.1	-1.1	-1.1	0.1	-0.1	-0.8
SR	-2.4	4.3	-1.6	-0.7	0.3	-1.4	2.8	-0.8	-0.4	1.1	0.1	0.1
LR	-3.1	5.6	-1.9	1.2	0.1	-1.2	0.5	-0.9	-0.6	0.6	0.2	-0.1
TR	2.5	-0.7	2.4	0.4	1.2	0.6	0.3	0.8	-1.4	-0.5	1.2	-0.9

From these results for the unrestricted VAR(4) model the following can be observed:

- The *eigenvalues and eigenvectors* of the unrestricted VAR(4) model from Table 20.15 consist of twelve pairs of complex conjugate eigenvalues. Groups of three, four and five of them can be attributed to respectively the Juglar, Kitchin and high frequency fluctuations. The modulus of these eigenvalues is much higher than for the VAR(1) model indicating more pronounced pseudo-periodic dynamic behavior. The relative modulus of the Juglar and Kitchin eigenvectors and their phases are again reasonably consistent with respectively the relative volatilities of the series shown in Table 20.11 and the phase numbers from Table 20.9.
- Broadly speaking, the *auto-spectra* of the unrestricted VAR(4) model, as shown in Figure 20.42, are more consistent with the reference spectra than was the case for the VAR(1) model in Figure 20.39. The spurious variance at the low frequencies is no longer there, while also most of the variance is correctly situated

at the Juglar and Kitchin frequencies. Unfortunately, new problems have occurred. Especially the peak at the Juglar frequency seems to have been split into two sharper peaks. This may be caused by the *feed-across effect* as described at the end of section 4.7.7. To see this, note that the third eigenvalue from Table 20.15 with a frequency of  $\omega=0.12$  is the most prominent one in the auto-spectrum of the consumer Price Index (PI) while in the auto-spectra of the other variables this is the first eigenvalue from Table 20.15 with a lower frequency of  $\omega=0.09$ . The difference in the relative importance of these eigenvalues in the dynamics of the PI variable can also be seen from the relative modulus of the corresponding first and third eigenvector. This specific use of these two eigenvalues indeed seems to give a good match with the reference auto-spectra of the variables. For the PI variable the Juglar peak is indeed situated more to the right than for the other variables. The feed-across effect now causes that, to a lesser extent, the  $\omega=0.09$  peak is also present in the auto-spectrum of the PI variable while, the other way around, the  $\omega=0.12$  peak is also present in the auto-spectra of the other variables. Thereby the auto-spectra show too many peaks compared to the reference spectra.

- Contrary to the VAR(1) model, the *coherence and phase spectra* of the VAR(4) model, as shown in Figure 20.43 and 20.44, are now more volatile than the reference spectra. This is caused by the fact that the six dimensional VAR(4) model has more instead of less eigenvalues ( $n \times p = 6 \times 4 = 24 = 12$ ) than the typical two dimensional VAR(6) models used for the Maximum Entropy estimates ( $n \times p = 2 \times 6 = 12$ ). The patterns are however very similar as for the reference spectra, including the return of the high coherence numbers at some individual frequency ranges. The phase spectra seem consistent as well, although the greater flexibility may cause differences at individual frequency ranges.

The results from this unrestricted VAR(4) model confirm finding number 13 from section 19.3. A high order model performs better in terms of describing the auto-spectra, which is in line with the high orders indicated by the order selection criteria from Table 20.10. However, the great number of eigenvalues can also introduce too much flexibility in the dynamics of the model in the sense of producing spectra with too many peaks that blur the “true” underlying spectrum and disturb the phase and coherence relations. One could also say that some of the great number of eigenvalues available in the model are used to model specific peaks in the auto-spectra of specific variables. Because of the feed-across effect, these specific spectral peaks from one variable also turn up in the spectra of other variables. In total, this causes spectral densities with many peaks. The intuitively more appealing *combining* of the individual auto-spectra to obtain a better estimate of the underlying business cycle dynamics therefore also does not work for high order unrestricted models. The problem is now that, in case of unrestricted estimates, on the hand high order models seem necessary to produce dynamic properties that are broadly speaking consistent with the empirical knowledge while on the other hand the same high orders introduce too much flexibility which disturbs the same dynamic properties.

An obvious way to solve these shortcomings is to put restrictions on the eigenvalues of a VAR model. This can be done for a low order model to enforce the Juglar and Kitchin dynamics. If this does not result in the required dynamics, and hence indeed a high order model is needed, than this can also be done to reduce the dynamics of a high order VAR model. The relevant EVR VAR models with such

restrictions are presented in section 20.5.3. First however, in the next section the effects of changing the eigenvalues and eigenvectors by hand are explored.

## 20.5.2 Changing eigenvalue and eigenvector parameters

Before applying restrictions on the eigenvalues it is good to first gain a better understanding of how exactly the eigenvalues and eigenvectors influence the dynamics of a VAR model. In this section this is investigated by starting with the unrestricted VAR(2) model for the series from Table 20.8 which has been estimated by the MLS procedure. The eigenvalues and eigenvectors of this model in terms of the polar coordinates as described in section 3.3.2 are reported in Table 20.16. The model has six pairs of complex conjugated eigenvalues. Groups of two, three and one of them can be attributed to respectively the Juglar, Kicthin and high frequency fluctuations. In the entire section we focus on the behavior of one of the six variables from the model, the Short Term Interest Rate (SR). Figure 20.45 shows the auto-spectrum from the basic VAR(2) model, indicated by model 0, while the peak power and coherence and phase numbers relative to the National Product index (NP) are shown in the first line of Table 20.17.

Table 20.16 Eigenvalues and eigenvectors in polar coordinates of unrestricted MLS estimates for six dimensional VAR(2) model for the variables from Table 20.8. The marked lines are the ones that need to be replaced for the changes of Table 20.18.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>					
	1	2	3	4	5	6
R	0.9	0.7	0.6	0.7	0.7	0.3
$\omega$	0.11	0.12	0.16	0.18	0.23	0.30
$1/\omega$	8.9	8.6	6.2	5.5	4.4	3.4
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>					
	1	2	3	4	5	6
NP	1.0	1.0	1.0	1.0	1.0	1.0
PI	0.4	0.6	3.0	1.2	0.7	0.2
WI	0.9	1.1	2.2	1.0	1.1	0.3
SR	0.7	0.9	1.4	0.9	1.3	1.0
LR	0.4	0.4	0.5	0.6	0.6	0.2
TR	4.6	10.1	12.8	2.9	8.7	3.9
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>					
	1	2	3	4	5	6
NP	0.0	0.0	0.0	0.0	0.0	0.0
PI	-4.2	-4.1	-1.3	-1.6	-0.6	-1.6
WI	-1.2	-2.9	-1.5	-1.4	-2.0	-0.7
SR	-2.0	0.9	-0.7	-1.1	-0.4	0.2
LR	-2.6	1.4	-0.1	2.0	-0.5	0.2
TR	1.1	-1.4	-2.5	0.3	1.1	1.7

Figure 20.45 Normalized auto-spectrum of Short Term Interest Rate (SR) from model 0 with eigenvalues and eigenvectors from Table 20.16 and 20.18.

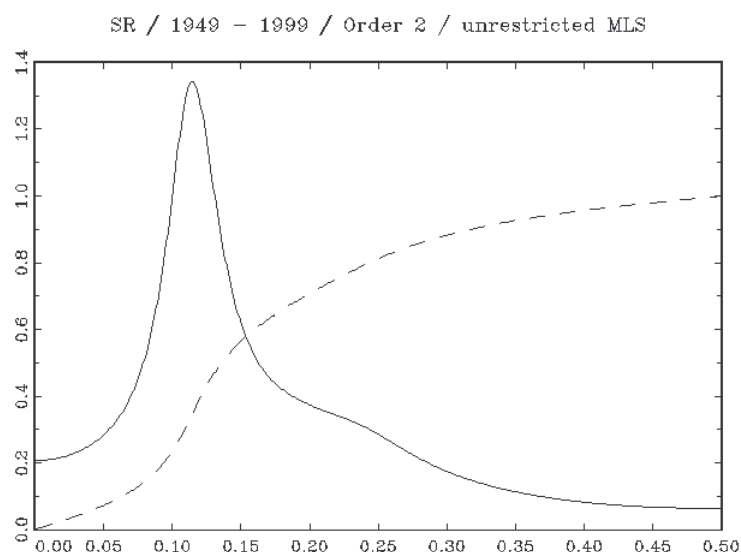


Table 20.17 Peak Power, coherence and phase in years relative to the NP index for the Short Term Interest Rate variable (SR) for the Juglar and Kitchin frequencies from VAR(2) models with eigenvalue and eigenvectors from Table 20.16 and 20.18.

	$\omega=1/10$			$\omega=1/4.5$		
	<i>PP</i>	<i>Coh</i>	<i>Phase</i>	<i>PP</i>	<i>Coh</i>	<i>Phase</i>
0	0.46	0.74	-2.4	0.29	0.58	-0.6
1	0.30	0.75	-0.3	0.28	1.00	0.2
2	0.24	0.97	-2.0	0.31	0.10	-1.9
3	0.35	0.97	-2.0	0.28	0.13	-1.7
4	0.50	1.00	-1.6	0.07	0.93	0.5
5	0.27	0.99	-1.9	0.27	0.64	-2.1
6	0.19	0.99	-2.3	0.39	0.60	0.6

In total, six alternative models are considered by combinations of changing the eigenvalues of the model and the eigenvector parameters that are related to the SR variable. The eigenvectors and eigenvalues from the alternative models, indicated as model 1 until 6, can be obtained by substituting the relevant lines from Table 20.18 into the marked lines of Table 20.16. The non-marked lines from Table 20.16 remain unchanged at the eigenvalue and eigenvector parameters of the basic VAR(2) model. The covariance matrix of the error terms also remains unchanged. For actually changing the eigenvalues and eigenvectors of the model, the procedure outlined in section 20.1.7 was used. A factor a little smaller than one was applied on the modulus of subsequent identical eigenvalues for reasons as described in section 20.1.7. The effects that are subsequently analyzed are:

1. The multiplicity of eigenvalues
2. The modulus of eigenvalues
3. The modulus of eigenvectors
4. The phase of eigenvectors

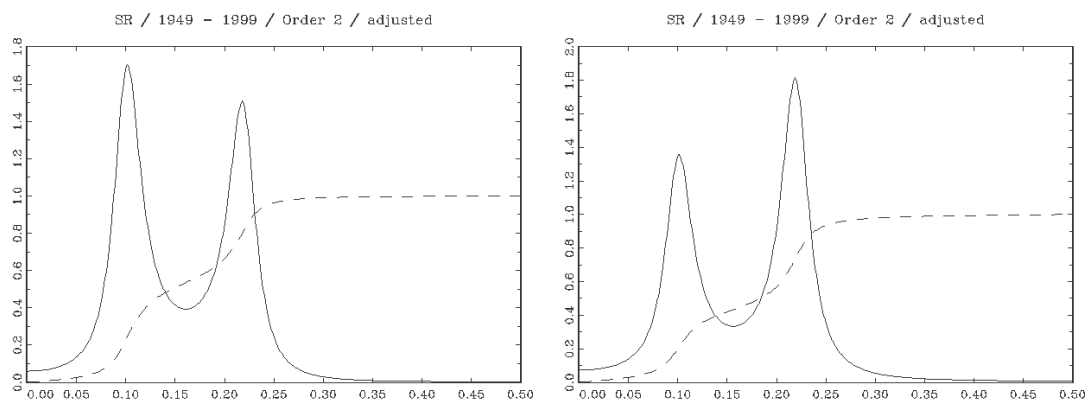
Table 20.18 Various cases of changes in eigenvalues and eigenvectors in polar coordinates of the six dimensional VAR(2) model from Table 20.16. All changes in the eigenvectors solely hold for the Short Term Interest Rate (SR). All other parameters remain unchanged at the values from Table 20.16. Only the marked lines in Table 20.16 need to be replaced by the relevant numbers from this table.

Model		<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>					
		1	2	3	4	5	6
0	R	0.9	0.7	0.6	0.7	0.7	0.3
	$\omega$	0.11	0.12	0.16	0.18	0.23	0.30
	$1/\omega$	8.9	8.6	6.2	5.5	4.4	3.4
1	R	0.9	0.9	0.9	0.9	0.9	0.9
	$\omega$	0.10	0.10	0.10	0.10	0.10	0.22
	$1/\omega$	10.0	10.0	10.0	10.0	10.0	4.5
2	R	0.9	0.9	0.9	0.9	0.9	0.9
	$\omega$	0.10	0.22	0.22	0.22	0.22	0.22
	$1/\omega$	10.0	4.5	4.5	4.5	4.5	4.5
3	R	0.9	0.8	0.8	0.8	0.8	0.8
	$\omega$	0.10	0.22	0.22	0.22	0.22	0.22
	$1/\omega$	10.0	4.5	4.5	4.5	4.5	4.5
4	R	0.9	0.9	0.9	0.9	0.9	0.9
	$\omega$	0.10	0.22	0.22	0.22	0.22	0.22
	$1/\omega$	10.0	4.5	4.5	4.5	4.5	4.5
5	R	0.9	0.9	0.9	0.9	0.9	0.9
	$\omega$	0.10	0.22	0.22	0.22	0.22	0.22
	$1/\omega$	10.0	4.5	4.5	4.5	4.5	4.5
6	R	0.9	0.9	0.9	0.9	0.9	0.9
	$\omega$	0.10	0.22	0.22	0.22	0.22	0.22
	$1/\omega$	10.0	4.5	4.5	4.5	4.5	4.5
Model		<i>Relative modulus (<math>\Delta</math>) of PI in eigenvectors</i>					
		1	2	3	4	5	6
0		0.7	0.9	1.4	0.9	1.3	1.0
1		0.7	0.9	1.4	0.9	1.3	1.0
2		0.7	0.9	1.4	0.9	1.3	1.0
3		0.7	0.9	1.4	0.9	1.3	1.0
4		0.7	0.1	0.1	0.1	0.1	0.1
5		0.7	10.0	10.0	10.0	10.0	10.0
6		0.7	10.0	10.0	10.0	10.0	10.0
Model		<i>Phase (<math>\phi</math>) in years of PI in eigenvectors</i>					
		1	2	3	4	5	6
0		-2.0	0.9	-0.7	-1.1	-0.4	0.2
1		-2.0	0.9	-0.7	-1.1	-0.4	0.2
2		-2.0	0.9	-0.7	-1.1	-0.4	0.2
3		-2.0	0.9	-0.7	-1.1	-0.4	0.2
4		-2.0	0.9	-0.7	-1.1	-0.4	0.2
5		-2.0	0.9	-0.7	-1.1	-0.4	0.2
6		-2.0	0.0	0.0	0.0	0.0	0.0

### 1. Multiplicity of eigenvalues

When specifying an EVR VAR model with only a few different eigenvalues, one needs to choose exactly how many eigenvalues should get one of these values. That is, one needs to decide on the *multiplicity* of the eigenvalues. The effects of this multiplicity are examined by comparing models 1 and 2 from Table 20.18. In these models there are two different pairs of complex conjugated eigenvalues, one with frequency  $\omega=0.10$  and one with frequency  $\omega=0.22$ , but each with modulus  $R=0.9$ . Model 1 has five eigenvalues with  $\omega=0.10$  and one with  $\omega=0.22$  while, the other way around, model 2 has one eigenvalue with  $\omega=0.10$  and five with  $\omega=0.22$ . The eigenvectors remain unchanged. Figure 20.46 shows the corresponding auto-spectra while the properties of the two spectral peaks are shown in Table 20.17.

Figure 20.46 Normalized auto-spectra of Short Term Interest Rate (SR) variable from model 1 (left) and model 2 (right) with eigenvalues and eigenvectors from Table 20.16 and 20.18.



From these results the following can be observed:

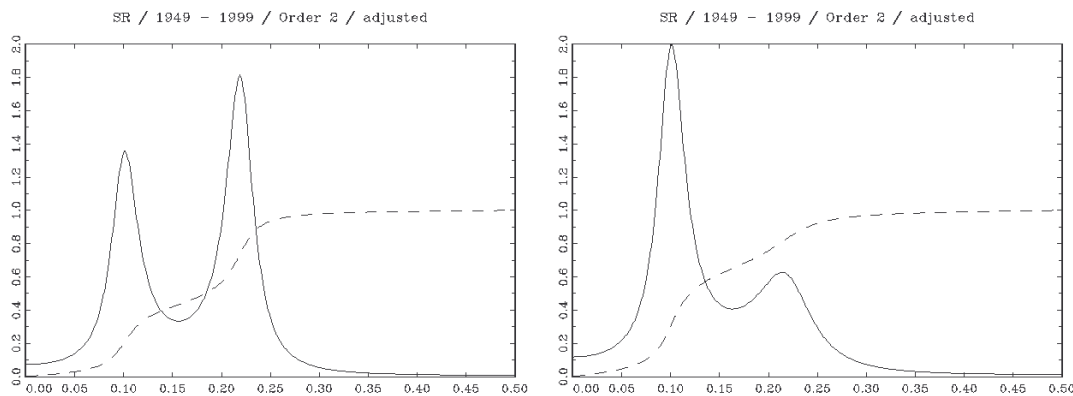
- The multiplicity of the eigenvalues influences the relative importance of the corresponding spectral peaks in the *auto-spectra*. The higher the multiplicity, the more important a spectral peak becomes in describing the total variance. This is the most clear from the *average* peak power (not shown here) of the  $\omega=0.10$  and  $\omega=0.22$  peaks. In model 1 these are respectively 0.40 and 0.18. In model 2 these are respectively 0.24 and 0.33.
- The lower the multiplicity, the higher the *coherence* at the corresponding frequency. It seems that when there are more identical eigenvalues, each with necessarily different eigenvectors, the coherence at their frequency gets more disturbed. Further on we will see that when the relevant (relative) amplitudes from the eigenvectors are made identical, the coherence increases again.
- Although the eigenvectors have not changed, the *phase* at the two frequencies has changed. This is logical because the distribution of the eigenvectors over the two frequencies has changed. In general, we see here that the eigenvalues not only influence the auto-spectra but also the coherence and phase spectra. This also shows that, in case of a multiplicity greater than one, the actual phase at the relevant frequency is a combination of the phase from the corresponding eigenvectors. Note that, in case of a multiplicity of one, the phase spectra and eigenvectors give identical information. For model 1, with one  $\omega=0.22$  eigenvalue

with the corresponding sixth eigenvector from Table 20.18, the eigenvector indicates a lead of 0.2 years for the SR variable which is identical to the value of the phase spectrum at this frequency from Table 20.17. For model 2, with one  $\omega=0.10$  eigenvalue with the corresponding first eigenvector from Table 20.18, the eigenvector indicates a lead of 2.0 years for the SR variable which is identical to the value of the phase spectrum at this frequency from Table 20.17.

## 2. Modulus of eigenvalues

The effects of changing the modulus of eigenvalues are examined by comparing models 2 and 3 from Table 20.18. Both models have one eigenvalue with frequency  $\omega=0.10$  and five eigenvalues with frequency  $\omega=0.22$ . In model 2 however the modulus of each eigenvalue is  $R=0.9$  while in model 3 the modulus of the eigenvalues with frequency  $\omega=0.22$  is lowered to  $R=0.8$ . The eigenvectors remain unchanged. Figure 20.47 shows the corresponding auto-spectra while the properties of the two spectral peaks are shown in Table 20.17.

Figure 20.47 Normalized auto-spectra of Short Term Interest Rate (SR) variable from model 2 (left) and model 3 (right) with eigenvalues and eigenvectors from Table 20.16 and 20.18.



From these results the following can be observed:

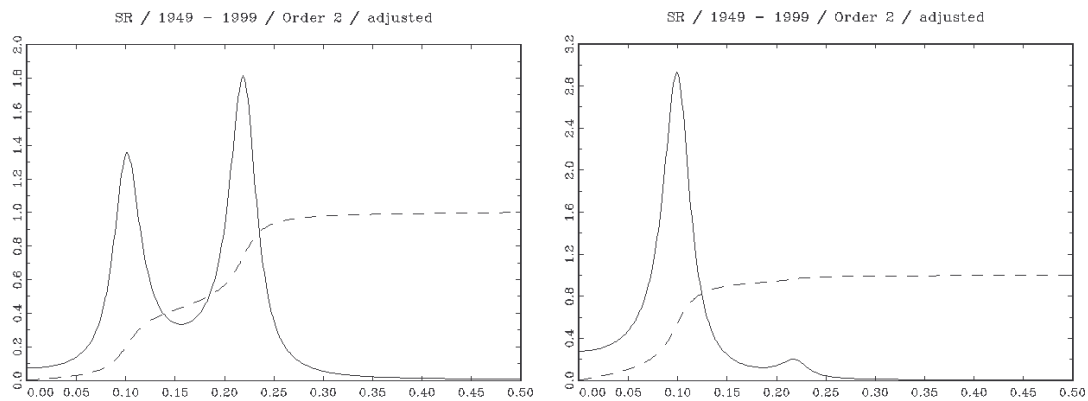
- Besides the multiplicity, the modulus of eigenvalues is an important parameter as well in determining the relative importance of the corresponding spectral peaks in the *auto-spectra*. By lowering the modulus of the  $\omega=0.22$  eigenvalue to 0.8, the  $\omega=0.10$  peak has become more important despite its much lower multiplicity. The peak power of the  $\omega=0.10$  peak has increased by 10% of the total variance while the peak power of the  $\omega=0.22$  peak remained unchanged.
- The *coherence* and *phase* numbers for the  $\omega=0.22$  frequency from Table 20.17 have only mildly changed because of the lowering of the modulus.



### 3. Modulus of eigenvectors

The effects of changing the (relative) modulus of eigenvectors are examined by comparing models 2 and 4 from Table 20.18. Both models have one eigenvalue with frequency  $\omega=0.10$  and five eigenvalues with frequency  $\omega=0.22$  with an identical modulus of  $R=0.9$ . The difference between the models is that in model 4 the (relative) modulus of the SR elements of the five eigenvectors of the  $\omega=0.22$  eigenvalue has been lowered to  $\Delta_{4,i}=0.1$  instead of the original value of around one. So in model 4, the  $\omega=0.22$  fluctuations have a ten times lower amplitude in the SR variable than in the (first) NP variable. All phase numbers  $\phi_{j,i}$  of the eigenvectors remain unchanged. Figure 20.48 shows the corresponding auto-spectra while the properties of the two spectral peaks are shown in Table 20.17.

Figure 20.48 Normalized auto-spectra of Short Term Interest Rate (SR) variable from model 2 (left) and model 4 (right) with eigenvalues and eigenvectors from Table 20.16 and 20.18.



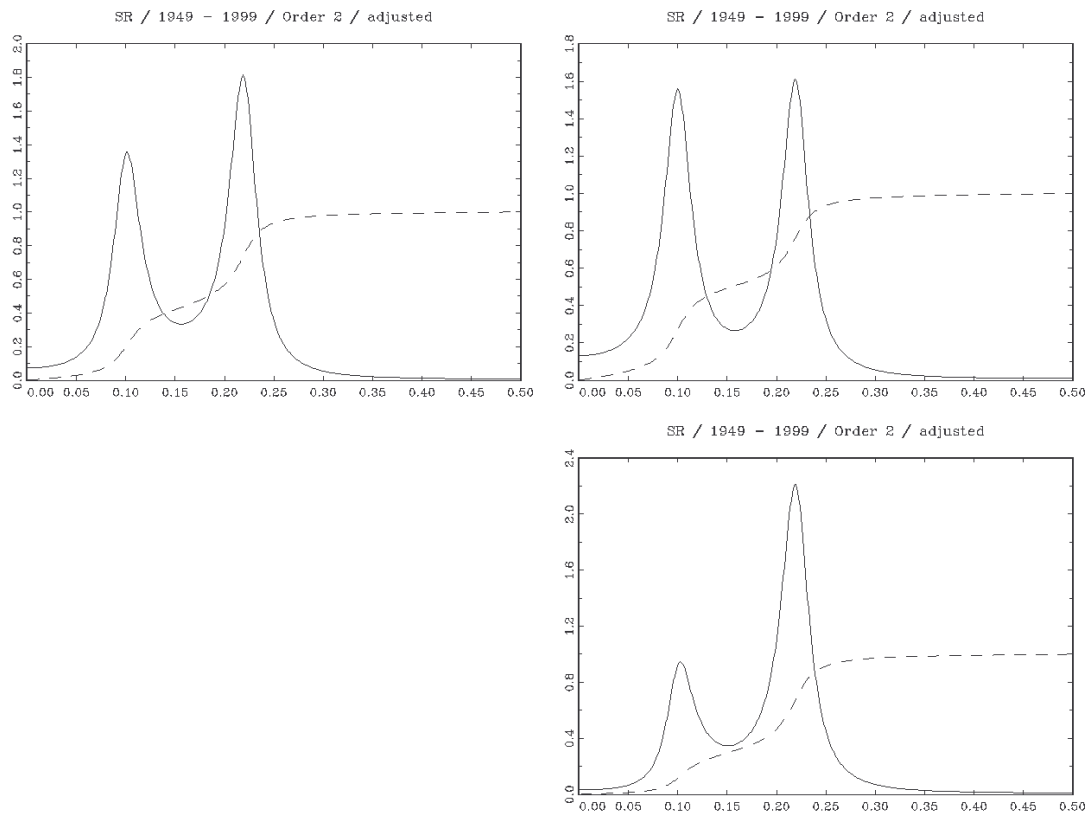
From these results the following can be observed:

- By lowering the (relative) modulus for a variable in the eigenvectors, the importance of the corresponding spectral peak in the *auto-spectrum* of this variable is reduced. This effect is stronger than the higher multiplicity of the  $\omega=0.22$  eigenvalue. The peak power of the  $\omega=0.10$  peak has increased by 25% of the total variance while the peak power of the  $\omega=0.22$  has reduced by the same amount. In general we see here that the eigenvectors not only influence the coherence and phase spectra but also the auto-spectra.
- Especially the *coherence* and to a lesser extent also the *phase* numbers for the  $\omega=0.22$  frequency from Table 20.17 have changed because of the lowering of the modulus of the relevant eigenvector elements. The coherence between the SR and NP variables has increased from 0.10 to a high 0.93. The results from models 5 and 6 will confirm that this strong increase in coherence is more due to the fact that the *same* relative amplitude between the SR and NP variables is used than that it is lowered or increased. Special cases are models 1 and 2 which have eigenvalues with a multiplicity of one and hence corresponding eigenvectors with a by definition “identical” relative modulus for which we also saw a high coherence.

#### 4. Phase of eigenvectors

Finally the effects of changing the (relative) phase of eigenvectors are examined by comparing models 2, 5 and 6 from Table 20.18. All models have one eigenvalue with frequency  $\omega=0.10$  and five eigenvalues with frequency  $\omega=0.22$  with an identical modulus of  $R=0.9$ . Model 5 does the opposite as model 4 by increasing the (relative) modulus of the SR elements of the five eigenvectors of the  $\omega=0.22$  eigenvalues to  $\Delta_{4,i}=10.0$  instead of lowering them to  $\Delta_{4,i}=0.1$ . In the model, the phase numbers  $\phi_{j,i}$  of the eigenvectors remain unchanged. Model 6 does the same but now combined with setting all (relative) phase numbers of the of the SR elements of the five eigenvectors of the  $\omega=0.22$  eigenvalues to  $\phi_{4,i}=0.0$ . Figure 20.49 shows the corresponding auto-spectra while the properties of the two spectral peaks are shown in Table 20.17.

Figure 20.49 Normalized auto-spectra of Short Term Interest Rate (SR) variable from model 2 (top left), model 5 (top right) and model 6 (bottom right) with eigenvalues and eigenvectors from Table 20.16 and 20.18.



From these results the following can be observed:

- Contrary to what one might have expected from the previous results, increasing the modulus of the relevant elements of the eigenvectors has *not* increased the relative importance of the  $\omega=0.22$  peak in the *auto-spectrum* of the SR variable in model 5. However when the corresponding phase numbers are set to zero as in model 6, this relative importance *has* increased. Compared to model 2, the  $\omega=0.22$  peak power has increased by 10% of the total variance. Here we see that also the phase number from the eigenvectors can influence the auto-spectra. The intuition here is that when the cosine functions from representation (3.3.11) corresponding to the  $\omega=0.22$  eigenvalues are brought into phase as is done in

model 6, increasing their amplitude will directly increase the amplitude and hence variance at this frequency. However, if these cosine functions are left out of phase as in model 5, it can be so that the effect of increasing the amplitude of one cosine function is dampened by increasing the amplitude of another cosine function with the same frequency. Note that a rather special property of this example is that the relative amplitudes of several eigenvalues are changed at the same time. Changing only one of these amplitudes might have the expected variance increasing effect.

- Again the *coherence* for the  $\omega=0.22$  frequency from Table 20.17 has increased because of setting the relative amplitude at identical values.
- Model 6 indeed shows a lower *phase* difference for the  $\omega=0.22$  frequency from Table 20.17 than model 5. However, it is not exactly zero, possibly because of the influence of the  $\omega=0.10$  eigenvalue.

In general, the results from this section show that the multivariate dynamic properties of a VAR model in terms of its auto-spectra, coherence spectra and phase spectra are determined by the eigenvalues and eigenvectors of the model in a rather complex and interrelated fashion. In some special cases the relations between the eigenvalues and eigenvectors on the one hand and the spectra on the other hand are very direct. In other cases the same effects can be achieved by different combinations of eigenvalue and eigenvector manipulations while different effects can also cancel or amplify each other. Furthermore, contrary to what one might have expected beforehand, eigenvalues can also influence coherence and phase spectra and, the other way around, eigenvectors can also influence auto-spectra. When changing eigenvectors and eigenvalues, the effects in terms of the spectral densities should always be checked. Finally, we have seen a great potential influence of the eigenvectors on the model dynamics. Restricting the eigenvalues at some pre-determined values, as will be done in the EVR VAR models of the next section, therefore still leaves a lot of room to change the model dynamics by changing the eigenvectors.

### 20.5.3 Six dimensional business cycle EVR VAR models

Given the confirmation in section 20.5.1 of shortcomings number 7 and 13 from section 19.3 of conventional VAR model applications in general, and unrestricted VAR models in particular, the obvious way to proceed is to apply restrictions on the dynamics of the VAR models in terms of their eigenvalues. In this section this is done by estimating EVR VAR models as described in section 20.1.6 and section 3.7.

#### *Models*

The models estimated here are EVR VAR models of the first type, as described in section 3.7.5. One model is estimated for each model order  $p=1,\dots,4$ . All eigenvalues of the models are restricted at one of the *three different eigenvalues* indicated at the top of Table 20.19. The first two of these eigenvalues describe the typical Juglar and Kitchen business cycle dynamics emerging from Part III and summarized in stylized facts *SF 47* through *SF 50* in section 17.2.2. The third eigenvalue describes irregular high frequency fluctuations with a relatively low modulus to cover the high frequency range. Its parameters are based on a rough assessment of the various univariate

Maximum Entropy spectral estimates from Part III. By fixing the eigenvalues, the Juglar and Kitchin dynamics are enforced on low order models. Unrestricted high order models already describe these types of dynamics, however with too much flexibility. For these high order models, fixing the eigenvalues is meant to reduce the dynamics.

Table 20.19 Multiplicity of different eigenvalues in estimated EVR VAR models.

R	0.9	0.8	0.7	
$\omega$	0.10	0.22	0.40	
$1/\omega$	10.0	4.5	2.5	Total
EVR VAR(1)	1	1	1	3
EVR VAR(2)	2	3	1	6
EVR VAR(3)	3	4	2	9
EVR VAR(4)	4	5	3	12

Besides specifying the possible values of the eigenvalues of the models, also the *multiplicity* of these eigenvalues needs to be specified. From the experiments in section 20.5.2 we know that similar effects on the auto-spectra of a VAR model can be achieved by different interrelated changes in its eigenvalues and eigenvectors. Furthermore, restricting the eigenvalues leaves enough room to change the model dynamics by changing the eigenvectors. It seems therefore of little use to estimate EVR VAR models for all possible combinations of multiplicities of the three eigenvalues. Instead we used one set of multiplicities for each model order, as indicated in Table 20.19. In principle, these are based on the number of eigenvalues of the unrestricted EVR VAR models of section 20.5.1 in the frequency intervals  $[0, 0.12]$ ,  $[0.12, 0.25]$  and  $[0.25, 0.50]$  for respectively the Juglar, Kitchin and high frequency type fluctuations. For the models with order  $p=3$  and  $p=4$  a slight modification has been made. In both cases using the base case multiplicities yielded too much variance in the high frequency ranges of the auto-spectra. For each of these models, two of the high frequency eigenvalues were therefore changed into one extra for the Juglar and one extra for the Kitchin eigenvalues. Finally note that no factor was applied on the modulus of subsequent identical eigenvalues as described in section 20.1.7.

### Estimation

The optimal parameters of the eigenvectors corresponding to the fixed eigenvalues are estimated by exactly the procedure as described in section 3.7.5. In short, this comes down to applying various numerical optimization routines for minimizing the natural logarithm of the multivariate residual sum of squares (the same objective as for the unrestricted models from section 20.5.1) using an analytical expression for the first order partial derivatives for the  $n-1$  true parameters of each of the  $np$  eigenvectors.

The models are estimated on the series from Table 20.8 from which the average has been removed. This is required for estimating an EVR VAR model. Note that these averages are practically zero anyway because the series used are filtered series. The constant vector of a VAR model can always be estimated in a second step from (3.2.34) by replacing the expected values with the sample averages.

In most cases the eigenvectors from the unrestricted models are used as a starting solution for the numerical estimation procedure. These unrestricted eigenvectors are appointed to the restricted eigenvalues based on the classification of the unrestricted eigenvalues in the three frequency intervals. In some cases, such as when some of the unrestricted eigenvalues are not complex and the attribution becomes troublesome, also the matrix approximation from section 3.7.4 with  $B=I$  is used for obtaining a starting solution. The results seem rather insensitive to the chosen starting solution which gives extra confidence that true optimal solutions have been found.

After a set of optimal eigenvectors has been estimated for a model, the vector of first order partial derivatives (3.7.26) in terms of the real and imaginary parts of the  $(n-1)$  true parameters of the  $np$  eigenvectors is checked for being sufficiently close to zero indeed. Also the matrix of second order partial derivatives (3.7.40) is checked for being positive definite, indicating that a minimum has indeed been found. Finally, the model standard deviations are set equal to the sample standard deviations without changing the model dynamics by using the procedure described in section 3.7.5.

#### *Judging model performance*

For judging the performance of the EVR VAR models in terms of describing business cycle dynamics that are consistent with the empirical knowledge, the same approach is followed as for the unrestricted VAR models in section 20.5.1. What follows are these results for each of the order  $p=1, \dots, 4$  models with the restricted eigenvalues from Table 20.19. At the end also some additional and summarizing information is given in terms of

- The multivariate  $R^2$  for both the unrestricted and EVR VAR models to judge to what extent this has been reduced by applying the eigenvalue restrictions.
- The average peak powers of the Juglar and Kitchin peaks in the auto-spectra of the variables, both for the unrestricted and EVR VAR models and the original bivariate Maximum Entropy estimates from Part III.
- The average coherence of all variables with the National Product index calculated across the entire frequency range  $[0, 0.5]$ , both for the unrestricted and EVR VAR models and the original bivariate Maximum Entropy estimates from Part III.

The results of the four models are not discussed separately. Instead, one global discussion of the results is given at the end of this section. The final objective is to select the EVR VAR model which performs best in terms of describing business cycle dynamics that are consistent with the empirical knowledge.

*EVR VAR(1) model*

The following figures and tables describe the dynamic properties of the estimated EVR VAR(1) model for the variables from Table 20.8 with the fixed eigenvalues indicated in Table 20.19. Figures 20.50, 20.51 and 20.52 respectively show the auto-spectra, coherence spectra and phase spectra. Table 20.20 shows the spectral properties at the Juglar and Kitchin frequencies. Table 20.21 reports the eigenvalues and eigenvectors of the model.

Figure 20.50 Normalized auto-spectra from estimated six dimensional EVR VAR(1) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

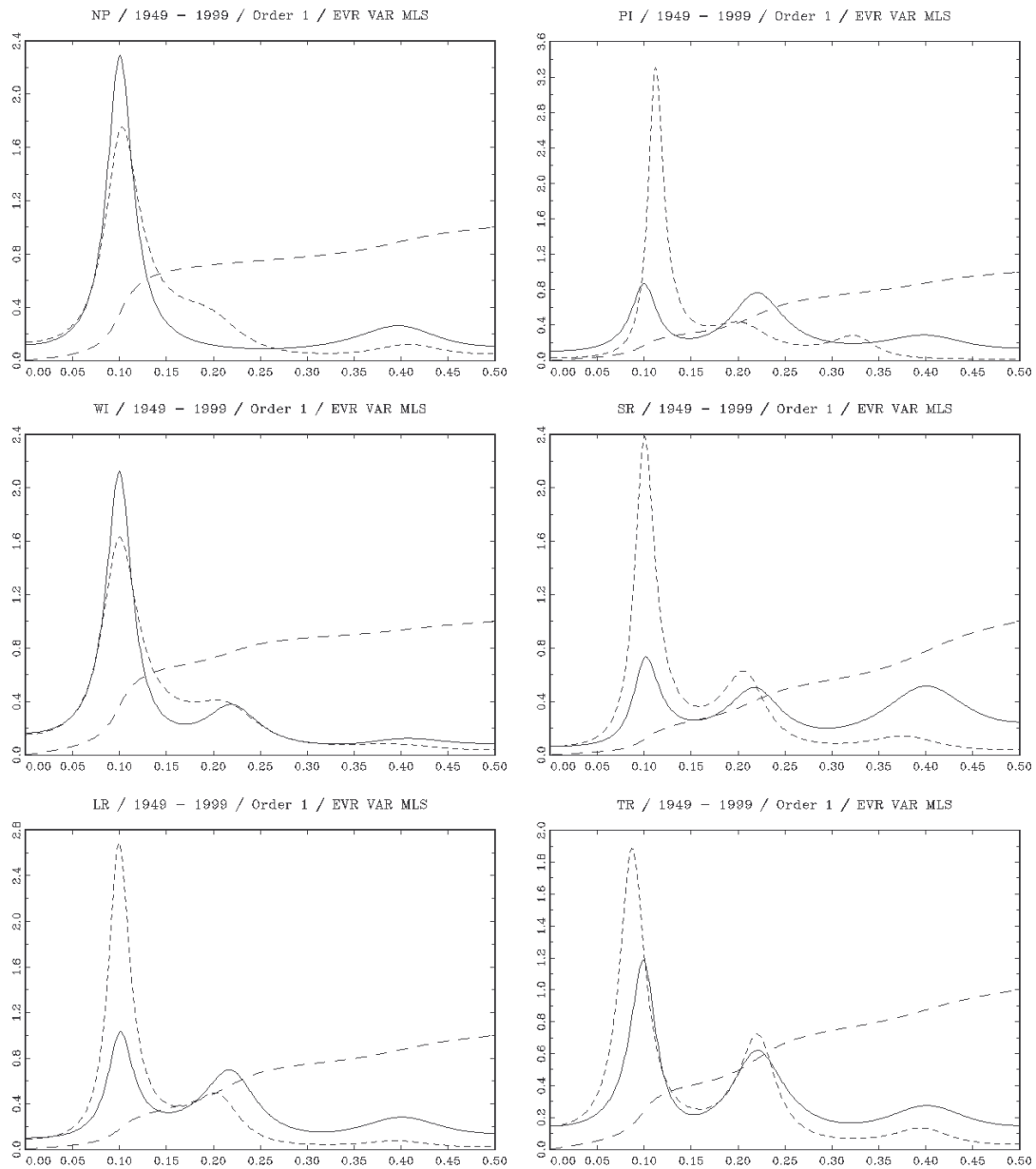


Figure 20.51 Coherence spectra relative to NP from estimated six dimensional EVR VAR(1) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

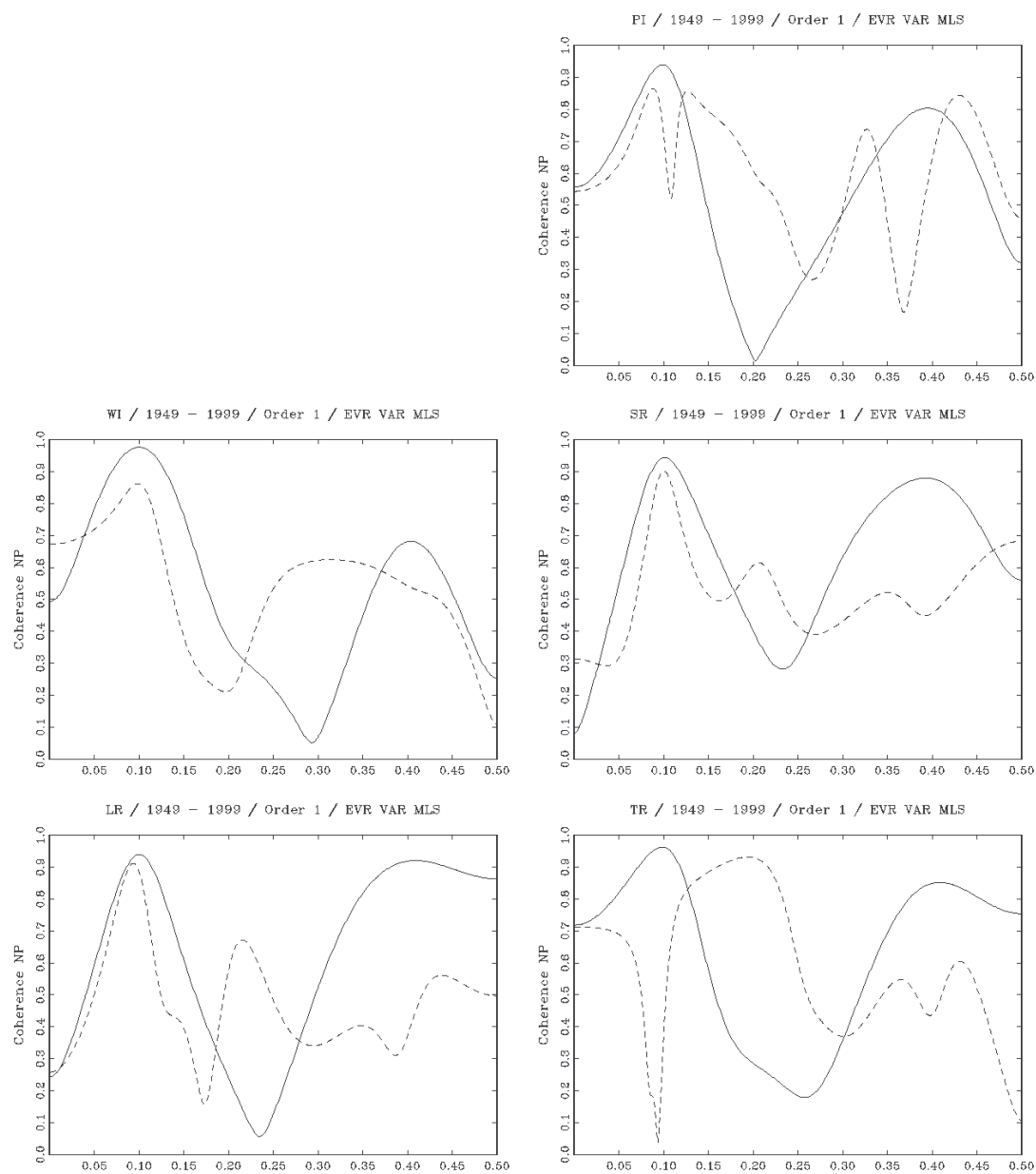




Figure 20.52 Phase spectra in years relative to NP from estimated six dimensional EVR VAR(1) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

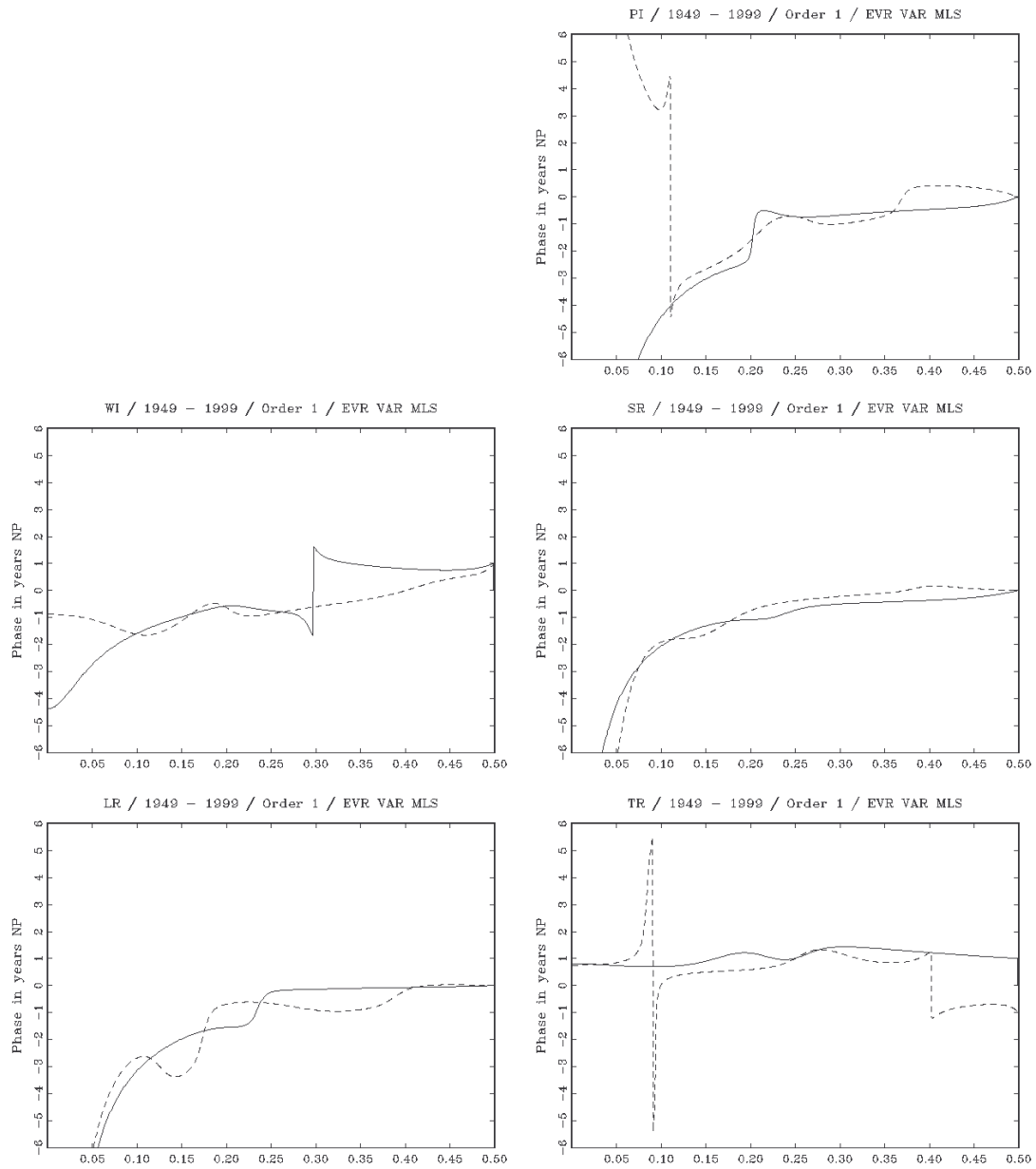


Table 20.20 Peak Power, coherence and phase in years relative to NP for Juglar and Kitchin frequencies from estimated six dimensional EVR VAR(1) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	<i>PP</i>	<i>Coh</i>	<i>Phase</i>	<i>PP</i>	<i>Coh</i>	<i>Phase</i>
NP	0.4			0.3		
PI	0.2	0.9	-4.4	0.3	0.1	-0.5
WI	0.4	1.0	-1.6	0.2	0.3	-0.6
SR	0.1	0.9	-2.1	0.2	0.3	-1.0
LR	0.2	0.9	-3.2	0.3	0.1	-1.5
TR	0.2	1.0	0.7	0.2	0.2	1.1
Average	0.2	1.0		0.3	0.2	

Table 20.21 Eigenvalues and eigenvectors in polar coordinates of estimated six dimensional EVR VAR(1) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>		
	<i>1</i>	<i>2</i>	<i>3</i>
R	0.9	0.8	0.7
$\omega$	0.10	0.22	0.40
$1/\omega$	10.0	4.5	2.5
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>		
	<i>1</i>	<i>2</i>	<i>3</i>
NP	1.0	1.0	1.0
PI	0.5	3229.3	0.8
WI	1.0	2499.4	0.6
SR	0.5	2352.9	1.2
LR	0.3	1513.9	0.4
TR	5.9	28409.5	8.1
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>		
	<i>1</i>	<i>2</i>	<i>3</i>
NP	0.0	0.0	0.0
PI	-4.5	-1.3	-0.5
WI	-1.7	-2.1	0.9
SR	-1.9	0.8	-0.4
LR	-3.1	0.4	0.0
TR	0.6	-0.7	-1.2

*EVR VAR(2) model*

The following figures and tables describe the dynamic properties of the estimated EVR VAR(2) model for the variables from Table 20.8 with the fixed eigenvalues indicated in Table 20.19. Figures 20.53, 20.54 and 20.55 respectively show the auto-spectra, coherence spectra and phase spectra. Table 20.22 shows the spectral properties at the Juglar and Kitchin frequencies. Table 20.23 reports the eigenvalues and eigenvectors of the model.

Figure 20.53 Normalized auto-spectra from estimated six dimensional EVR VAR(2) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

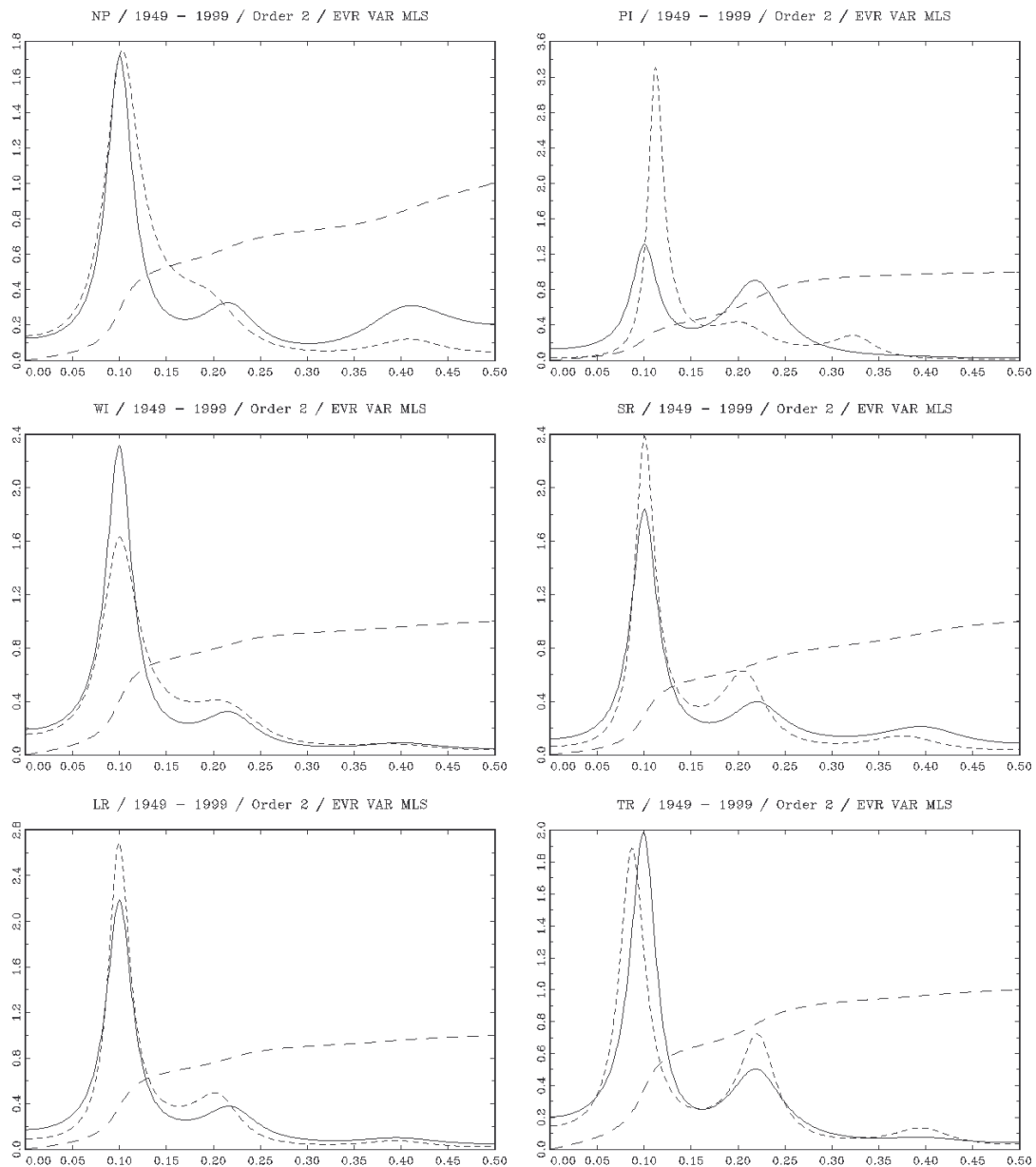


Figure 20.54 Coherence spectra relative to NP from estimated six dimensional EVR VAR(2) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

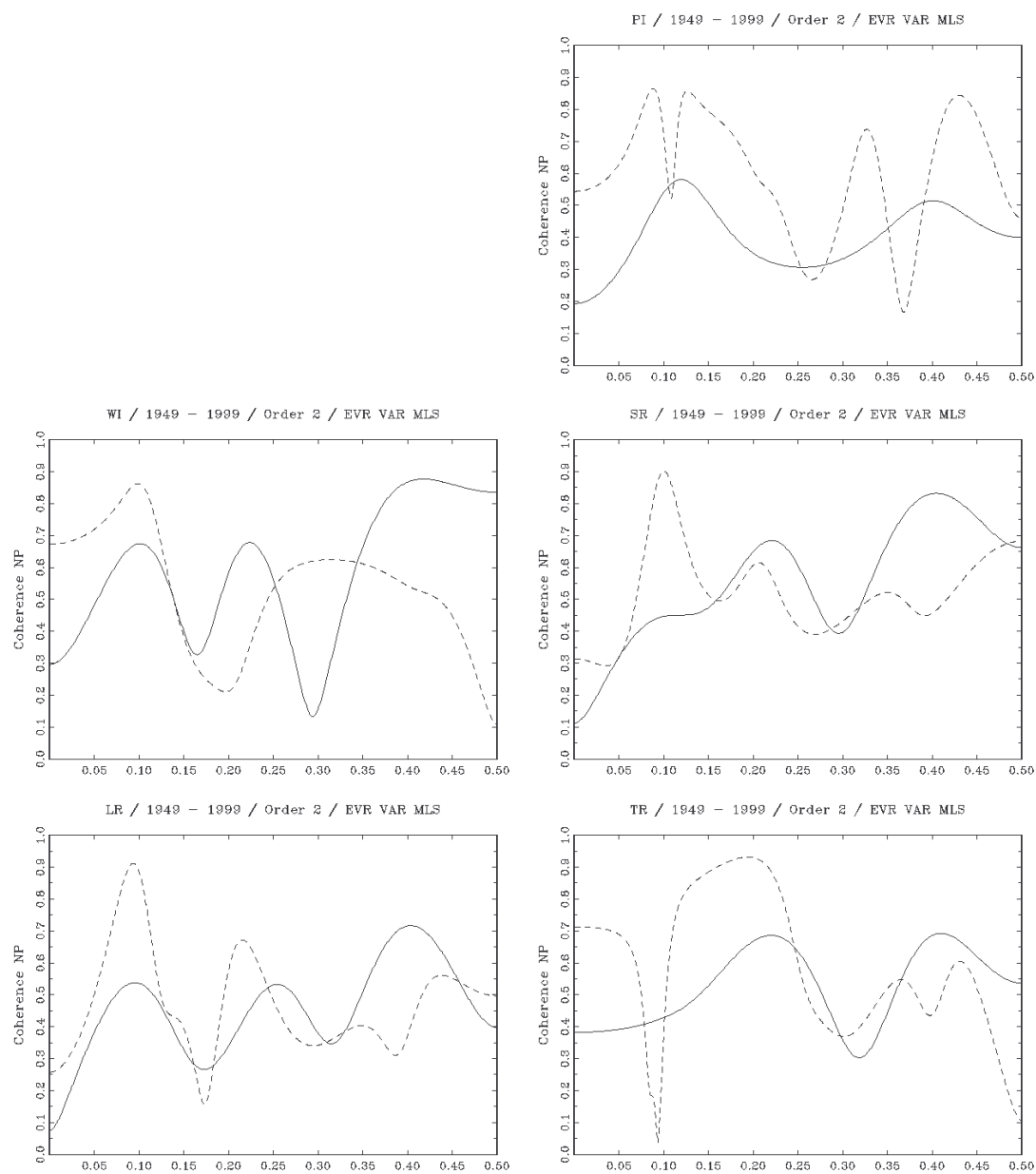


Figure 20.55 Phase spectra in years relative to NP from estimated six dimensional EVR VAR(2) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

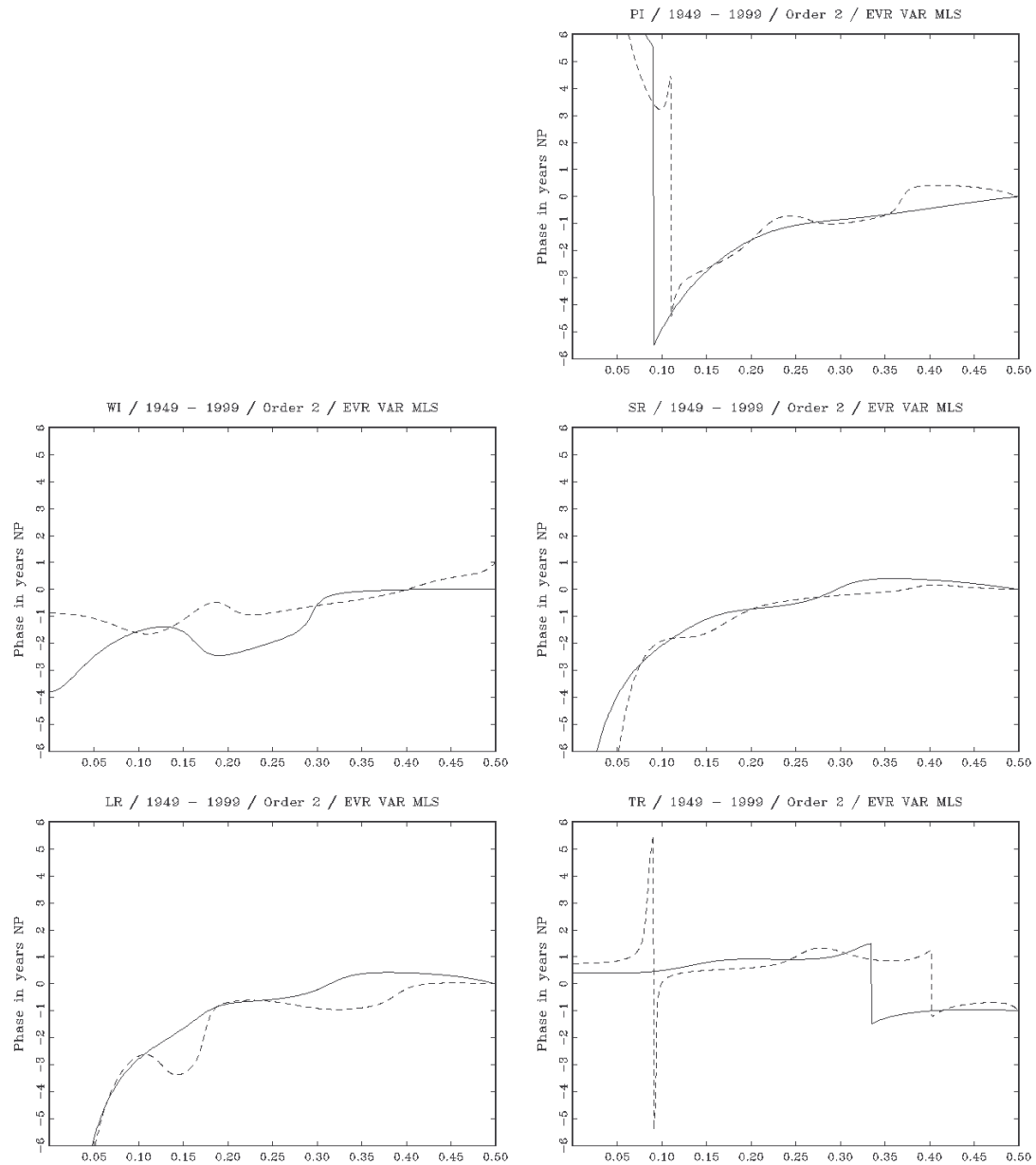


Table 20.22 Peak Power, coherence and phase in years relative to NP for Juglar and Kitchin frequencies from estimated six dimensional EVR VAR(2) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	<i>PP</i>	<i>Coh</i>	<i>Phase</i>	<i>PP</i>	<i>Coh</i>	<i>Phase</i>
NP	0.3	0.0	0.0	0.2	0.0	0.0
PI	0.2	0.5	-4.9	0.3	0.3	-1.3
WI	0.4	0.7	-1.5	0.2	0.7	-2.3
SR	0.3	0.4	-2.1	0.2	0.7	-0.7
LR	0.4	0.5	-2.8	0.2	0.4	-0.7
TR	0.3	0.4	0.5	0.2	0.7	0.9
Average	0.3	0.5	0.0	0.2	0.6	0.0

Table 20.23 Eigenvalues and eigenvectors in polar coordinates of estimated six dimensional EVR VAR(2) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>					
	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>
R	0.9	0.9	0.8	0.8	0.8	0.7
$\omega$	0.10	0.10	0.22	0.22	0.22	0.40
$1/\omega$	10.0	10.0	4.5	4.5	4.5	2.5
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>					
	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>
NP	1.0	1.0	1.0	1.0	1.0	1.0
PI	0.4	0.3	3.5	0.8	1.3	0.1
WI	0.7	0.6	0.3	0.9	1.0	0.5
SR	0.5	0.8	1.7	1.1	1.0	0.8
LR	0.3	0.5	1.2	0.9	0.3	0.2
TR	2.0	2.1	1.3	1.5	8.5	3.8
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>					
	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>
NP	0.0	0.0	0.0	0.0	0.0	0.0
PI	4.8	4.0	-0.5	-0.9	-1.5	-0.6
WI	-1.7	-0.9	1.9	-1.7	-2.2	-0.1
SR	-1.5	-1.9	-0.1	-1.3	-0.7	0.3
LR	-2.3	-2.5	-0.7	0.9	-1.1	0.4
TR	-0.2	2.6	-0.8	-0.7	1.2	-1.0

*EVR VAR(3) model*

The following figures and tables describe the dynamic properties of the estimated EVR VAR(3) model for the variables from Table 20.8 with the fixed eigenvalues indicated in Table 20.19. Figures 20.56, 20.57 and 20.58 respectively show the auto-spectra, coherence spectra and phase spectra. Table 20.24 shows the spectral properties at the Juglar and Kitchin frequencies. Table 20.25 reports the eigenvalues and eigenvectors of the model.

Figure 20.56 Normalized auto-spectra from estimated six dimensional EVR VAR(3) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

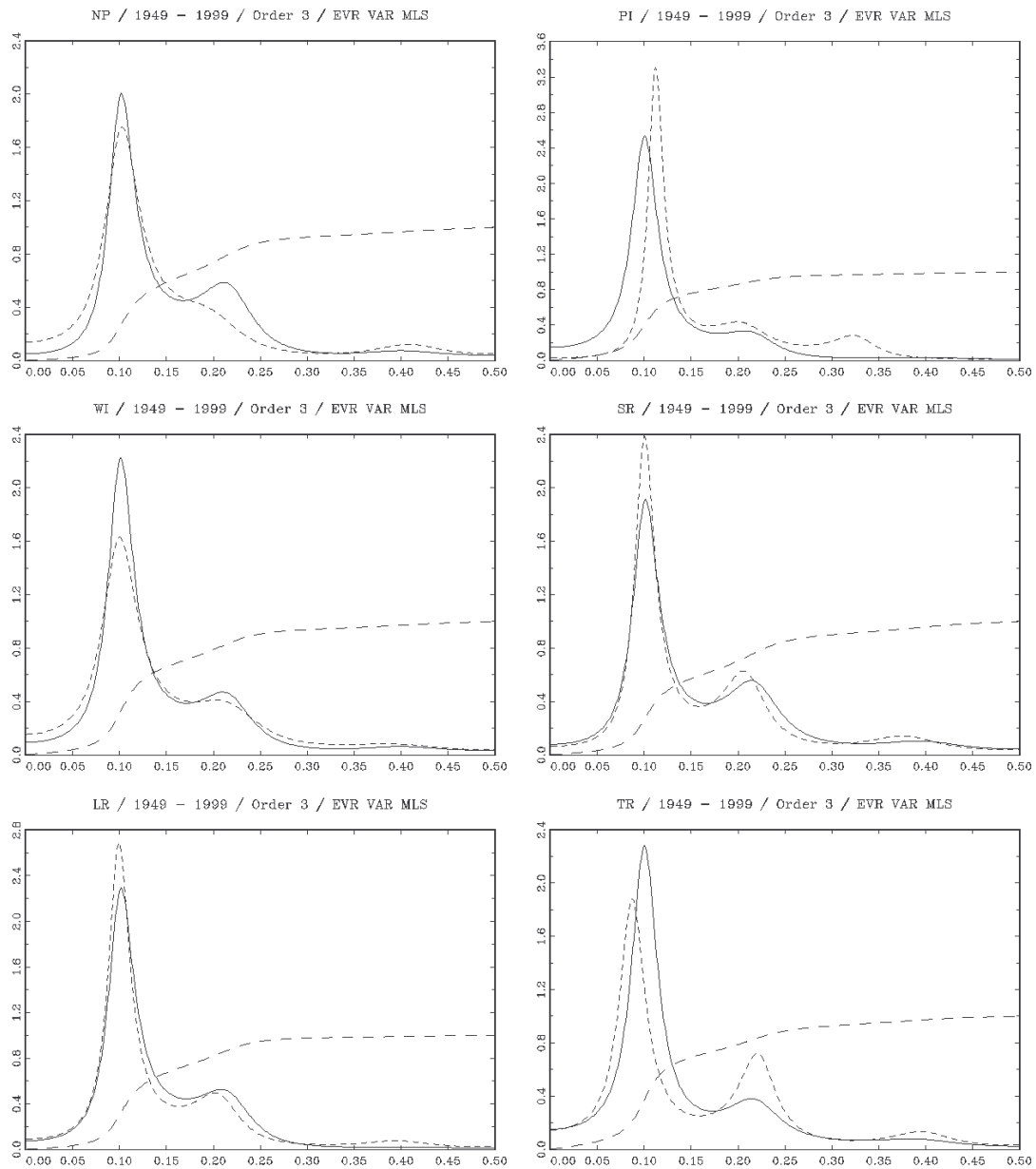




Figure 20.57 Coherence spectra relative to NP from estimated six dimensional EVR VAR(3) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

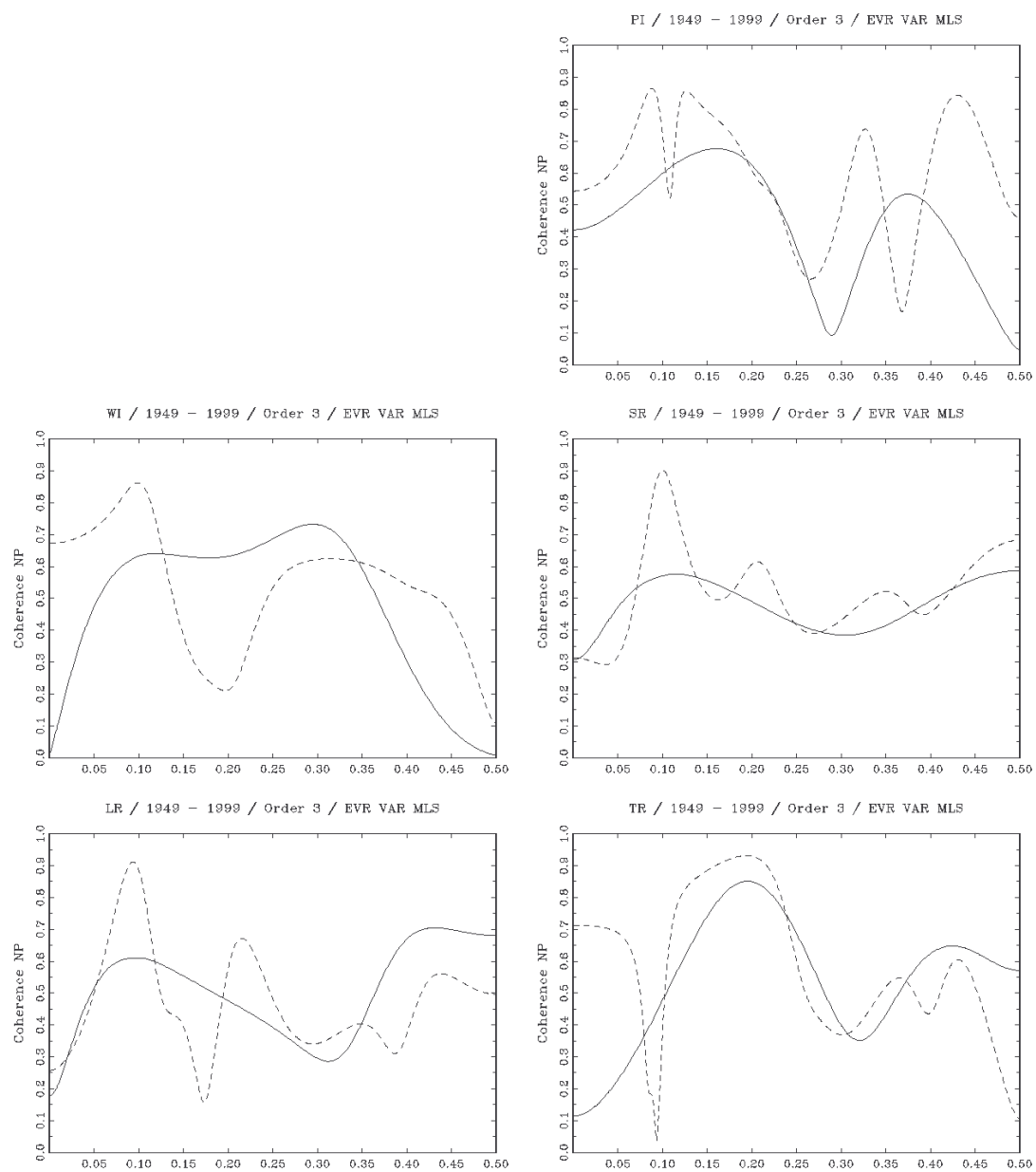


Figure 20.58 Phase spectra in years relative to NP from estimated six dimensional EVR VAR(3) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

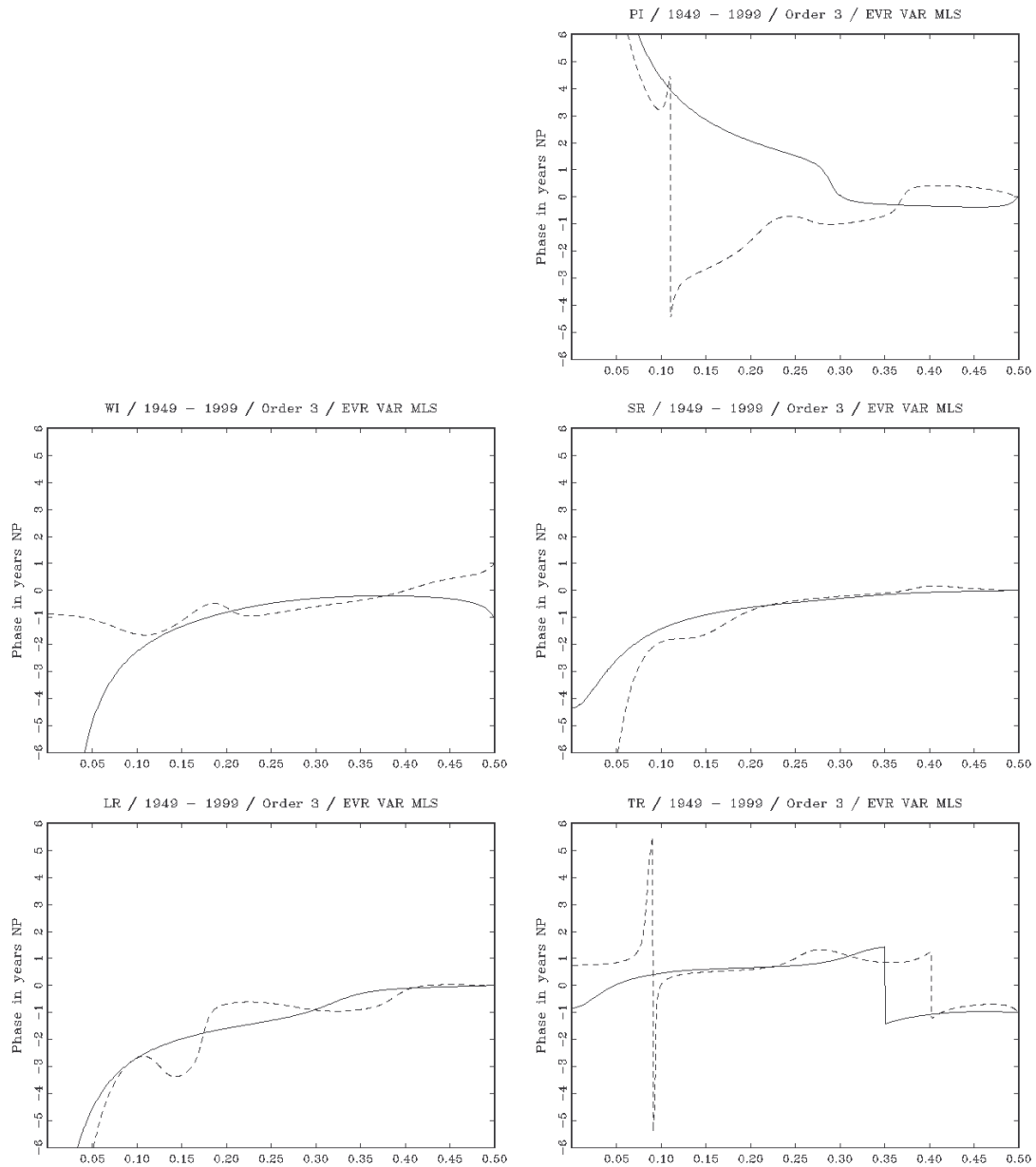


Table 20.24 Peak Power, coherence and phase in years relative to NP for Juglar and Kitchin frequencies from estimated six dimensional EVR VAR(3) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	PP	Coh	Phase	PP	Coh	Phase
NP	0.4			0.3		
PI	0.4	0.6	4.4	0.2	0.6	1.8
WI	0.4	0.6	-2.3	0.2	0.6	-0.7
SR	0.3	0.6	-1.4	0.3	0.5	-0.5
LR	0.4	0.6	-2.7	0.2	0.4	-1.5
TR	0.4	0.5	0.5	0.2	0.8	0.7
Average	0.4	0.6		0.2	0.6	

Table 20.25 Eigenvalues and eigenvectors in polar coordinates of estimated six dimensional EVR VAR(3) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>								
	1	2	3	4	5	6	7	8	9
R	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.7	0.7
$\omega$	0.10	0.10	0.10	0.22	0.22	0.22	0.22	0.40	0.40
1/ $\omega$	10.0	10.0	10.0	4.5	4.5	4.5	4.5	2.5	2.5
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>								
	1	2	3	4	5	6	7	8	9
NP	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
PI	0.4	5.0	3.6	4.6	6.0	1.4	0.8	1.2	1.1
WI	0.5	3.9	2.4	4.0	1.2	0.8	1.0	2.1	1.2
SR	0.7	2.8	2.3	4.6	3.3	1.0	1.3	2.6	0.5
LR	0.3	1.0	1.4	5.4	1.3	0.4	0.5	0.3	0.1
TR	1.5	5.2	16.2	48.6	9.2	0.9	1.3	18.4	2.2
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>								
	1	2	3	4	5	6	4	5	6
NP	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
PI	4.6	3.6	4.6	1.6	2.0	-1.6	-1.6	1.1	-0.4
WI	-2.5	4.4	4.6	1.1	0.4	-1.2	-0.6	1.1	-0.4
SR	-1.0	0.1	-0.6	-1.0	-1.3	-0.4	0.5	0.3	-0.8
LR	-2.0	-1.9	-1.6	-1.0	-1.6	1.6	0.1	0.1	-0.1
TR	-1.0	-0.3	3.9	0.8	0.2	1.2	-0.2	-0.9	-1.1

*EVR VAR(4) model*

The following figures and tables describe the dynamic properties of the estimated EVR VAR(4) model for the variables from Table 20.8 with the fixed eigenvalues indicated in Table 20.19. Figures 20.59, 20.60 and 20.61 respectively show the auto-spectra, coherence spectra and phase spectra. Table 20.26 shows the spectral properties at the Juglar and Kitchin frequencies. Table 20.27 reports the eigenvalues and eigenvectors of the model.

Figure 20.59 Normalized auto-spectra from estimated six dimensional EVR VAR(4) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) from Figure 20.38.

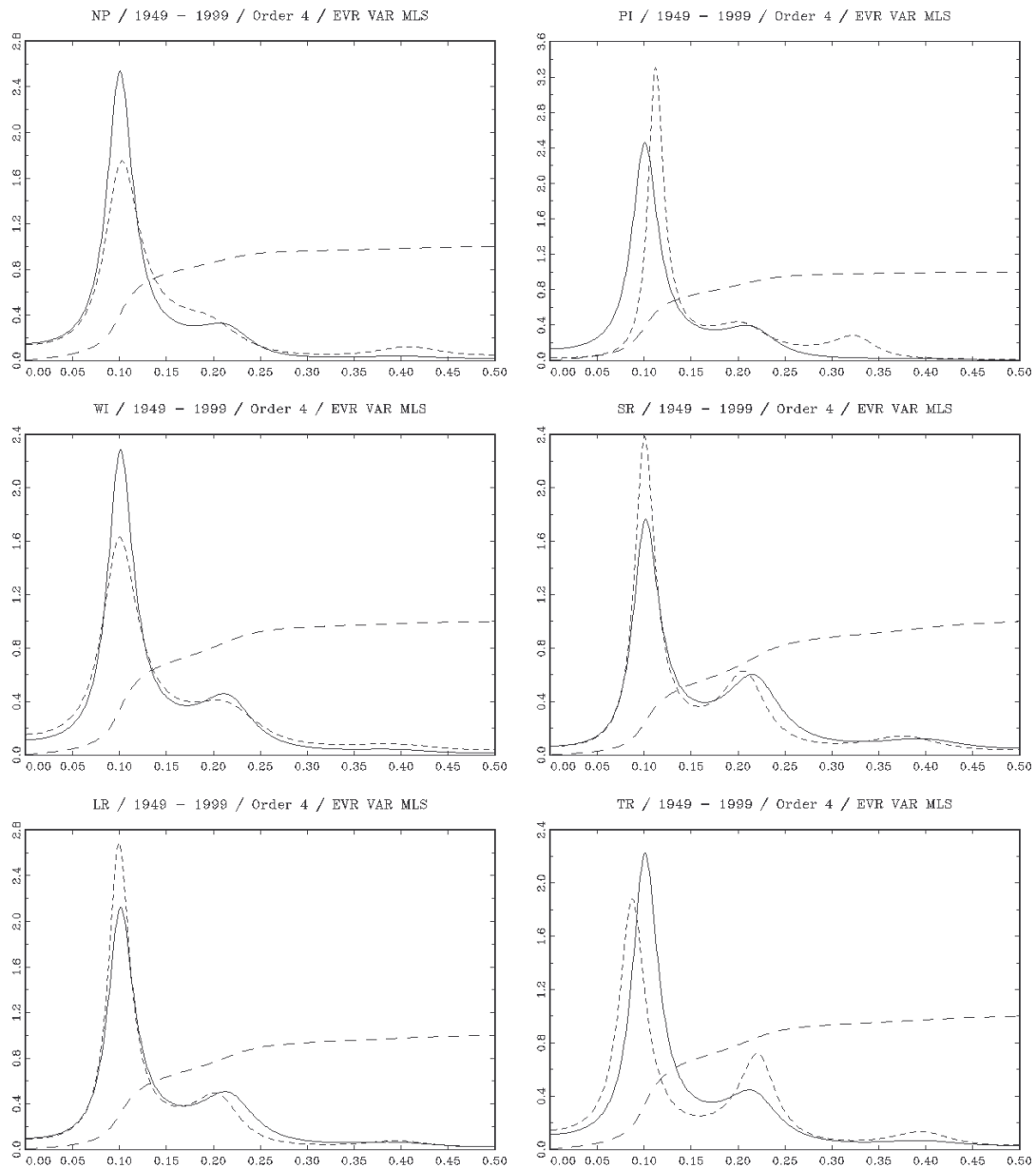


Figure 20.60 Coherence spectra relative to NP from estimated six dimensional EVR VAR(4) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

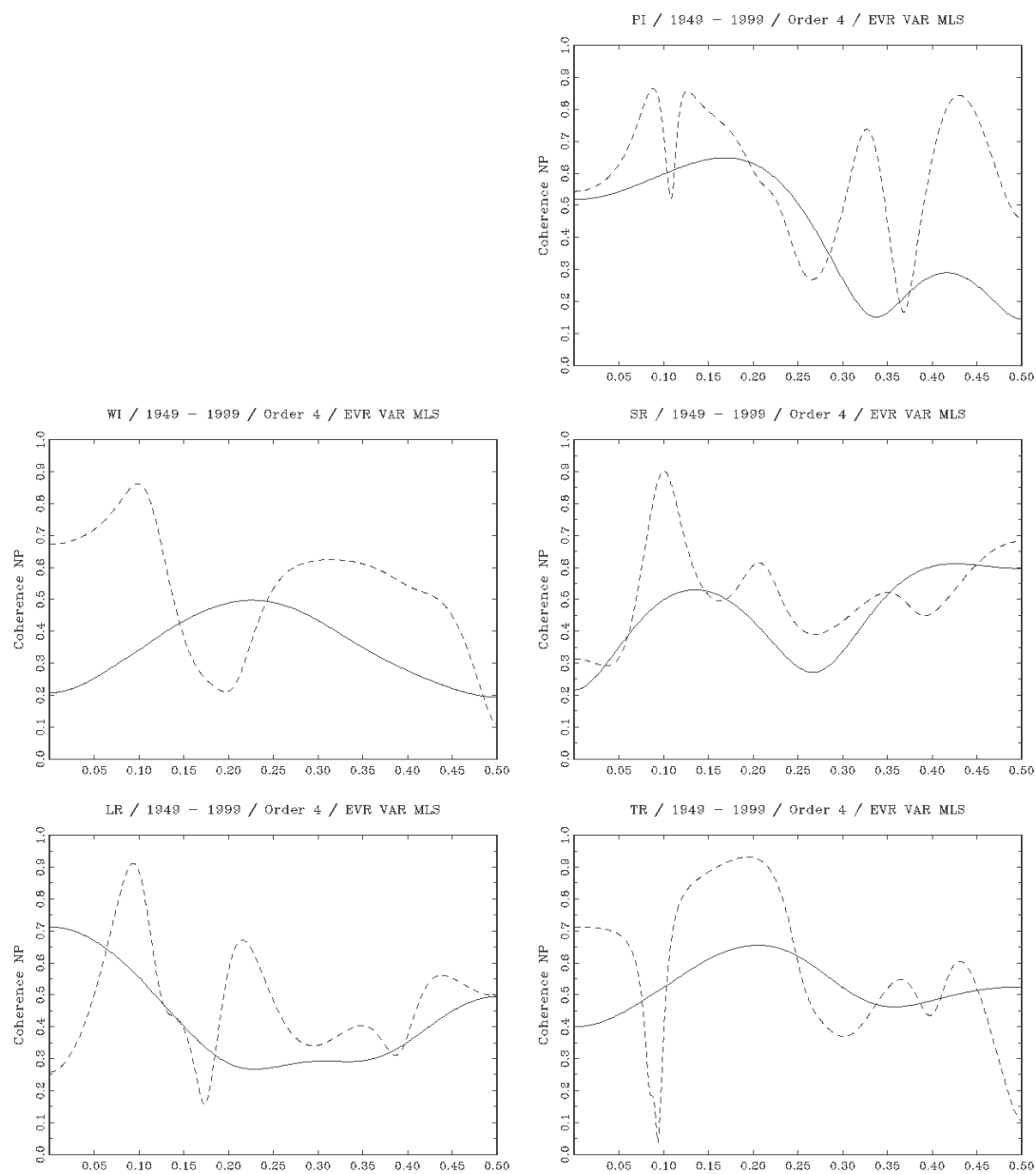


Figure 20.61 Phase spectra in years relative to NP from estimated six dimensional EVR VAR(4) model with eigenvalues from Table 20.19 for the variables from Table 20.8 with reference spectra (fine dotted lines) corresponding to Figure 20.38.

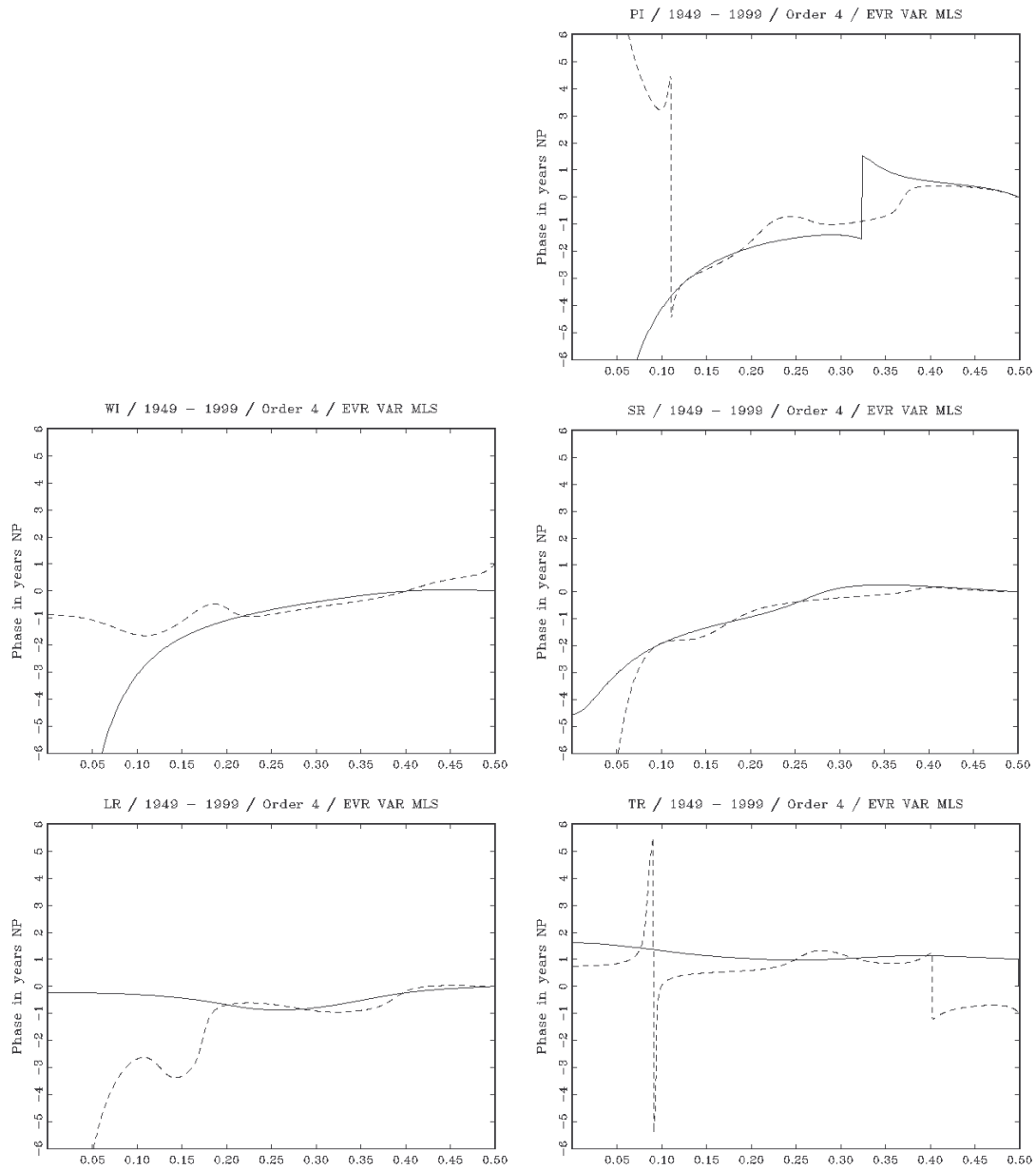


Table 20.26 Peak Power, coherence and phase in years relative to NP for Juglar and Kitchin frequencies from estimated six dimensional EVR VAR(4) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	$\omega=1/10$			$\omega=1/4.5$		
	PP	Coh	Phase	PP	Coh	Phase
NP	0.4			0.2		
PI	0.4	0.6	-4.1	0.2	0.6	-1.7
WI	0.4	0.3	-3.1	0.2	0.5	-0.9
SR	0.3	0.5	-1.9	0.3	0.4	-0.8
LR	0.4	0.6	-0.3	0.2	0.3	-0.8
TR	0.4	0.5	1.3	0.2	0.7	1.0
Average	0.4	0.5		0.2	0.5	

Table 20.27 Eigenvalues and eigenvectors in polar coordinates of estimated six dimensional EVR VAR(4) model with eigenvalues from Table 20.19 for the variables from Table 20.8.

	<i>Modulus (R) and frequency (<math>\omega</math>) of eigenvalues</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
R	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	0.8	0.7	0.7	0.7
$\omega$	0.10	0.10	0.10	0.10	0.22	0.22	0.22	0.22	0.22	0.40	0.40	0.40
$1/\omega$	10.0	10.0	10.0	10.0	4.5	4.5	4.5	4.5	4.5	2.5	2.5	2.5
	<i>Relative modulus (<math>\Delta</math>) of eigenvectors</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
NP	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
PI	1.3	1.9	0.7	0.5	0.7	0.4	1.2	1.8	0.9	3.6	0.5	0.2
WI	2.2	1.2	0.1	0.7	1.3	0.6	0.6	0.8	1.2	5.6	0.8	0.6
SR	0.6	1.8	1.0	0.3	2.1	0.4	1.0	0.4	0.6	5.5	1.5	0.7
LR	0.5	1.0	0.3	0.2	0.9	0.5	0.2	1.0	0.3	2.3	0.5	0.1
TR	5.1	1.3	1.0	1.9	2.4	2.9	1.6	1.7	4.2	33.9	2.0	3.5
	<i>Phase (<math>\phi</math>) in years of eigenvectors</i>											
	1	2	3	4	5	6	7	8	9	10	11	12
NP	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
PI	-3.8	-3.6	3.4	-3.1	-1.4	-1.1	-1.9	-1.9	-1.6	-0.2	0.1	0.9
WI	-4.4	-1.6	-4.0	-3.8	0.6	-1.1	0.7	-0.3	-2.0	-0.9	1.2	0.4
SR	-2.4	-0.8	-1.4	-2.3	1.0	-0.1	-1.2	-1.2	-0.1	-0.2	-0.2	0.4
LR	2.4	-1.6	-1.9	1.7	1.0	1.3	1.6	-1.1	1.3	-0.2	-0.6	-1.2
TR	-4.4	-2.1	-4.3	1.4	-0.3	0.6	1.7	1.3	2.2	-1.1	0.9	1.2



### Summary

The following tables provide some additional and summarizing information about the EVR VAR models for the variables from Table 20.8 with the fixed eigenvalues indicated in Table 20.19. Table 20.28 compares the multivariate  $R^2$  of the models with that of the unrestricted models from section 20.5.1 in order to judge to what extent this  $R^2$  has been reduced by applying the eigenvalue restrictions. Table 20.29 summarizes the average peak powers of the Juglar and Kitchin peaks in the auto-spectra of the variables. Finally, Table 20.30 reports the average coherence of all variables with the National Product index calculated across the entire frequency range  $[0, 0.5]$ . In the last two tables also the reference results from the original bivariate Maximum Entropy estimates from Part III are given.

Table 20.28 Multivariate  $R^2$  of unrestricted VAR models from section 20.5.1 and EVR VAR models with eigenvalues from Table 20.19 for the variables from Table 20.8.

<i>Order</i>	<i>Unrestricted MLS</i>	<i>EVR VAR</i>
1	0.32	-0.21
2	0.48	0.43
3	0.70	0.49
4	0.76	0.51

Table 20.29 Average Peak Power and Coherence with NP at the Juglar and Kitchin frequencies from the spectra of the variables from Table 20.8 in case of the EVR VAR models with eigenvalues from Table 20.19 and from the spectra from the bivariate Maximum Entropy estimates of Part III as also shown in Table 20.9.

<i>Order</i>	$\omega=1/10$		$\omega=1/4.5$	
	<i>Avg. Peak Power</i>	<i>Avg. Coherence</i>	<i>Avg. Peak Power</i>	<i>Avg. Coherence</i>
1	0.2	1.0	0.3	0.2
2	0.3	0.5	0.2	0.6
3	0.4	0.6	0.2	0.6
4	0.4	0.5	0.2	0.5
Bivariate ME	0.4	0.7	0.2	0.6

Table 20.30 Average Coherence with NP over all frequencies from the spectra of the variables from Table 20.8 in case of the EVR VAR models with eigenvalues from Table 20.19 and from the spectra of the bivariate Maximum Entropy estimates from Part III.

<i>Order</i>	<i>PI</i>	<i>WI</i>	<i>SR</i>	<i>LR</i>	<i>TR</i>	<i>Average</i>
1	0.5	0.5	0.6	0.6	0.6	0.6
2	0.4	0.6	0.6	0.5	0.5	0.5
3	0.4	0.5	0.5	0.5	0.5	0.5
4	0.4	0.3	0.5	0.4	0.5	0.4
Bivariate ME	0.6	0.5	0.5	0.5	0.6	0.5

### Discussion of results

From the results of the EVR VAR models with the restricted eigenvalues from Table 20.19, in terms of Figures 20.50 until 20.61, Table 20.20 until 20.27 and the summarizing Tables 20.28 until 20.30, the following can be observed with respect to the ability of the EVR VAR models to describe business cycle dynamics that are consistent with the empirical knowledge.

- Although the *auto-spectra* of the EVR VAR(1) model are definitely more consistent with the reference spectra than was the case for the unrestricted VAR(1) model in Figure 20.39, there are still some major differences. The auto-spectra from Figure 20.50 describe too much high frequency variance. Moreover the Kitchin peak is missing from the NP spectrum while the Juglar peak in the PI, SR, LR and to a lesser extent also in the TR spectrum, is too small. As shown in Table 20.20, the Juglar peak power in these auto-spectra is about 10 to 20% of the total variance while, as shown in Table 20.9, in the reference spectra it is 30 to 40%. This shows that fixing the eigenvalues at the appropriate values is no guarantee for a model to produce the required auto-spectra. From section 20.5.2 we already know that also the eigenvectors influence the shape of the auto-spectra. Apparently, also with restrictions on the eigenvalues, a VAR(1) model is not suitable in this case for adequately describing the multivariate properties of the series from Table 20.8.
- That the EVR VAR(1) model does not perform well can also be seen from the extreme drop in the multivariate  $R^2$  from 0.32 for the unrestricted VAR(1) model to -0.21 for the restricted model, as shown in Table 20.28. The model therefore gives a poor explanation of the total sample variance of the series. For the higher order models there is also a loss of  $R^2$  compared to the unrestricted models, but to a much lesser extent. Striking from Table 20.28 is that for the restricted models the  $R^2$  seems to converge to a value of around 0.50, while for the unrestricted models it keeps increasing as the order increases. In this sense, the restrictions on the eigenvalues reduce the danger of overfitting the sample data.
- The *auto-spectra* of the EVR VAR(2) model in Figure 20.53 are a lot better than the EVR VAR(1) spectra, but still not good enough. The NP auto-spectrum still contains a lot of high frequency variance while the Juglar peak in the PI auto-spectrum still has a relatively low peak power of 20% of the total variance, as shown in Table 20.22. The *auto-spectra* of the EVR VAR(3) and EVR VAR(4) models, as shown in respectively Figure 20.56 and 20.59, are virtually identical and both very consistent with the reference spectra. Their consistency in terms of the peak power of the Juglar and Kitchin peaks is clear from Table 20.29. Therefore, in terms of consistency in the auto-spectra, also for the EVR VAR models a high order model is required, which is consistent with the results from the order selection criteria from Table 20.10 and the results for the unrestricted VAR models from section 20.5.1.
- Table 20.30 shows that the average *coherence* in the EVR VAR models decreases as the order increases. This can be understood from the fact that, as Table 20.19 indicates, the models of a higher order have higher multiplicities of the three eigenvalues while in section 20.5.2 we saw that this can lead to a lower coherence at the related frequencies. Until order  $p=3$  the average coherence is still in line with the reference spectra while for order  $p=4$  it starts to get a little too low (0.4 instead of 0.5). This can also be seen from the average coherence at the Juglar and Kitchin frequencies in Table 20.29. There, the average coherence is the

highest (0.6) for the EVR VAR model of order  $p=3$  while for order  $p=4$  it decreased below the values of the reference spectra (0.5 instead of 0.7 and 0.6 for the Juglar and Kitchin peaks in the reference spectra).

- The *coherence spectra* of the EVR VAR models are clearly more smooth than their counterparts from the unrestricted VAR models in section 20.5.1 and the reference spectra from the bivariate Maximum Entropy estimates from Part III, also for high order models. To see this, compare for example the coherence spectra for the EVR VAR(4) model from Figure 20.60 to those for the unrestricted VAR(4) model from Figure 20.43. This shows that restricting the number of different eigenvalues also influences the shape of the coherence spectra. This is logical from the definition of the cross spectrum (4.4.10), from which the coherence and phase spectra are derived, which is very similar to the definition of the auto-spectrum (4.3.7). Because of the smoother shape of the coherence spectra, the coherence of the EVR VAR models at individual frequencies can be substantially lower than was the case for the reference spectra. The high coherence numbers of 0.9 in Table 20.9 are never observed for the EVR VAR models of order  $p=2, 3$  and 4. Note that the bivariate Maximum Entropy estimates from Part III were also unrestricted VAR models with a large number of different eigenvalues, which, as we know from section 20.5.2, can cause high coherence numbers. It then comes down to which coherence estimates is believed to be the most reliable. Personally we have more confidence in coherence estimates with a stable value around 0.5 across all frequencies than in coherence estimates that wildly fluctuate between a very low coherence of 0.1 for one frequency and a very high coherence of 0.9 for another (adjacent) frequency. Furthermore a coherence of 0.9 seems a bit too good to be true. In this sense, one could argue that the EVR VAR models also improve the estimates of the coherence spectra. The typical coherence of 0.8 and 0.7 for respectively the Juglar and Kitchin fluctuations found in Part III and described in stylized facts *SF 49* and *SF 50* of section 17.2.2, might then be too high and values of around 0.5 or 0.6 might be more appropriate.
- Just as for the results in Chapter 19 and section 20.5.1, the *phase spectra* are robustly estimated for all EVR VAR models. Of course, the specific phase number at individual frequencies can be different, but the general picture is always the same. Just as for the coherence spectra, restricting the eigenvalues also results in more smooth phase spectra, also for high order models. To see this, compare for example the phase spectra for the EVR VAR(4) model from Figure 20.61 to those for the unrestricted VAR(4) model from Figure 20.44.

In total, the EVR VAR models perform better in terms of the consistency with the empirical knowledge from Part III than the unrestricted VAR models from section 20.5.1. They offer a remedy for the shortcomings 7 and 13 of conventional VAR model applications as described in section 19.3. They especially produce better estimates of the auto-spectra and coherence spectra. The phase spectra are estimated well by almost all models and estimation procedures. Of the four EVR VAR models considered here, the EVR VAR(3) model is found to perform best. The order 1 and 2 models are not good enough, mainly because of their aberrant auto-spectra. The order 4 model shows a similar (good) performance in terms of the auto-spectra as the order 3 model but the coherence spectra of this model start getting too low.

Also, the order 4 model adds only very little to the order 3 model in terms of its explanatory power of the sample variance ( $R^2$ ). Finally, the phase spectra are sufficiently consistent for all model orders, so also for order 3. If desired, the dynamic properties of this EVR VAR(3) model can be further refined by means of changing its eigenvalues and eigenvectors as described in section 20.5.2.

The original reasons for applying EVR VAR models were (a) to enforce the required dynamics on a low order model or (b) to reduce the dynamics of a high order model that were generally speaking already pretty good. The results from this section clearly lead to the conclusion that in this specific case, application (b) leads to the best models. This is logical in the light of the high orders indicated by the order selection criteria and the fact that also for the unrestricted VAR models, the high order models performed best. Finally, the preference for high order models seems intuitively logical because the models need to describe fluctuations with period lengths of some 10 and 4.5 years, while they are based on annual time series. A sufficiently long time frame is needed to be able to pick up the appropriate dynamics that cover several time periods.

## 21 Summary and Concluding Remarks

### 21.1 Introduction

This research deals with macroeconomic scenarios. A scenario is a possible description of the future evolution of macroeconomic variables that are relevant in the context of their application such as the national product, unemployment rates, price inflation, interest rates and stock prices. Amongst other applications, scenarios are often applied in Asset & Liability Management for pension funds and insurance companies. In ALM, the consequences of strategic policy alternatives, in terms of for example asset allocation and duration policy, are calculated in simulation models for a large number of (say a thousand) stochastic scenarios. For each policy alternative, this results in the corresponding scenarios of variables that are important for assessing the consequences in terms of the objectives and constraints of the different stakeholders. Boards of directors, trustees and other policymakers use these results to support their strategic decision making. It is not hard to imagine that the properties of the economic scenarios that are used, can have a large impact on not only the absolute but also on the relative risk and return results of such simulations. Hence, the properties of the scenarios have a great influence on the final strategic policy choices that are made. It is therefore of the utmost importance to proceed as careful as possible when generating and evaluating such economic scenarios. The general objective of this research is therefore to contribute to a higher quality of economic scenarios and thereby also to a higher quality of the strategic decision making that is based on these scenarios. The two specific objectives of this research are precisely summarized in one specific scenario definition given by Bunn and Salo (1993) who state that a scenario is a possible evolution of the future that should be consistent with a clear set of assumptions.

The first specific research objective is related to the clear set of assumptions. In both practice and science this is often filled in by the desire to let the properties of the scenarios in a statistical sense resemble the empirical behavior that economic variables have shown the past. At first sight, this certainly seems like a clear and adequate assumption. At further inspection however, it is not clear at all what this assumption actually beholds. In other words: What empirical behavior of economic variables are we actually referring to here? The first specific research question is therefore to obtain broad knowledge about the empirical stochastic behaviour of macroeconomic variables of developed western countries, based on a uniform methodology that is as pure and correct as possible in a methodological sense. This question concerns variables from both the real as well as from the financial sector of the economy and their interaction, the behavior ranging from the very long run to shorter term business cycles and the possible changes that have occurred in this behavior over time. Based on about 150 years of data for the Netherlands, United Kingdom and United States and using a new developed method of analysis, this part of the research results in almost ninety so called stylized facts about the behavior of macroeconomic variables. We speak of stylized facts if the observed empirical behavior of economic variables turns out to be robust with respect to the country and the historical time period for which it is observed.

In both practice and science, the class of Vector AutoRegressive (VAR) time series models is often used for generating scenarios. The second specific research objective is to find out to what extent VAR models, and especially also the way in which they are applied, indeed lead to scenarios that are consistent with the empirical behavior that is present in the time series that are used for estimating the models. The models are tested on the empirical knowledge that is obtained in terms of the stylized facts. Hence, these stylized facts are considered as the “benchmark” empirical knowledge that the VAR models should be able to describe properly. Because the test results indicate a number of shortcomings of conventional VAR model applications, a new VAR scenario framework is developed and applied, that is based on the frequency domain rather than on the time domain. It is important to note that, although the two specific research objectives as formulated here, directly come from the world of ALM and scenario analysis, the results can have relevance for many other types of economic research and applications. The stylized facts are for example very relevant for the building and testing of macroeconomic models in general while the results of testing the VAR models are also relevant for applications in the field of optimal investment portfolio choices.

The research is organized in four parts. Part I serves as an introduction in which also an overview is given of all available techniques and paradigms. From this list, three central techniques have been chosen that seem the best suited for the research objectives. Part II describes the theoretical and technical background as well as the performed experiments of these three chosen central techniques. Part III describes the results on the first specific research question (“clear set of assumptions”) as obtained by means of a new developed method of analysis. Part IV describes the results on the second specific research question (“consistent with”) and the new developed scenario framework.

The research resulted in an extensive book of almost eight hundred pages. Partially, because a lot of results have been obtained. Partially, because a lot of results have been obtained. Partially, also the result of attempting, given some assumed level of basic knowledge, to be as *complete* and *consistent* as possible in terms of the data, techniques and notation used. Moreover, this has been done because the central technique that is applied here, Spectral Analysis, contrary to the natural sciences, is not well known in the economic sciences. The intention is that the results can be used without having to resort to too much additional literature. The setup is as modular as possible to offer readers with different levels of knowledge and different interests an easy access to the for them relevant parts of the research. Furthermore, especially Part III is meant to serve as a broad and consistent reference for studying the empirical behavior of macroeconomic variables. Next, we provide a separate summary of each of the four parts in which the research is organized. We end with a number of directions for future research and future applications of the result from this research.



## 21.2 Summary Part I: Introduction

Part I consists of Chapter 1 and 2 and is meant as an introduction. In the first place, this part covers the background and relevance of the research in terms of ALM and scenario analysis as well as the two specific research objectives. It is known from the literature that the methods that are used for empirical research of time series of economic variables can have a great impact on the results that are found. It is even the case that spurious results can be found that are a direct consequence of the techniques that are used instead of being true empirical behavior. To prevent this as much as possible we developed and tested a methodology that is as correct and consistent as possible for performing the empirical research in Part III. As a first step in constructing such a methodology, we made an inventory of the paradigms and techniques that are and have been used in the macroeconomic sciences. Next, from this list three central techniques have been chosen that at a fundamental level are found the best suitable for the research objectives formulated here. These three central techniques are:

1. Filtering in the frequency domain
2. Maximum Entropy Spectral Analysis
3. Vector AutoRegressive (VAR) time series models

It is important to stress that these techniques are combined into one methodology specifically suited for the research on both the first research question (“clear set of assumptions”, Part III) and the second research question (“consistent with”, Part IV) instead of being compared or used in some other competitive sense. This combined methodology is described in the first chapter of Part III (Chapter 6) and the first chapter of Part IV (Chapter 18). The reasons for choosing precisely these three techniques are as follows.

The choice for filtering techniques first stems from recognizing the useful decomposition in the line of Jan Tinbergen by decomposing economic processes and time series into trends, business cycles, seasonal effects and random fluctuations. Second, a fruitful analysis of empirical economic time series can only be done when trends have been adequately removed. Noting the variety of available detrending techniques and the potential dangers of obtaining spurious results, the first requirement of such filtering techniques is that they are as “correct” as possible. Given the objective to analyze the evolution of economic variables at different frequencies, ranging from the very long run to short term business cycles, the second requirement for the filtering techniques is that they offer enough flexibility to “zoom in” on the various components of a time series. The only way to meet these requirements is to define and arrive at filtering techniques in the frequency domain instead of in the time domain. In the absence of knowing the true Data Generating Process (DGP), filtering time series should “merely” be seen as applying a set of adequate statistical techniques to extract the components of interest from the time series. Designing filters directly based on their properties in the frequency domain comes closest to doing this in an “optimal” manner. The for this research relevant theory of (linear) filter techniques is further described, developed and tested in Chapter 5 in Part II



The choice for Spectral Analysis techniques on the one hand stems from the chosen approach for designing filters. On the other hand, Spectral Analysis also matches very naturally with the different types of fluctuations in macroeconomic time series in terms of for example long waves, business cycle fluctuations and seasonal effects. For collecting stylized facts about economic fluctuations for the purpose of the first research question, estimating (multivariate) spectral densities therefore appears to be a very efficient way to report information. The most important reason why nevertheless Spectral Analysis has been applied with only limited success in economic research thus far, is that classical (non parametric) spectral techniques require a lot of data to provide sufficiently informative results. However, by using parametric Maximum Entropy spectral estimation techniques it is possible to also efficiently extract information from relatively short time series. The for this research relevant theory of Spectral Analysis is further described, developed and tested in Chapter 4 in Part II.

The choice for VAR time series models on the one hand stems from the use of the Maximum Entropy spectral techniques because these are closely related to calculating spectra from estimated VAR models. On the other hand, as said before, VAR models are often used in ALM for generating economic scenarios and hence these are the types of models that are tested in Part IV. The for this research relevant theory of VAR models is further described, developed and tested in Chapter 3 in Part II.

### 21.3 Summary Part II: Technical Background

Part II consists of the Chapters 3 until 5 that together describe the theoretical and technical background as well as the performed experiments of the three chosen central techniques. At the start of Part III (Chapter 6) and at the start of Part IV (Chapter 18) where the actual research with respect to the two specific questions is described, these techniques are combined into one specific methodology especially suited for these research objectives.

Chapter 3 in the first place contains the relevant general theory of autoregressive models. Besides general issues such as the available (not always well known) estimation methods and order selection criteria and the way in which confidence intervals can be calculated, also special attention is given to showing how the dynamic stochastic behavior of such models is caused by the eigenvalues and eigenvectors of the parameter matrices. This dynamic behavior is a fundamental property of autoregressive models that is very relevant because of the multi-period character of economic scenarios. Furthermore, two new types of VAR models are introduced that are both used in the new scenario framework as described in Part IV. The simplest is the class of Truncated VAR (TVAR) models that, instead of on symmetrical normal distributions, are based on truncated normal distributions while maintaining the same first and second moment matrices (averages and covariances). TVAR models can be used for the modeling of variables with skewed distributions of which the value is not allowed to reach a value above or below some critical level. Think for example of interest rates that should stay strictly positive. Much more complex is the class of EigenValue Restricted VAR (EVR VAR) models. In the literature many types of VAR models have been proposed that by applying restrictions on the model parameters try to offer a solution for the well known

problem of estimating very flexible and richly parameterized VAR models on in general limited time series data of economic variables. EVR VAR models offer the possibility to apply restrictions in terms of the parameters that at the very foundation determine the dynamic behavior of a VAR model: the eigenvalues and eigenvectors. Because of the strong non-linear character of the resulting models, a numerical estimation procedure is described that uses analytical first order derivatives as much as possible.

Chapter 4 contains the for this research fundamentally important theory of Spectral Analysis. As said before, this technique is not applied very much in the economic sciences as a counterpart of the conventional analysis in the time domain and is therefore in general not well known. We therefore chose to build up the relevant theory starting from the basic definitions of the underlying sine and cosine functions, through the Fourier transform and the periodogram, until the definitions of univariate and multivariate spectral densities of stochastic processes. A univariate spectrum (auto-spectrum) can be compared with a probability density function (pdf). In this case however, not a probability mass of one is distributed over a range of possible outcomes but the total variance of a process is distributed over fluctuations varying between the lowest frequency zero with an infinite period length and fluctuations with the, in discrete time, highest frequency of one half with a period length of two (years). From a spectrum it is easy to see what type of fluctuations dominate the behavior of a stochastic process. In a similar fashion also the correlations between variables in a multivariate stochastic process can be attributed to different frequencies. For each frequency, the correlation is split between an absolute correlation number (coherence spectrum) and a phase number (phase spectrum). Because VAR models are also descriptions of stochastic processes, it is possible to calculate the spectra of the corresponding processes directly from the model parameters. Next, Chapter 4 deals with how spectra can be efficiently estimated from the in general short time series of economic variables. Based on the information theoretical concept of Entropy it has been proven that the most “neutral” estimator of a spectrum is obtained by first estimating an autoregressive model on the data and then to calculate the spectrum of this estimated model. Here “neutral” means that this estimator adds the minimum additional information about the stochastic process to the information that is available in the sample, under the restriction that the spectrum obtained is consistent with the information from the sample. The resulting parametric estimator is called the Maximum Entropy estimator of a spectrum. Because the Maximum Entropy concept does not provide any information on this, Chapter 4 also describes an extensive Monte Carlo experiment that we performed to determine what estimation method and what model order can best be used in practice. The results indicate that the Yule-Walker estimation method leads to the best results. It is of crucial importance that this uses the biased estimators of the autocovariances because this results in imposing a valuable stationarity restriction on the estimated process. With respect to the model order it appears that it is better to use order selection criteria that tend to overestimate rather than criteria that tend to underestimate the model order. The intuition is here that the harmful effects of choosing too low a model order can never be corrected by the subsequent estimation procedure. Furthermore, selecting the model order should never become some fully automated process. Best is to keep increasing the model order until the information in the spectrum does not significantly change anymore

and only “noise” is added. Finally, in Chapter 4 it is indicated how the statistical significance and the confidence intervals of a Maximum Entropy spectral estimate can be calculated.

Chapter 5 contains the theory of linear filters. In principle, these are simple filters that construct a new, filtered, time series by replacing each value from a time series by a weighted average of past and future values from the original time series. To assess the effects that different filters have on the properties of a time series, it is possible to calculate for each filter the extent to which fluctuations of different frequencies are suppressed or amplified by the filter (Power Transfer Function). Furthermore, it is possible to calculate what (phase) shifts in time are caused by the filter for each individual frequency. These two effects are calculated for a number of well known filters. In this research, the application of such filters is to decompose time series into the desired frequency components (trend, long waves and business cycle fluctuations) with as little distortion as possible. By this we mean that ideal (sharp) pass-bands are used, that no phase shifts occur and that no data is lost at the beginning or end of the samples that are already relatively short in most cases. Furthermore, it needs to be possible to easily adjust the parameters of the filter such that any desired decomposition can be accomplished. Virtually all filters known from the literature are originally defined in the time domain. Assessing the effects of these filters in the frequency domain shows that these filters do not meet these requirements. However, Chapter 5 contains the theory of a filter that does meet these requirements. This filter is based on existing ideas from the literature. We further refined and extensively tested this filter. The only disadvantage of this frequency domain filter is that the results from previously filtered time series are not exactly the same once new data comes available. The filter is a further perfected and tested version of a filter from (less well known) literature. Because of the numerical character of the filter, just as for the Maximum Entropy spectral estimator, simulation experiments have been performed to test the filter and to find the optimal parameter settings for practical applications. The Maximum Entropy estimator and the frequency domain filter have in common that, based on the very foundations of the frequency domain methodology, both first make an extrapolation in the time domain for the pre and post sample period before making the transition into the frequency domain. In this way, in both cases the fundamental (leakage) problem of applying spectral techniques on short sample time series is avoided. In case of the Maximum Entropy spectral estimator this is done by extrapolating the sample autocovariances based on an autoregressive model and then transforming these autocovariances into the frequency domain by the Fourier transform. In case of the frequency domain filter this is done by fitting the sample data with a large number of sine and cosine functions which in principle extend into infinity and then determining which of the sine and cosine functions should be maintained or suppressed according to the pass-band of the filter.

## 21.4 Summary Part III: Stylized Facts

Part III consists of Chapter 6 until 17 that together describe the results on the first specific research question as formulated in the introduction. The objective is to gather broad empirical information about the behavior of macroeconomic variables based on one and the same methodology. In case such information is found to be robust with respect to for example the historical time period or the country of origin, we speak of stylized facts. The chapters of Part III can be divided into three parts.

The first part consists of Chapter 6 which describes the specific methodology that is used to conduct the empirical research. This methodology is constructed as an optimal combination and parameterization of the three techniques from Part II. Annual time series dating as far back as possible have been collected for nine economic variables for both the Netherlands, the United Kingdom (UK) and the United States (US). The variables are: national product, industrial production, unemployment, consumer prices, wages, short and long term interest rates, credit spreads and stock prices. Most of the samples run from the middle of the nineteenth century (1850) until the end of the twentieth century (1999). Where possible, the time series have been corrected for the evolution of the general price level and therefore hold in real terms. The actual time series that have been used and their sources are described in Appendix F. The analysis that has been applied on each of these time series as well as on a number of interesting transformations of these time series, such as for example the equity risk premium, consists of the following three steps:

1. Data analysis
2. Filtering results
3. Spectral Analysis

The data analysis (step 1) starts with a simple graphical display of the time series that is combined with a number of simple statistics such as the sample averages, standard deviations and correlations, calculated on different sub-samples. In the filtering process (step 2), by using the frequency domain filter from Chapter 5, time series are decomposed into the following three components that are based on thinking about economic fluctuations in the frequency domain:

- (a) Trend component: This is defined as all fluctuations in a time series with a period length longer than the sample size. So, this are all components of a time series that are not able to complete a full cycle within the sample size, including the average value of the time series.
- (b) Periodic component: This is defined as all fluctuations in a time series with a period length between the sample size and fifteen years. So, this are all long term fluctuations that have occurred around the underlying trend component. The limit of fifteen years is primarily chosen as a comfortable upper limit on all the types of alleged business cycle fluctuations that are contained in the third component.
- (c) Stochastic component: This is defined as all fluctuations in a time series with a period length shorter than fifteen years. This component is supposed to contain all business cycle fluctuations that have occurred around the sum of the underlying trend and periodic component.

The resulting trend and periodic component can subsequently be analyzed directly by using simple graphical and numerical overviews. The third component is called stochastic because the higher frequency of these fluctuations render it useful to consider it as stochastic and then learn more about its underlying dynamic behavior by using the Maximum Entropy spectral analysis technique from Chapter 4 to estimate multivariate spectral densities (step 3) on these filtered series. On the one hand, these spectra are estimated in a static sense on three sub-periods, the pre World War I period, the period between World War I and II and the period after World War II. On the other hand, for the postwar period, also rolling window estimates are performed to trace possible changes that might have occurred in the business cycle behavior in the course of time. The results from the Spectral Analysis are on the one hand contained in a graphical presentation. On the other hand, the properties of the resulting spectral densities are also summarized in several representative numbers. The results from all the analyses are complemented by discussions in a bulleted format in order to present the information as compact and clear as possible. Furthermore, it is important to note that the total methodology is not meant to generate results in fully automated sense. At various stages of the analysis, (expert) judgement is needed, for example by defining the appropriate sub-samples, adjusting the filter boundaries or choosing the appropriate model order in the Maximum Entropy spectral estimator.

The second part of Part III consists of Chapter 7 until 16 which contain the results that are obtained by applying the uniform methodology from Chapter 6. Each chapter contains the analysis of one of the economic variables mentioned or some transformation of these variables. The graphical results most of the times concern the Dutch data. Where possible these are compared to the UK and US results. The complete graphical results for the UK and US are contained in respectively Appendix D and E. Contrary to the graphical results, the complete numerical results for all three countries are contained in the various chapters. The outline of the analysis in each of the chapters is identical and corresponds to the steps of the methodology described in Chapter 6. Some chapters contain some additional and specific analyses that are only interesting for the relevant variable. Examples are the long term evolution of real interest rates in Chapter 13, the relation between the level of interest rates and their business cycle volatility (level effect) in Chapter 12 and 13, the relative importance of price fluctuations and dividend yields in equity returns and the short and long term inflation hedging capacities of equities, both in Chapter 16. Together with the aforementioned Appendix D and E, the results in these chapters constitute a detailed and consistent reference to study the empirical behavior of macroeconomic variables in the countries mentioned.

The third part of Part III consists of Chapter 17 in which the results from Chapter 7 until 16 are summarized in terms of almost ninety stylized facts. We speak of stylized facts if the observed empirical behavior of economic variables appears to be robust with respect to the country and time period for which the behavior can be observed. The stylized facts are grouped into the long run results in terms of the trends and the long term fluctuations, the business cycle fluctuations and the changes in these business cycle fluctuations in the course of time. Chapter 17 thereby constitutes the “answer” to the first specific research question as formulated in the introduction (“clear set of assumptions”). Because of the magnitude of the results it is not possible to include all the stylized facts in this summary. Instead, we



limit ourselves to a description of the most important general results. This means that various interesting specific results that we found are not included in this summary. Examples are the positive relation between the level of nominal interest rates and their business cycle volatility ("level effect"), how business cycle fluctuations can imply the empirical mean reversion patterns in stock prices and the long term (longer than fifteen years) inflation hedging capacities of stocks versus their poor short term inflation hedging performance and how this can be understood from the business cycle behavior. With respect to the trends the general results are as follows:

- The long term averages in case of level variables (for example interest rates) or average growth rates in case of trending index variables (for example national product) match well with economic intuition and long term growth theories.
- The long term average real growth rate of the national product is approximately 3% per annum. Approximately 1% of this growth is caused by population growth.
- The long term average unemployment rate is approximately 5%.
- Until 1900, the average consumer price inflation was approximately 0%, thereafter 3.5% per annum.
- The long term average nominal short term interest rate is approximately 4%.
- The long term average nominal long term interest rate is approximately 5%. The average term spread (premium) is therefore approximately 1%.
- The long term average real long term interest rate is approximately 3% which is approximately the same as the long term average real growth of the national product.
- The long term arithmetic average real return on equities is approximately the sum of a 3% real increase in equity prices and a 5% dividend yield. The long term average real increase in equity prices is therefore approximately the same as the long term average real long term interest rate and real economic growth rate while the arithmetic equity risk premium is approximately the same as the average dividend yield, the compensation for the business risk taken.

With respect to the long term fluctuations (periodic component) in the evolution of economic variables, the most important general results are as follows:

- Around the long term trends or averages, the development of both real and financial macroeconomic variables is without exception for a substantial part characterized by worldwide long wave or Kondratieff type fluctuations with an average period length of approximately fifty years.
- These long wave components on average describe approximately 60% of the volatility of the broader periodic components (36% of the variance).
- The approximate dating of the subsequent peaks and troughs of these long waves with the corresponding peak-to-peak and trough-to-trough periods matches well with what is known in the literature and is as follows:

	1826	42	1868	52	1920	52	1972	
1790	57	1847	47	1893	54	1947	47	1994

- These long wave fluctuations cause that significant deviations from the previously described long term trends and averages can occur for longer periods of time.

- The long wave fluctuations occur in the various economic variables with specific lead and lag periods.
- Because of the limited number of cycles that have been completed, finding these fluctuations in historical time series does not necessarily mean that they are caused by some self generating endogenous economic mechanism. An alternative explanation could be that they are the consequence of specific historical economic circumstances (i.e. and exogenous mechanism).

With respect to the business cycle fluctuations (stochastic component) in the evolution of economic variables, the most important general results are as follows:

- Juglar and Kitchin type business cycle fluctuations are consistently and to a large extent also statistically significant present in all economic variables for as long as data has been collected for these variables.
- The Juglar fluctuations have an average period length of approximately ten years and describe approximately 30% of the variance of the broader stochastic component.
- The Kitchin fluctuations have an average period length of approximately four and a half years and describe approximately 20% of the variance of the broader stochastic component.
- The business cycle fluctuations occur in the various economic variables with specific lead and lag periods.
- The for these phase differences corrected correlations are high, both within countries as between countries, which indicates a strong connection between business cycle fluctuations.

With respect to the changes in the behavior of the business cycle fluctuations (stochastic component) in the evolution of economic variables, the most important general results are as follows:

- The variance of business cycle fluctuations was the greatest during the interwar period. During the postwar period it was smaller than during the prewar period. The absolute variance can therefore change over time.
- However, the relative volatilities and the dynamics of business cycle fluctuations in terms of period lengths and phase relations have remained remarkably stable during the past centuries.
- During the postwar period, the period length of the Kitchin fluctuations has increased by approximately one year.
- Furthermore, the business cycle volatility (standard deviation) has increased by approximately 60%.
- The importance of the Juglar fluctuations in describing the total business cycle variance has increased by approximately 5%. The importance of the Kitchin fluctuations has decreased by approximately 5%.
- Business cycle fluctuations have become approximately 5 to 10% more periodic.
- The international correlations in terms of the Juglar and Kitchin business cycle fluctuations have increased by respectively approximately 30 and 10%.
- Nationally, these increases are respectively approximately 10 and 5%
- The lagging of the Dutch business cycle on that of the UK and US has decreased by approximately one year. Apart from that, all phase relations have remained pretty much the same.



With respect to an overall decomposition of the variance in the evolution of economic variables around the underlying long term trends, the most important general results are as follows:

- The long wave or Kondratieff type fluctuations on average describe approximately 25% of the total variance of the level of macroeconomic variables. For the first order differences this is approximately 5%.
- The Juglar type business cycle fluctuations on average describe approximately 10% of the total variance of the level of macroeconomic variables. For the first order differences this remains approximately 10%.
- The Kitchin type business cycle fluctuations on average describe approximately 5% of the total variance of the level of macroeconomic variables. For the first order differences this is approximately 25%.

## 21.5 Summary Part IV: Scenarios and Stylized Facts

Part IV consists of Chapter 18, 19 and 20 that together describe the results on the second specific research question as formulated in the introduction. To what extent do conventional VAR scenario applications indeed lead to scenarios that are consistent with the “benchmark” empirical knowledge as found in Part III and summarized in terms of the stylized facts in Chapter 17 and, if this is not the case, how can these models be modified to do so?

Chapter 18 describes the specific methodology to test this “consistency”. The methodology is linked to the methodology used for the research of Part III as described in Chapter 6 and therefore also heavily relies on the models and techniques from Part II. First, based on a number of actual scenario applications of VAR models from the literature, a set of stylized applications is constructed in terms of data representation, estimation method and model order. The in total twenty six resulting six dimensional VAR models are estimated for the Netherlands based on the same time series as used in Part III. Because in scenario applications of VAR models it is easy to separately change the averages or trends, the models are then especially tested on the empirical knowledge in terms of the long wave and business cycle fluctuations. The testing is done by comparing the multivariate spectra of the estimated models with the properties of the spectra found in Part III.

Chapter 19 describes the results of the testing of the VAR models. The results are summarized in section 19.3. which thereby constitutes the “answer” to the first part of the second specific research question as formulated in the introduction (“consistency”). Because of the magnitude of the results it is not possible to include all results in this summary. Instead, we limit ourselves to a description of the most important general results. These are as follows:

- Conventional applications of VAR models in many respects do not result in scenarios that are consistent with the empirical knowledge as found in Part III.
- In general, too low model orders ( $p=1$ ) are used. Higher order models often give better results.
- Partially because Spectral Analysis is rarely used, one is often not well aware of the for multi-period scenario applications important dynamic properties of the estimated models. This for example holds for the presence of long wave or business cycle fluctuations and the phase and correlation properties at those frequencies.

- It appears not to be possible to use one model on a single set of data to simultaneously give a good description of both the low and high frequency properties of time series. By the choice of a data representation and estimation procedure often implicitly one of the frequency regions is considered as more important. However, the dynamics at all frequencies are in principle equally important.
- The data used for estimating models and the frequencies to be modeled are often unintentionally not well in line with each other. It is for example strange that “only” fifty years of data is used for giving information on the scenario behavior in terms of long waves with a period length of fifty years. After all, this sample in principle contains only one observation for the behavior at such a low frequency.
- By using unrestricted multivariate VAR models, both of low and high orders, it is difficult to appropriately describe the univariate spectra of the relevant variables. Low order models fail to pick up the right dynamics, although they do have enough flexibility to describe these dynamics. High order models offer too much flexibility which “blurs” the true dynamics.
- Phase relations appear to be estimated well (robust) in almost any situation.

These results show that by using conventional applications of VAR models it is in general in many ways difficult or even impossible to generate scenarios that are consistent with the empirical knowledge as found in Part III. Therefore, the final chapter of this research is devoted to describing and illustrating alternative models with which it is possible to generate such scenarios. Thereby, Chapter 20 constitutes the “answer” to the second part of the second specific research question as formulated in the introduction. How can VAR models be modified to solve for the shortcomings that we found?

Chapter 20 can be divided into two parts. In the first part, section 20.1, a formal description is given of a new frequency domain VAR scenario framework that solves many of the shortcomings of conventional models that were found. This new framework does not result in a fully automated process that by pressing a button results in a set of scenarios with the desired properties. Instead, the framework offers a flexible modeling environment in which the complex behavior of economic variables can be adequately described. The most important components of this newly developed framework are:

- (a) Multifactor VAR (MVAR) models: The fundamental idea of these models is to construct a separate VAR model (stochastic factor) for every frequency range that is considered relevant. For this, first the time series to be modeled need to be filtered into these frequency ranges by using the frequency domain filter from Chapter 5. Next, a separate model can be estimated for each filtered series of which the resulting scenarios can be simply added up. This approach can be justified by the fact that, because of a fundamental property of the frequency domain methodology, time series that are adequately filtered into non-overlapping frequency ranges are uncorrelated and that the filtered time series constitute a simple additive decomposition of the original time series. These MVAR models provide a better description of the behavior of economic variables in different frequency ranges, offer the possibility to use different observation frequencies and sample periods for each component model and offer the possibility to model complex relations between the behavior in different frequency ranges, such as for example the level effect in interest rates. The models can be extended with a

stochastic trend modeling for the behavior at the ultra low frequencies. Finally, these models create a better awareness of the model and therefore also of the scenario properties. This does mean that for some aspects of the behavior of economic variables some difficult choices need to be made, for example about the variance and correlations in the long term trends. In conventional VAR model applications, these assumptions are unintended and implicitly made by the chosen modeling approach.

- (b) Backtesting procedure: By a specific combination of the confidence intervals and the multivariate spectra of the estimated models it can be tested to what extent the models indeed are consistent, or are on purpose not consistent, with the behavior of historical time series.
- (c) Truncated VAR (TVAR) models: These are the adjusted VAR models as described in Part II. They can be used to model skewed distributions of variables of which the value is not allowed to reach above or below some critical level. Think for example of interest rates that should remain strictly positive.
- (d) EigenValue Restricted VAR (EVR VAR) models: These are the adjusted VAR models as described in Part II. They can be used to estimate VAR models with restrictions on the eigenvalues in order to adequately describe the univariate spectra of the relevant variables. Think for example of the behavior at the business cycle frequencies.
- (e) Adjustment possibilities: By means of the Yule-Walker equations and the representation of VAR models in terms of the eigenvalues and eigenvectors, it is possible to by hand directly adjust volatilities and averages as well as the important dynamic behavior of the models. In terms of the conventional parameters of VAR models the latter cannot be done easily. These possibilities for adjusting the models can for example be used to deviate on purpose from the historical behavior of the relevant variables by incorporating forward looking information or for performing a sensitivity analysis.

Especially the MVAR and EVR VAR models are completely new types of (restricted) VAR models that will hopefully find many other useful applications. In the second part of Chapter 20, section 20.2 until 20.5, four examples of this new scenario framework are presented and discussed that separately illustrate the workings of the different components. Furthermore, constructing the examples is meant for gaining experience with the new framework. An example in which all components are used at the same time is not given here. However, combining the different components of the framework is relatively straightforward.

## 21.6 Directions for future research and applications

We end this chapter, and thereby this research, with a number of directions for future research and applications of the results from this research. In the first place, the empirical behavior of the studied macroeconomic variables at higher frequencies needs to be investigated. In this research the focus has been on the trending behavior, the long term fluctuations covering several decades and the business cycle fluctuations covering several years. Based on the same methodology and by using the same models and techniques, it is possible to extend the research to for example seasonal fluctuations at monthly frequencies and the behavior at a weekly or even daily basis. Issues such as stochastic volatilities will probably also become important here.

A second direction for future research is to extend the number of studied variables and countries. Difficult variables such as exchange rates have for example been left out of consideration here. Furthermore, one can think of extensions towards more types of investment classes than only stocks and towards the different components of the national product such as consumption, export and import and government expenditure. By on the one hand extending the analyzed frequencies and on the other hand extending the variables and countries involved, based on a uniform methodology, an even more complete and consistent view can be created of the empirical behavior of macroeconomic variables that can be used for many different purposes.

A third and final important direction for future research is to investigate the consequences for strategic policy decisions when the newly developed scenario framework is actually applied. Because these decisions are in principle conditional on the properties of the scenarios that are used and hence cannot be used for testing scenarios, this aspect has deliberately been left out of consideration here. However, now that scenarios are available that can describe the “correct” empirical behavior, such research can take place by applying scenarios that are generated by the new scenario framework in existing ALM models for example for pension funds and insurance companies. The example in section 1.2 illustrates the enormous impact scenario properties can have on both the absolute and relative outcomes of ALM models and thereby also on the policy decisions that are based on these model outcomes. Besides investigating the sensitivity for the different empirical properties, an interesting line of thought is to what extent different levels of decision making can be connected to the scenario properties in the different frequency ranges. For example, the very long run low frequency behavior of economic variables could be used for long term strategic decision making while the short term high frequency behavior could be used for more short term tactical types of decision making.

In the end, by applying and further developing the new scenario framework, the results from this research will hopefully contribute to the higher level of strategic decision making, we originally set out to achieve.

## Samenvatting / Summary in Dutch

### Macro Economische Scenario's en de Werkelijkheid

#### Introductie

Dit onderzoek handelt over macro economische scenario's. Een scenario is hierbij een mogelijke toekomstige ontwikkeling van in de context van hun toepassing relevante macro economische variabelen als het nationaal product, werkloosheid, prijsinflatie, rentes en aandelenkoersen. Scenario's worden onder andere veelvuldig toegepast in Asset & Liability Management (ALM) voor bijvoorbeeld pensioenfondsen en verzekeraars. In ALM worden de gevolgen van strategische beleidsalternatieven, in termen van bijvoorbeeld asset allocatie en duration beleid, in een simulatiemodel doorgerekend voor een groot aantal (bijvoorbeeld duizend) stochastische economische scenario's. Het resultaat bestaat uit de bij de beleidsvarianten behorende scenario's van variabelen als bijvoorbeeld de solvabiliteitspositie welke van belang zijn om de consequenties van het beleid in termen van de doelstellingen en randvoorwaarden van de verschillende stakeholders te kunnen beoordelen. Dergelijke uitkomsten ondersteunen directies en bestuurders bij het nemen van strategische beleidsbeslissingen. Het is niet moeilijk voor te stellen dat de eigenschappen van de gebruikte economische scenario's van grote invloed kunnen zijn op niet alleen de absolute maar ook op de relatieve risico- en rendements-uitkomsten van dergelijke simulaties. Hierdoor hebben de eigenschappen van de scenario's enorm veel invloed op de uiteindelijk gemaakte strategische beleids-keuzes. Het is daarom van groot belang om zeer zorgvuldig om te gaan met het genereren en evalueren van dergelijke economische scenario's. De algemene doelstelling van dit onderzoek is daarom om bij te dragen aan een hogere kwaliteit van economische scenario's en daarmee ook aan een hogere kwaliteit van de strategische besluitvorming die op deze scenario's wordt gebaseerd. De twee specifieke doelstellingen van dit onderzoek worden precies samengevat in één specifieke scenario definitie van Bunn en Salo (1993) welke stellen dat een scenario een mogelijke ontwikkeling van de toekomst is welke consistent moet zijn met een heldere set van aannamen.

De eerste specifieke doelstelling heeft betrekking op de heldere set van aannamen. In zowel praktijk als wetenschap wordt deze vaak ingevuld door de eigenschappen van de scenario's statistisch gezien overeen te willen laten komen met het empirische gedrag dat economische variabelen in het verleden hebben vertoond. Op het eerste gezicht lijkt dit een redelijk heldere en adequate aanname. Bij nadere inspectie blijkt echter dat vaak onvoldoende bekend is wat deze aanname concreet inhoudt. Met andere woorden: Welk empirisch gedrag van economische variabelen wordt hiermee eigenlijk bedoeld? De eerste specifieke doelstelling van dit onderzoek is daarom om op basis van een uniforme en zo zuiver mogelijke methodologie zo breed mogelijke empirische kennis te verzamelen over het empirische stochastische gedrag van macro economische variabelen in ontwikkelde westerse economieën. Het gaat hierbij om variabelen uit zowel de reële als uit de financiële sector van de economie



en de interactie daartussen, om zowel lange als korte termijn (conjunctuur cyclus) gedrag en om zowel het overall gedrag van de variabelen als om mogelijke veranderingen van dit gedrag door de tijd heen. Gebaseerd op ongeveer 150 jaar data en gebruik makend van een nieuw ontwikkelde analyse methode, resulteert dit deel van het onderzoek in bijna negentig zogenaamde stylized facts over het gedrag van macro economische variabelen. We spreken van stylized facts wanneer het gevonden empirische gedrag van economische variabelen robuust blijkt te zijn voor het land en de tijdsperiode waarvoor het gedrag kan worden geobserveerd.

In zowel praktijk als wetenschap wordt voor het modelleren en genereren van scenario's vaak gebruik gemaakt van de klasse van Vector AutoRegressieve (VAR) tijdreeks modellen. De tweede specifiek doelstelling is te onderzoeken in hoeverre deze VAR modellen, en met name ook de wijze waarop ze in conventionele scenario toepassingen worden gebruikt, inderdaad tot scenario's leiden die consistent zijn met het empirische gedrag dat aanwezig is in de data die voor het schatten van de modellen wordt gebruikt. De modellen worden hierbij getoetst aan de als "benchmark" beschouwde empirische kennis die in termen van de stylized facts is verkregen. Omdat uit dit onderzoek een aantal tekortkomingen van conventionele scenario toepassingen van VAR modellen naar voren komen, wordt een nieuw op VAR modellen gebaseerd raamwerk voor het genereren van scenario's ontwikkeld en toegepast, dat is gebaseerd op het frequentiedomein in plaats van op het tijd domein. Het is belangrijk op te merken dat, ook al zijn de twee bovenstaande doelstellingen van het onderzoek geplaatst in de vrij specifieke context van ALM scenario modellen, de resultaten relevant zijn voor veel andere typen economisch onderzoek en toepassingen. De stylized facts zijn bijvoorbeeld zeer relevant voor het bouwen en toetsen van macro economische modellen in het algemeen terwijl de resultaten van het toetsen van de VAR modellen ook relevant zijn voor onderzoek naar optimale asset allocatie beslissingen.

Het onderzoek is georganiseerd in vier delen. Deel I fungeert als introductie waarin ook een overzicht wordt gegeven van alle beschikbare technieken en paradigma's. Uit deze lijst zijn drie centrale technieken gekozen welke het best geschikt lijken voor de onderzoeksdoelstellingen. Deel II beschrijft zowel de theoretische en technische achtergrond alsmede de uitgevoerde experimenten van deze drie gekozen centrale technieken. Deel III beschrijft resultaten met betrekking tot de eerste specifieke vraagstelling ("heldere set van aannamen") zoals verkregen met behulp van een nieuwe ontwikkelde analyse methode. Deel IV beschrijft de resultaten met betrekking tot de tweede specifieke vraagstelling ("consistent met") en het nieuw ontwikkelde scenario raamwerk.

Het onderzoek heeft geresulteerd in een lijvig boekwerk van bijna achthonderd pagina's. Ten dele komt dit doordat veel resultaten zijn verkregen. Ten dele komt dit ook doordat gepoogd is om, gegeven een zeker verondersteld niveau van basiskennis, zo volledig en consistent mogelijk te zijn in termen van de gebruikte data, technieken en notatie, temeer omdat de centrale techniek binnen dit onderzoek, die van Spectraal Analyse, in tegenstelling tot de natuurwetenschappen, binnen economisch onderzoek minder bekend is. De bedoeling is dat het resultaat hierdoor zonder veel extra literatuur kan worden geraadpleegd. Er is daarbij gezorgd voor een zo modulair mogelijke opzet om lezers met verschillende niveau's van voorkennis en verschillende interesses eenvoudig toegang te bieden tot de voor hen relevante delen van het onderzoek. Met name Deel III is verder bedoeld om als breed en consistent

naslagwerk te dienen voor het bestuderen van het empirische gedrag van macro economische variabelen. Hieronder volgt een samenvatting van elk van de vier delen waarin het onderzoek is beschreven. We eindigen met een beschrijving van een aantal richtingen voor vervolgonderzoek en toepassingen van de resultaten uit dit onderzoek.

## Samenvatting Deel I: Introductie

Deel I bestaat uit Hoofdstuk 1 en 2 fungeert als introductie. Hierin wordt in de eerste plaats de achtergrond en relevantie van het onderzoek in termen van ALM en scenario analyse beschreven alsmede de twee specifieke doelstellingen van het onderzoek. Uit de literatuur is bekend dat de methoden die worden gebruikt voor empirisch onderzoek van macro economische tijdreeksen van grote invloed kunnen zijn op de gevonden resultaten. Het is zelfs zo dat resultaten kunnen worden gevonden die puur het gevolg zijn van de gebruikte technieken in plaats van het echte empirische gedrag van tijdreeksen. We spreken in dit geval van “spurious results”. Om dit zoveel mogelijk te voorkomen een zo zuiver en consistent mogelijke methodologie opgesteld en getest voor het verrichten van het empirische onderzoek in Deel III. Als eerste stap in het opstellen van een dergelijke methodologie hebben we een inventarisatie gemaakt van nu en in het verleden binnen de macro economische wetenschap toegepaste paradigma’s en technieken. Uit deze lijst zijn vervolgens drie centrale technieken gekozen welke in de basis het meest geschikt zijn bevonden voor de hier gestelde onderzoeksdoelen. Deze drie technieken zijn:

1. Filteren in het frequentiedomein
2. Maximum Entropy Spectraal Analyse
3. Vector AutoRegressieve (VAR) tijdreeks modellen

Het is belangrijk te benadrukken dat deze technieken in zowel het onderzoek naar de eerste vraagstelling (“heldere set van aannamen”, Deel III) als naar de tweede vraagstelling (“consistent met”, Deel IV) in één specifiek voor dit onderzoek toegesneden gecombineerde methodologie worden toegepast in plaats dat ze met elkaar worden vergeleken of elkaar op andere wijze beconcurreren. Deze gecombineerde methodologie wordt beschreven in het eerste hoofdstuk van Deel III (Hoofdstuk 6) en in het eerste hoofdstuk van Deel IV (Hoofdstuk 18). De redenen om juist deze drie technieken te kiezen zijn als volgt.

De keuze van filter technieken is in de eerste plaats het gevolg van het inzien van het nut van een decompositie in de lijn van Jan Tinbergen door economische ontwikkelingen op te splitsen in trends, conjunctuur cycli, seizoenseffecten en random fluctuaties. In de tweede plaats is een zinnige analyse van economische tijdreeksen alleen mogelijk onder de voorwaarde dan eventuele trends uit de tijdreeksen zijn verwijderd. Gegeven de grote hoeveelheid aan beschikbare detrending technieken en de mogelijke “spurious results” die deze kunnen opleveren, is een eerste vereiste dat de filtertechnieken zo zuiver en correct mogelijk zijn. Gegeven de doelstelling om het gedrag van economische variabelen op verschillende frequenties te bestuderen, variërend van de hele lange termijn tot kortere termijn conjunctuur fluctuaties, is een tweede vereiste dat de filtertechnieken voldoende flexibiliteit bieden om in te kunnen zoomen op de verschillende componenten van de tijdreeksen. De enige manier om aan deze twee vereisten te voldoen is door gebruik te maken van filter technieken in het frequentiedomein. Zonder het werkelijke Data Genererende



Proces (DGP) van de tijdreeksen te kennen moet het filteren van tijdreeksen “slechts” gezien worden als het toepassen van een set van geschikte statistische technieken om de relevante componenten uit de tijdreeksen zichtbaar te maken. Het direct ontwerpen en toepassen van deze technieken in het frequentiedomein komt het dichtst bij dit zo “optimaal” mogelijk te doen. De voor dit onderzoek relevante theorie van (lineaire) filter technieken wordt verder beschreven, ontwikkeld en getest in Hoofdstuk 5 in Deel II.

De keuze van Spectraal Analyse technieken is enerzijds afkomstig van de gekozen aanpak voor het filteren van tijdreeksen. Anderzijds, sluit Spectraal Analyse heel natuurlijk aan bij de verschillende typen van fluctuerende gedrag in macro economische tijdreeksen in termen van bijvoorbeeld lange golven, conjunctuur fluctuaties en seizoenseffecten. Voor het verzamelen van stylized facts over fluctuaties in macro economische tijdreeksen als in de eerste onderzoeksdoelstelling, blijkt het schatten van (multivariate) spectra daarom een heel efficiënte manier te zijn om informatie te rapporteren. De belangrijkste reden waarom Spectraal Analyse desondanks tot nu toe maar met weinig succes is toegepast in economisch onderzoek is dat klassieke (niet parametrische) spectraal technieken veel data vereisen om voldoende informatieve gegevens te kunnen verschaffen. Door parametrische Maximum Entropy spectraal technieken te gebruiken blijkt het echter ook mogelijk om efficiënt informatie te verkrijgen uit relatief korte tijdreeksen. De voor dit onderzoek relevante theorie van Spectraal Analyse wordt verder beschreven, ontwikkeld en getest in Hoofdstuk 4 in Deel II.

De keuze van VAR tijdreeks modellen komt enerzijds doordat de Maximum Entropy spectraal technieken nauw samenhangen met het berekenen van spectra op basis van geschatte VAR modellen. Anderzijds, worden VAR modellen zoals gezegd in onder andere ALM veelvuldig toegepast voor het genereren van economische scenario's waardoor dit ook het type modellen is dat in Deel IV getest wordt. De voor dit onderzoek relevante theorie van VAR modellen wordt verder beschreven, ontwikkeld en getest in Hoofdstuk 3 in Deel II.

## Samenvatting Deel II: Technische Achtergrond

Deel II bestaat uit de Hoofdstukken 3 tot en met 5 welke tezamen de theoretische en technische achtergrond alsmede de uitgevoerde experimenten beschrijven van de drie gekozen centrale technieken. Elk van deze hoofdstukken behandelt één van de drie genoemde technieken. Aan het begin van Deel III (Hoofdstuk 6) en aan het begin van Deel IV (Hoofdstuk 18) waarin het daadwerkelijke onderzoek naar de twee specifieke doelstellingen wordt besproken, worden deze technieken gecombineerd tot één specifiek op deze doelstellingen toegesneden methodologie.

Hoofdstuk 3 bevat in de eerste plaats de relevante algemene theorie van autoregressieve modellen. Naast algemene zaken als de beschikbare (deels minder bekende) schattingsmethoden en ordeselectie criteria en de manier waarop betrouwbaarheidsintervallen kunnen worden berekend, gaat daarbij ook speciale aandacht uit naar het geven van inzicht in hoe het dynamische stochastische gedrag van deze modellen tot stand komt als gevolg van de eigenwaarden en eigenvectoren van de parametermatrices. Dit dynamische gedrag is een fundamentele eigenschap van autoregressieve modellen welke zeer relevant is vanwege het meer periode karakter van economische scenario's. Verder worden twee nieuwe typen VAR

modellen geïntroduceerd welke beide worden gebruikt in het nieuwe scenario raamwerk als beschreven in Deel IV. De eenvoudigste hiervan is de klasse van Truncated VAR (TVAR) modellen welke, in plaats van op symmetrisch normaal verdeelde kansprocessen met de bekende dunne staarten, zijn gebaseerd op afgekapte normale verdelingen met behoud van de eerste twee momenten matrices (gemiddelden en covarianties). TVAR modellen kunnen worden gebruikt voor het modelleren van scheef verdeelde variabelen waarvan de waarde niet onder of boven een bepaalde grens mag komen. Denk bijvoorbeeld aan rentes die strikt positief moeten blijven. Een stuk complexer is de klasse van EigenValue Restricted VAR (EVR VAR) modellen. In de literatuur zijn vele varianten van VAR modellen bekend welke middels het aanbrengen van restricties op de parameters een oplossing proberen te bieden voor het algemeen bekende probleem van het schatten van erg flexibele en rijk geparameteriseerde VAR modellen op in het algemeen beperkte tijdreeksen data van economische variabelen. EVR VAR modellen bieden de mogelijkheid om restricties aan te brengen op de parameters die heel fundamenteel het dynamisch gedrag van een VAR model bepalen: de eigenwaarden en eigenvectoren. Vanwege het sterk niet lineaire karakter van de resulterende modellen wordt een numerieke schattingsmethode beschreven welke zoveel mogelijk gebruik maakt van analytisch bepaalde eerste orde afgeleiden.

Hoofdstuk 4 bevat de voor dit onderzoek van fundamenteel belang zijnde theorie van Spectraal Analyse. Omdat deze techniek zoals gezegd in de economische wetenschappen weinig wordt toegepast als tegenhanger van de conventionele analyses in het tijddomein en daardoor ook in het algemeen niet goed bekend is, is ervoor gekozen deze theorie vanaf de basale definitie van de onderliggende sinus en cosinus functies via de Fourier transformatie en het periodogram op te bouwen tot aan de definities van univariate en multivariate spectraal dichtheden van stochastische processen. Een univariaat spectrum (auto-spectrum) kan worden vergeleken met een kansdichtheidsfunctie (pdf). Nu wordt echter niet een kansmassa van één verdeeld over een range van mogelijke uitkomsten maar wordt de totale variantie van een proces verdeeld over fluctuaties variërend tussen de allerlaagste frequentie nul met een oneindige periodelengte en fluctuaties met de, in discrete tijd, allerhoogste frequentie van een half met een periodelengte van twee (jaren). Op basis van een spectrum is eenvoudig te zien welk type fluctuaties het meest bepalend zijn voor het gedrag van een stochastisch proces. Op soortgelijke wijze kunnen ook correlaties tussen variabelen in een multivariaat stochastisch proces worden ontleed naar verschillende frequenties. Per frequentie wordt de samenhang gesplitst in een absoluut correlatie getal (coherence spectrum) en een fase getal (phase spectrum). Omdat VAR modellen ook beschrijvingen zijn van stochastische processen, kunnen op basis van de parameters van deze modellen direct de spectra worden berekend van de processen die ze beschrijven. Vervolgens behandelt Hoofdstuk 4 hoe spectra efficiënt kunnen worden geschat op basis van de in het algemeen korte tijdreeksen van economische variabelen. Op basis van het informatie theoretische concept Entropy is bewezen dat de meest “neutrale” schatter van een spectrum wordt verkregen door eerst een autoregressief model op de data te schatten en vervolgens het spectrum van dit geschatte model te berekenen. Met “neutraal” wordt hier bedoeld dat deze schatter zo min mogelijk extra informatie over het stochastische proces toevoegt aan de informatie die in de gebruikte steekproef aanwezig is, onder de restrictie dat het gevonden spectrum wel consistent is met de informatie uit de

steekproef. De resulterende parametrische schatter wordt de Maximum Entropy schatter van een spectrum genoemd. Omdat het Maximum Entropy concept hier nog niets over zegt, bevat Hoofdstuk 4 verder een uitgebreid Monte Carlo experiment dat we hebben uitgevoerd om te achterhalen welke schattingsmethode en welke modelorde er in de praktijk het beste kan worden gebruikt. Hieruit blijkt dat de Yule-Walker schattingsmethode tot de beste resultaten leidt. Van belang is hierbij dat hiervoor de onzuivere autocovariantie schatters worden gebruikt omdat deze een waardevolle stationariteitsrestrictie opleggen aan het geschatte proces. Wat betreft de te gebruiken modelorde blijkt dat het beste ordeselectie criteria kunnen worden gebruikt die de orde eerder overschatten dan onderschatten. De intuïtie hierachter is dat de schadelijke effecten van een te lage orde nooit meer kunnen worden gecorrigeerd door de daarop volgende schattingsmethode. Verder moet ordeselectie nimmer een volledig geautomatiseerd proces worden. Er beste is te kijken vanaf welke orde de informatie in het geschatte spectrum niet meer wezenlijk veranderd maar feitelijk alleen nog “ruis” worden toegevoegd. Tot slot wordt in Hoofdstuk 4 aangegeven op welke wijze statistische significantie en betrouwbaarheidsintervallen van Maximum Entropy spectraal schattingen kunnen worden berekend.

Hoofdstuk 5 bevat de theorie van lineaire filters. Dit zijn in principe eenvoudige filters welke uit een gegeven tijdreeks een nieuwe, gefilterde, tijdreeks construeren door iedere waarde te vervangen door een gewogen gemiddelde van zowel historische als toekomstige waarden van de oorspronkelijke tijdreeks. Om inzichtelijk te maken welke effecten verschillende filters hebben op de eigenschappen van een tijdreeks kan van elk filter worden berekend in welke mate fluctuaties van de verschillende frequenties worden onderdrukt door het filter of juist versterkt (Power Transfer Function). Daarnaast kan worden berekend welke (fase) verschuivingen in de tijd per frequentie worden veroorzaakt door een filter. Van een aantal bekende filters zijn deze beide effecten inzichtelijk gemaakt. De toepassing van dergelijke filters in dit onderzoek bestaat eruit dat tijdreeksen met zo min mogelijk verstoring moeten kunnen worden gefilterd of gedeclineerd naar fluctuaties in onderscheiden frequentie gebieden (trend, lange golf en conjunctuurfluctuaties). Met zo min mogelijk verstoringen wordt bedoeld het gebruik van ideale (scherpe) passbands zonder dat fase verschuivingen optreden en zonder verlies van data aan het begin of einde van de vaak toch al relatief korte steekproef. Tevens dient elke gewenste decompositie eenvoudig via de parameters van de filtertechniek tot stand te kunnen worden gebracht. Vrijwel alle in de literatuur bekende filters zijn in de basis gedefinieerd in het tijddomein. Beoordeling in het frequentiedomein toont aan dat deze filters niet aan de gestelde eisen kunnen voldoen. Hoofdstuk 5 bevat echter de theorie van een filter waarvoor dit wel het geval is. Dit filter is gebaseerd op bestaande ideeën uit de literatuur. We hebben dit filter verder verfijnd en uitgebreid getest. Het enige nadeel van dit frequentiedomein filter is dat het eerder gefilterde data niet perfect intact laat zodra meer data beschikbaar komt. Het filter is een verder geperfectioneerde en geteste versie van een filter afkomstig uit (minder bekende) literatuur. Vanwege het numerieke karakter van het filter zijn, net als bij de Maximum Entropy spectraal schatter, simulatie experimenten uitgevoerd om het filter te testen en de optimale instellingen te achterhalen. De Maximum Entropy schatter en het frequentiedomein filter hebben gemeenschappelijk dat ze beide eerst een onderbouwde en op de fundamenteën van het frequentiedomein denken aansluitende extrapolatie van de steekproefgegevens in het tijddomein maken naar

de periode voorafgaand aan de steekproefperiode en de periode volgende op de steekproefperiode alvorens de overstap te maken naar het frequentiedomein. Hierdoor wordt in beide gevallen het fundamentele (leakage) probleem van spectraal analyse toepassingen op korte tijdreeksen van economische data omzeild. Bij de Maximum Entropy spectraal schatter gebeurt dit door de autocovarianties van de steekproef te extrapoleren op basis van een autoregressief model en dan de informatie in deze autocovarianties naar het frequentiedomein te transformeren met behulp van de Fourier transformatie. Bij het frequentiedomein filter gebeurt dit door de steekproefgegevens te fitten met een groot aantal sinus en cosinus functies welke in principe tot in het oneindige doorlopen en vervolgens per sinus of cosinus te bepalen of de desbetreffende frequentie door het filter behouden moet blijven of moet worden verwijderd volgens de pass-band van het filter.

### Samenvatting Deel III: Stylized Facts

Deel III bestaat uit de Hoofdstukken 6 tot en met 17 welke tezamen de uitwerking van de eerste specifieke vraagstelling vormen als in de inleiding beschreven. De doelstelling is om zo breed mogelijke empirische kennis te verzamelen over het gedrag van macro economische variabelen gebaseerd op één en dezelfde methodologie. Als deze informatie robuust blijkt te zijn voor de historische periode of het land waarvoor het is geobserveerd, spreken we van “stylized facts”. De hoofdstukken van Deel III vallen uiteen in de volgende drie delen.

Het eerste gedeelte bestaat uit Hoofdstuk 6 waarin de voor het empirische onderzoek toegepaste specifieke methodologie beschreven wordt. Deze methodologie wordt gevormd door een zo optimaal mogelijke combinatie en invulling van de drie technieken uit Deel II. Hiertoe zijn zo lang mogelijke jaarlijkse tijdreeksen verzameld van negen economische variabelen voor zowel Nederland, het Verenigd Koninkrijk (VK) en de Verenigde Staten (VS). De variabelen zijn: nationaal product, industriële productie, werkloosheid, consumenten prijzen, loonontwikkeling, korte en lange rentes, credit spreads en aandelenkoersen. De meeste steekproeven lopen vanaf midden negentiende eeuw (1850) tot eind twintigste eeuw (1999). Waar mogelijk zijn de reeksen gecorrigeerd voor ontwikkelingen in het algemene prijspeil en gelden de reeksen dus in reële termen. De werkelijke tijdreeksen die zijn gebruikt en hun bronnen zijn beschreven in Appendix F. De analyse die is toegepast op elk van deze tijdreeksen alsmede op een aantal interessante transformatie van deze tijdreeksen als bijvoorbeeld de aandelen risicopremie, bestaat uit de volgende drie stappen:

1. Data analyse
2. Filter resultaten
3. Spectraal Analyse

Als start worden in de data analyse (stap 1) een eenvoudige grafische weergave van een tijdreeks gecombineerd met een aantal eenvoudige statistieken als steekproef gemiddelden, standaard deviaties en correlaties, berekend voor verschillende steekproef perioden.

Bij het filteren (stap 2) worden de tijdreeksen met behulp het frequentiedomein filter uit Hoofdstuk 5 gedeclineerd naar de volgende drie componenten welke sterk zijn gebaseerd op het denken over economische fluctuaties in het frequentiedomein:

- (a) Trend component: Deze is gedefinieerd als alle fluctuaties in een tijdreeks met een periodelengte langer dan de beschikbare steekproeflengte. Dit zijn dus alle componenten in de tijdreeks welke geen volledige golfbeweging kunnen doorlopen binnen de lengte van de steekproef, inclusief het gemiddelde van de tijdreeks.
- (b) Periodieke component: Deze is gedefinieerd als alle fluctuaties in een tijdreeks met een periodelengte tussen de steekproeflengte en vijftien jaar. Dit zijn dus alle lange termijn fluctuaties die zich rondom de onderliggende trend component hebben voorgedaan. De grens van vijftien jaar is hoofdzakelijk gekozen als een ruime bovengrens voor wat in het algemeen nog als conjunctuur fluctuaties wordt beschouwd welke in de derde component zijn opgenomen.
- (c) Stochastische component: Deze is gedefinieerd als alle fluctuaties in een tijdreeks met een periodelengte korter dan vijftien jaar. Deze wordt geacht alle conjunctuur fluctuaties te bevatten die zich rondom de som van de trend en de periodieke component hebben voorgedaan.

De resulterende trend en periodieke component kunnen vervolgens op basis van eenvoudige grafische en numerieke overzichten direct worden geanalyseerd. De derde component wordt stochastisch genoemd omdat het vanwege de hogere frequentie van deze fluctuaties zinvol is deze als stochastisch te beschouwen en vervolgens meer informatie over de dynamische aard van deze fluctuaties te verkrijgen door er met behulp van de Maximum Entropy spectraal analyse techniek uit Hoofdstuk 4 multivariate spectra op te schatten (stap 3). Deze spectra worden enerzijds statisch geschat voor de sub-perioden voor Wereld Oorlog (WO) I, tussen WO I en WO II en na WO II. Anderzijds, worden voor de na WO II periode ook dynamische rolling window spectraal schattingen uitgevoerd op basis waarvan wijzigingen van het conjunctuur gedrag door de tijd heen kunnen worden opgespoord. De resultaten van de Spectraal Analyse worden enerzijds grafisch weergegeven. Anderzijds worden de kenmerken van de resulterende spectra ook in een aantal kerngetallen samengevat. De resultaten van alle analyses worden in een puntsgewijs format voorzien van commentaar teneinde de informatie zo compact en helder mogelijk weer te geven. Het is verder belangrijk op te merken dat de totale methodologie niet bedoeld is om volledig automatisch resultaten te genereren. Op verschillende plekken is inbreng van de (ervaren) gebruiker vereist, bijvoorbeeld bij het definiëren van gedeeltelijke steekproeven, het aanpassen van de filtergrenzen of de keuze van de modelorde bij de Maximum Entropy spectraal schattingen.

Het tweede gedeelte van Deel III bestaat uit de Hoofdstukken 7 tot en met 16 welke de resultaten bevatten die met behulp van de uniforme methodologie uit Hoofdstuk 6 zijn verkregen. Elk hoofdstuk betreft de analyse van één van de genoemde economische variabelen of een transformatie daarvan. De grafische resultaten betreffen grotendeels die voor de Nederlandse data welke waar mogelijk met die van het VK en de VS worden vergeleken. De complete grafische resultaten voor het VK en de VS zijn weergegeven in respectievelijk Appendix D en E. De numerieke resultaten zijn wel voor alle drie de landen gezamenlijk in de hoofdstukken opgenomen. De hoofdlijn van de analyses is in elk van de hoofdstukken identiek en conform de stappen van de methodologie uit Hoofdstuk 6. Sommige hoofdstukken bevatten daarnaast een aantal specifiek voor de



desbetreffende variabele interessante analyses. Voorbeelden hiervan zijn de lange termijn ontwikkelingen van reële rentes in Hoofdstuk 13, de relatie tussen het renteniveau en de volatiliteit van de conjunctuur fluctuaties (level effect) in rentes in Hoofdstuk 12 en 13, het relatieve belang van koersontwikkelingen en dividend opbrengsten in aandelenrendementen en de korte en lange termijn inflatie hedge capaciteiten van aandelen, beide in Hoofdstuk 16. Tezamen met de genoemde Appendices D en E vormen de resultaten in deze hoofdstukken een gedetailleerd en consistent naslagwerk om het empirisch gedrag van macro economische variabelen in de genoemde landen te onderzoeken.

Het derde gedeelte van Deel III bestaat uit Hoofdstuk 17 waarin de resultaten uit de Hoofdstukken 7 tot en met 16 worden samengevat in termen van bijna negentig stylized facts. We spreken van stylized facts wanneer het gevonden empirische gedrag van economische variabelen robuust blijkt te zijn voor het land en de tijdsperiode waarvoor het gedrag kan worden geobserveerd. De stylized facts zijn gegroepeerd naar de gevonden lange termijn ontwikkelingen in termen van zowel trends als lange termijn fluctuaties, de conjunctuur ontwikkelingen en de veranderingen in deze conjunctuur ontwikkelingen door de tijd heen. Hoofdstuk 17 vormt daarmee het “antwoord” op de eerste specifieke onderzoeksvraag als geformuleerd in de inleiding (“heldere set van aannamen”). Vanwege de omvang van de resultaten is het niet mogelijk al de gevonden stylized facts in deze samenvatting op te nemen. In plaats daarvan volstaan we met een beschrijving in termen van de belangrijkste algemene resultaten. Dit betekent dat verschillende interessante specifieke resultaten die we hebben gevonden niet zijn opgenomen in deze samenvatting. Voorbeelden zijn de positieve relatie tussen het niveau van nominale rentes and hun conjunctuur volatiliteit (“level effect”), hoe conjunctuur fluctuaties de empirische mean reversion patronen in aandelenprijzen kunnen impliceren en de lange termijn (langer dan vijftien jaar) inflatie hedge van aandelen versus hun slechte korte termijn inflatie hedge prestaties en hoe dit kan worden begrepen uit het conjunctuur gedrag. Met betrekking tot de trends luiden de algemene resultaten als volgt:

- De lange termijn gemiddelden in geval van niveau variabelen (bijvoorbeeld rentes) of gemiddelde groei in geval van trendmatige index variabelen (bijvoorbeeld nationaal product) komen goed overeen met economische intuïtie en lange termijn groei theorieën.
- De lange termijn gemiddelde reële groei van het nationaal product bedraagt ongeveer 3% per jaar. Hiervan wordt ongeveer 1% veroorzaakt door bevolkingsgroei.
- Het lange termijn gemiddelde werkeloosheidspercentage bedraagt ongeveer 5%.
- Tot 1900 bedroeg de gemiddelde consumenten prijsinflatie ongeveer 0%, daarna ongeveer 3.5% per jaar.
- De lange termijn gemiddelde reële loongroei bedraagt ongeveer 1% per jaar.
- De lange termijn gemiddelde nominale korte rente bedraagt ongeveer 4%.
- De lange termijn gemiddelde nominale lange rente bedraagt ongeveer 5%. De gemiddelde termijn spread (premie) bedraagt dus ongeveer 1%.
- De lange termijn gemiddelde lange reële rente bedraagt ongeveer 3% wat ongeveer gelijk is aan de gemiddelde reële groei van het nationaal product.

- Het lange termijn rekenkundige gemiddelde reële rendement op aandelen is ongeveer gelijk aan 3% reël koersrendement plus 5% dividendrendement. Het lange termijn gemiddelde koersrendement is dus ongeveer gelijk aan de lange termijn gemiddelde reële rente en reële economische groei terwijl de rekenkundige risicopremie op aandelen ongeveer gelijk is aan het gemiddelde dividendrendement, de beloning voor het genomen extra risico.

Met betrekking tot de lange termijn fluctuaties (periodieke component) in de ontwikkelingen van economische variabelen luiden de belangrijkste algemene resultaten als volgt:

- Rondom de lange termijn groei of gemiddelden wordt de ontwikkeling van zowel reële als financiële economische variabelen vrijwel zonder uitzondering voor een substantieel gedeelte gekenmerkt door wereldwijde lange golven of Kondratieff golven met een gemiddelde periodelengte van ongeveer vijftig jaar.
- Deze lange golven beschrijven ongeveer 60% van de volatiliteit van de bredere periodieke componenten (36% van de variantie).
- De gevonden benaderende datering van de pieken en dalen van deze lange golven met de bijbehorende piek tot piek en dal tot dal periodelengten komt overeen met hetgeen hierover in de literatuur bekend is en luidt als volgt:

	1826	42	1868	52	1920	52	1972	
1790	57	1847	47	1893	54	1947	47	1994

- Deze lange golven zorgen ervoor dat gedurende langere perioden significante afwijkingen in gemiddelden en trends kunnen bestaan ten opzichte van de lange termijn trends.
- De lang golven komen met specifieke voor en achterloop periodes voor in de verschillende economische variabelen.
- Vanwege het beperkte aantal doorlopen golven wil het vinden van deze lange golven in historische tijdreeksen nog niet zeggen dat ze zijn veroorzaakt door een zichzelf genererend endogeen economisch mechanisme. Een alternatieve verklaring kan zijn dat ze het gevolg zijn van specifieke historische economische omstandigheden (een exogeen mechanisme).

Met betrekking tot de conjunctuur fluctuaties (stochastische component) in de ontwikkelingen van economische variabelen zijn de belangrijkste algemene resultaten de volgende:

- Juglar en Kitchin conjunctuur fluctuaties zijn consistent en grotendeels ook statistisch significant aanwezig in alle economische variabelen vanaf het eerste moment dat er data over deze variabelen werd verzameld.
- De Juglar fluctuaties hebben een gemiddelde periode lengte van ongeveer tien jaar en beschrijven ongeveer 30% van de variantie van de bredere stochastische componenten.
- De Kitchin fluctuaties hebben een gemiddelde periode lengte van ongeveer viereneenhalf jaar en beschrijven ongeveer 20% van de variantie van de bredere stochastische componenten.
- Deze conjunctuur fluctuaties komen met specifieke voor en achterloop periodes voor in de verschillende economische variabelen.



- De voor deze fase verschillen gecorrigeerde correlaties zijn zowel binnen landen als tussen landen hoog wat duidt op een grote samenhang in conjunctuur fluctuaties.

Met betrekking tot veranderingen in het gedrag van de conjunctuur fluctuaties (stochastische component) zijn de belangrijkste algemene resultaten de volgende:

- De variantie van conjunctuur fluctuaties was gedurende de periode tussen de twee wereld oorlogen het grootst. Gedurende de periode na WO II was deze kleiner dan gedurende de periode voor WO I. De absolute variantie kan dus variëren in de tijd.
- De relatieve volatiliteiten en de dynamiek van de conjunctuur fluctuaties in termen van periodelengten en fase relaties zijn echter verrassend stabiel gebleven door de eeuwen heen.
- Na WO II is de periodelengte van de Kitchin fluctuaties met ongeveer een half jaar toegenomen.
- Verder is de conjunctuur volatiliteit (standaard deviatie) gedurende deze periode met ongeveer 60% toegenomen.
- Het belang van de Juglar fluctuaties in het beschrijven van de totale conjunctuur variantie is met ongeveer 5% toegenomen. Het belang van de Kitchin fluctuaties is met ongeveer 5% afgenomen.
- Conjunctuurontwikkelingen zijn ongeveer 5 tot 10% meer periodiek geworden.
- De internationale correlaties op het gebied van de Juglar en Kitchin conjunctuur fluctuaties zijn respectievelijk met ongeveer 30 en 10% toegenomen.
- Nationaal, bedragen deze toenames respectievelijk ongeveer 10 en 5%.
- Het achterlopen van de Nederlandse conjunctuur op die van het VK en de VS is met ongeveer één jaar afgenomen. Verder zijn alle fase relaties redelijk constant gebleven.

Met betrekking tot een overall decompositie van de variantie in de ontwikkelingen van economische variabelen rondom de onderliggende lange termijn trends zijn de belangrijkste algemene resultaten de volgende:

- In termen van niveaus beschrijft de lange golf of Kondratieff golf gemiddeld ongeveer 25% van de totale variantie van de fluctuaties rondom de lange termijn trends of gemiddelden. In termen van groei of rendementen bedraagt dit ongeveer 5%.
- In termen van niveaus beschrijven de Juglar fluctuaties gemiddeld ongeveer 10% van de totale variantie van de fluctuaties rondom de lange termijn trends of gemiddelden. In termen van groei of rendementen blijft dit ongeveer 10%.
- In termen van niveaus beschrijven de Kitchin fluctuaties gemiddeld ongeveer 5% van de totale variantie van de fluctuaties rondom de lange termijn trends of gemiddelden. In termen van groei of rendementen bedraagt dit ongeveer 25%.

## Samenvatting Deel IV: Scenario's en Stylized Facts

Deel IV bestaat uit de Hoofdstukken 18, 19 en 20 welke tezamen de uitwerking van de tweede specifieke vraagstelling vormen als in de inleiding weergegeven. In welke mate resulteren conventionele scenario toepassingen van VAR modellen inderdaad in scenario's welke consistent zijn met de als "benchmark" beschouwde empirische kennis die is verkregen in Deel III en samengevat in termen van de stylized facts in Hoofdstuk 17 en, indien dit niet het geval is, hoe kunnen de modellen worden aangepast om dit te realiseren?

Hoofdstuk 18 beschrijft de specifieke methodologie die is gevolgd om deze "consistentie" te toetsen. De gebruikte methodologie sluit aan op de methodologie van het onderzoek in Deel III als beschreven in Hoofdstuk 6 en maakt dus ook zwaar gebruik van de modellen en technieken uit Deel II. Op basis van een aantal werkelijke scenario toepassingen van VAR modellen uit de literatuur worden als eerste een aantal gestileerde toepassingen opgesteld in termen van data representatie, schattingsmethode en modelorde. De in totaal zesentwintig resulterende zes dimensionale VAR modellen worden voor Nederland geschat op basis van dezelfde tijdreeksen als gebruikt in Deel III. Omdat in scenario toepassingen van VAR modellen de gemiddelden of trends eenvoudig los kunnen worden ingestuurd, worden deze modellen vervolgens vooral getoetst aan de empirische kennis in termen van de lange golven en de conjunctuur fluctuaties. Het toetsen gebeurt door de multivariate spectra van de geschatte modellen te vergelijken met de kenmerken van de spectra uit Deel III.

Hoofdstuk 19 beschrijft de resultaten van het toetsen van de VAR modellen. De resultaten zijn samengevat in paragraaf 19.3 welke daarmee het "antwoord" vormt op het eerste deel van de tweede specifieke onderzoeksvraag als geformuleerd in de inleiding ("consistentie"). Door de omvang is het niet mogelijk alle bevindingen in deze samenvatting op te nemen. In plaats daarvan volstaan we met een beschrijving in termen van de belangrijkste algemene resultaten. Deze zijn:

- Conventionele toepassingen van VAR modellen resulteren in vele opzichten niet in scenario's welke consistent zijn met de empirische kennis als gevonden in Deel III.
- Over het algemeen wordt een te lage model orde ( $p=1$ ) gebruikt. Hogere orde modellen geven vaak betere resultaten.
- Onder andere doordat geen gebruik wordt gemaakt van Spectraal Analyse technieken, is men zich vaak slecht bewust van de voor meer periode scenario toepassingen belangrijke dynamische eigenschappen van de geschatte modellen. Dit geldt bijvoorbeeld voor de aanwezigheid van lange golf of conjunctuur fluctuaties en de fase en correlatie eigenschappen op die frequenties.
- Het blijkt niet goed mogelijk om met één model op een zelfde set van data tegelijkertijd een goede beschrijving te geven van zowel de lage als de hoge frequentie eigenschappen van tijdreeksen. Door de gekozen data representatie in combinatie met de schattingsmethode wordt vaak onbewust de nadruk gelegd op een van beide frequentie gebieden. Vanwege het gebruik van mogelijke transformaties van de modellen zijn alle frequentie gebieden echter in principe even belangrijk.

- De voor het schatten gebruikte data en de te modelleren frequenties zijn vaak onbewust niet goed op elkaar afgestemd. Het is bijvoorbeeld vreemd dat “slechts” vijftig jaar data wordt gebruikt om als informatie te dienen voor het scenario gedrag in termen van lange golven met een periodelengte van vijftig jaar. Deze steekproef bevat immers in principe maar één waarneming voor het gedrag op deze lage frequentie.
- Met ongerestricteerde multivariate VAR modellen, zowel van lage als van hoge orde, is het moeilijk om de univariate spectra van de desbetreffende variabelen goed te beschrijven. Lage orde modellen pikken de dynamiek niet goed op, ook al hebben ze wel voldoende flexibiliteit om deze dynamiek te beschrijven. Hoge orde modellen bieden te veel flexibiliteit waardoor de dynamiek als het ware wordt vertroebeld.
- De fase relaties blijken vrijwel altijd goed (robuust) geschat te kunnen worden.

Uit deze resultaten blijkt dus dat het met conventionele toepassingen van VAR modellen in het algemeen op meerdere vlakken moeilijk of zelfs onmogelijk is om scenario's te genereren welke consistent zijn met het in Deel III gevonden empirische gedrag van macro economische variabelen. Daarom is het laatste hoofdstuk van dit onderzoek gewijd aan het beschrijven en illustreren van alternatieve modellen waarmee dit wel mogelijk is. Hiermee vormt Hoofdstuk 20 het “antwoord” op het tweede deel van de tweede specifieke onderzoeksvraag als geformuleerd in de inleiding. Hoe kunnen VAR modellen worden aangepast om de tekortkomingen die we hebben gevonden op te lossen?

Hoofdstuk 20 valt uiteen in twee delen. In het eerste deel, paragraaf 20.1, wordt een formele beschrijving gegeven van een nieuw op het frequentiedomein gebaseerde VAR scenario raamwerk waarmee een groot aantal van de gevonden tekortkomingen van de huidige modellen kan worden opgelost. Dit nieuwe raamwerk resulteert niet in een volledig geautomatiseerd proces waarmee met één druk op de knop een scenarioset met de gewenste eigenschappen kan worden gegenereerd. In plaats daarvan biedt het raamwerk een flexibele modelleer omgeving waarmee het complexe gedrag van macro economische variabelen goed kan worden beschreven. De belangrijkste componenten van dit nieuw ontwikkelde raamwerk zijn de volgende:

- (a) Multifactor VAR (MVAR) modellen: Het basisidee van deze modellen is om een afzonderlijk VAR model (stochastische factor) te construeren voor ieder relevant geacht frequentie gebied. Hiervoor dienen te modelleren tijdreeksen eerst naar deze frequentie gebieden te worden gefilterd met behulp van het frequentiedomein filter uit Hoofdstuk 5. Vervolgens kan per gefilterde reeks een afzonderlijk model worden geschat waarvan de scenario's daarna eenvoudig bij elkaar kunnen worden opgeteld. Deze aanpak kan worden verantwoord uit het feit dat naar niet overlappende frequentiegebieden gefilterde reeksen door een fundamentele eigenschap van het frequentiedomein denken ongecorreleerd zijn en dat de gefilterde reeksen een eenvoudige additieve decompositie van de tijdreeksen geven. Deze MVAR modellen leveren een betere beschrijving van het gedrag van economische variabelen in verschillende frequentie gebieden, bieden de mogelijkheid om verschillende steekproeven met verschillende observatie frequenties te gebruiken per model en bieden de mogelijkheid om complexe relaties tussen het gedrag op verschillende frequentiegebieden, zoals bijvoorbeeld het level effect in rentes, eenvoudig te modelleren. De modellen kunnen worden uitgebreid met een stochastische trend modellering voor het gedrag op de

allerlaagste frequenties. Tenslotte creëren deze modellen een betere bewustwording van de model en daarmee van de scenario eigenschappen. Dit brengt wel met zich mee dat voor sommige aspecten van het gedrag van economische variabelen moeilijke uitspraken moeten worden gedaan, zoals bijvoorbeeld over de variantie en correlaties in de lange termijn trends. In conventionele toepassingen van VAR modellen worden deze aannamen onbedoeld impliciet gemaakt door de gekozen modellering.

- (b) Backtesting procedure: Door een specifieke combinatie van de betrouwbaarheidsintervallen en de multivariate spectra van de geschatte modellen kan worden getoetst in hoeverre deze inderdaad overeenstemmen, of bewust juist niet overeenstemmen, met het gedrag van historische tijdreeksen.
- (c) Truncated VAR (TVAR) modellen: Dit zijn de aangepaste VAR modellen als beschreven in Deel II. Hiermee kunnen scheef verdeelde variabelen worden gemodelleerd waarvan de waarde niet onder of boven een bepaalde grens mag komen. Denk bijvoorbeeld aan rentes die strikt positief moeten blijven.
- (d) EigenValue Restricted VAR (EVR VAR) modellen: Dit zijn de aangepaste VAR modellen als beschreven in Deel II. Hiermee is het mogelijk om met restricties op de eigenwaarden multivariate VAR modellen te schatten waarmee de univariate spectra van de desbetreffende variabelen goed kunnen worden beschreven. Denk hierbij aan het gedrag op de conjunctuur frequenties.
- (e) Aanpassingsmogelijkheden: Middels de Yule-Walker vergelijkingen en de representatie van VAR modellen in termen van de eigenwaarden en eigenvectoren is het mogelijk om met de hand enerzijds volatiliteiten en gemiddelden en anderzijds het belangrijke dynamische gedrag van de modellen direct te beïnvloeden. In termen van de conventionele parameters van VAR modellen is het laatste niet goed mogelijk. Deze aanpassingsmogelijkheden kunnen bijvoorbeeld worden gebruikt om bewust af te wijken van het historische gedrag van de variabelen door het inbrengen van toekomstgerichte informatie of ten behoeve van gevoeligheidsanalyses.

Voor de MVAR en EVR VAR modellen zijn volledig nieuwe typen van (gerestricteerde) VAR modellen welke hopelijk vele andere nuttige toepassingen zullen kennen. In het tweede deel van Hoofdstuk 20, paragraaf 20.2 tot en met 20.5, worden vier voorbeelden van dit nieuwe raamwerk behandeld welke de werking van verschillende componenten los van elkaar illustreren. De uitwerking van de voorbeelden zijn verder bedoeld om ervaring met het nieuwe raamwerk op te doen. Eén totaal voorbeeld waarin alle componenten tezamen zijn opgenomen is niet gegeven. Het combineren van de verschillende componenten is echter relatief eenvoudig.

## Richtingen voor vervolgonderzoek en toepassingen

We eindigen deze samenvatting, en daarmee dit onderzoek, met een aantal richtingen voor vervolgonderzoek en toepassingen van de resultaten uit dit onderzoek. In de eerste plaats dient het empirische gedrag van de bestudeerde macro economische variabelen op hogere frequenties in kaart te worden gebracht. In dit onderzoek lag de focus op trendmatige ontwikkelingen, lange termijn fluctuaties in termen van decennia en conjunctuur fluctuaties in termen van jaren. Op basis van dezelfde methodologie en met gebruikmaking van dezelfde modellen en technieken is het mogelijk het onderzoek uit te breiden naar bijvoorbeeld seizoensontwikkelingen op maandfrequenties en het gedrag op week of zelf dagbasis. Hierin zullen zaken als stochastische volatiliteiten waarschijnlijk ook een rol gaan spelen.

Een tweede richting voor vervolgonderzoek is om het aantal bestudeerde variabelen en landen uit te breiden. Lastige variabelen als wisselkoersen zijn hier bijvoorbeeld nog buiten beschouwing gelaten. Verder kan gedacht worden aan het uitbreiden naar meer typen beleggingsmogelijkheden dan alleen aandelen en aan de verschillende componenten van het nationaal product als de consumptie, export en import en de overheidsbestedingen. Door enerzijds de geanalyseerde frequenties en anderzijds de bestudeerde variabelen en landen uit te breiden wordt op basis van een uniforme methodologie een nog completer en consistent beeld gecreëerd van het empirische gedrag van macro economische variabelen dat voor veel verschillende doeleinden kan worden gebruikt.

Een derde en laatste belangrijke richting voor vervolgonderzoek is het onderzoeken van de consequenties voor strategische beleidsbeslissingen wanneer het nieuw ontwikkelde scenario raamwerk daadwerkelijk wordt toegepast. Omdat de beleidsbeslissingen in principe conditioneel zijn op de eigenschappen van de gehanteerde scenario's en dus niet voor toetsing van de scenario's kunnen worden gehanteerd is dit aspect in dit onderzoek in eerste instantie bewust buiten beschouwing gelaten. Nu de scenario's die de "juiste" empirische eigenschappen beschrijven in principe beschikbaar zijn, kan dergelijk onderzoek plaatsvinden door met het nieuwe scenario raamwerk gegenereerde scenario's toe te passen in bestaande ALM modellen voor bijvoorbeeld pensioenfondsen en verzekeraars. Het voorbeeld in paragraaf 1.2 illustreert de enorme invloed die scenario eigenschappen kunnen hebben op zowel absolute als relatieve uitkomsten van ALM modellen en daarmee op de beleidsbeslissingen die op de modeluitkomsten worden gebaseerd. Naast het onderzoeken van de gevoeligheden voor de verschillende empirische kenmerken is een interessante denkrichting om na te gaan in hoeverre verschillende niveau's van besluitvorming kunnen worden gekoppeld aan de eigenschappen van de scenario's in verschillende frequentiegebieden. Het hele lange termijn lage frequentie gedrag van economische variabelen zou bijvoorbeeld kunnen worden gekoppeld aan lange termijn strategische beslissingen terwijl het korte termijn hoge frequentie gedrag zou kunnen worden gekoppeld aan meer korte termijn tactische beslissingen.

Uiteindelijk zullen de resultaten uit dit onderzoek, door het toepassen van het nieuwe scenario raamwerk, hopelijk bijdragen aan het hogere niveau van strategische besluitvorming, wat we ons oorspronkelijk tot doel hadden gesteld.



## *Appendices*

Appendix A contains some basic mathematics on sine and cosine functions, complex numbers and matrix operations. These basics are of fundamental importance for the (frequency domain) approach followed in the research presented here. Appendix B gives the proofs and derivations of various relations given in the main text. Appendix C specifies the parameters of VAR models that are used for testing purposes in section 4.7. Appendix A, B and C relate mostly to the theoretical background presented in Part II. Appendix D and E contain the results of the filtering and spectral analyses of the time series for respectively the United Kingdom and the United States corresponding to the analyses in Part III. Finally, Appendix F contains the time series that are used for the analyses in Part III and IV as well as a description of their sources.





# Appendix A: Mathematics

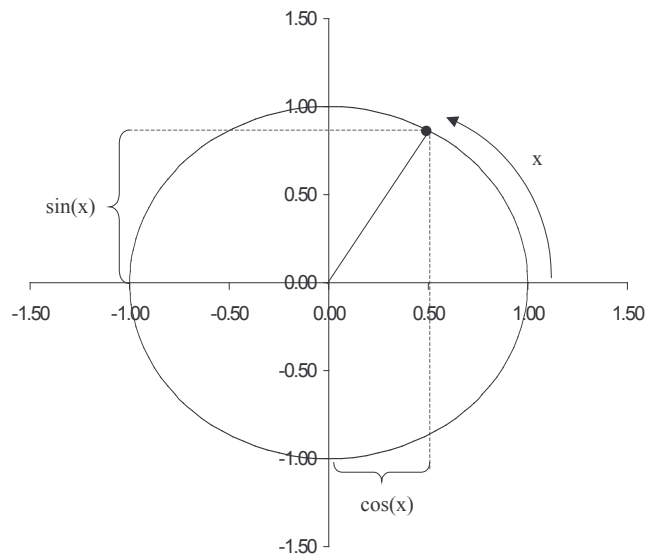
## A.1 Sine and Cosine

This section gives some basic properties of the sine and cosine functions

$$\begin{aligned} f(x) &= \cos(x) \\ g(x) &= \sin(x) \end{aligned} \tag{A.1.1}$$

These functions play a central role in everything that has to do with the spectral analysis of time series. They are defined as the coordinates of a point following the perimeter of a circle with a radius of one as shown in Figure A.1. The argument of the functions  $x$  is the length along this *unit circle* starting from coordinates  $(1,0)$ .  $\cos(x)$  is the coordinate of the point on the horizontal axis and  $\sin(x)$  is the coordinate on the vertical axis.

Figure A.1 Definition of sine and cosine function.



Figures A.2 and A.3 show the sine and cosine functions for values of  $x$  between 0 and  $2\pi$ . Because the perimeter of a unit circle is  $2\pi=2\cdot3.1415927\dots\approx6.28$  both the sine and cosine function repeat themselves every interval of length  $2\pi$ . This is why these functions are said to have a *period length* of  $2\pi$ . The inverse of the period length is called the *frequency* and indicates how many periods the functions cover for each increase of one in  $x$ .

Figure A.2 Sine function

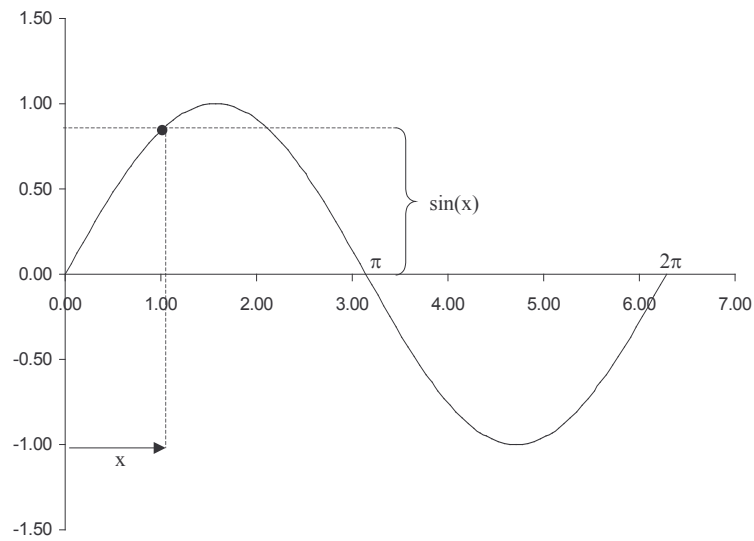
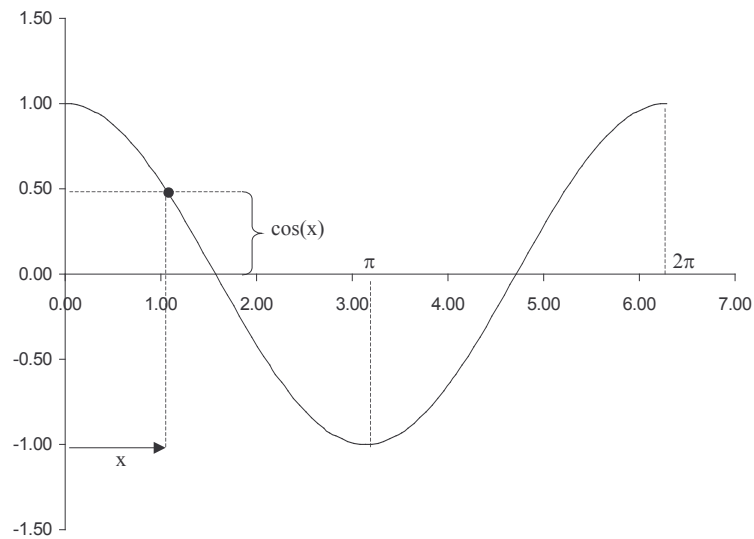


Figure A.3 Cosine function



The maximum value of the sine and cosine functions in an absolute sense is called the *amplitude*. Both functions have a standard amplitude of one. For the  $\sin(x)$  and  $\cos(x)$  functions a great number of relations hold. From the definitions it is for example easily seen that

$$\cos(-x) = \cos(x) \quad (\text{A.1.2})$$

$$\sin(-x) = -\sin(x) \quad (\text{A.1.3})$$

$$\cos^2(x) + \sin^2(x) = 1 \quad (\text{A.1.4})$$

$$\cos(x + k2\pi) = \cos(x) \quad \forall \quad k \in N \quad (\text{A.1.5})$$

$$\sin(x + k2\pi) = \sin(x) \quad \forall \quad k \in N$$

Less obvious but very useful are the so called doubling formula

$$\begin{aligned}\sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\end{aligned}\tag{A.1.6}$$

Now consider the following function

$$f(t) = R \cos(\omega t + \phi)\tag{A.1.7}$$

For the intuition it is instructive to see  $t$  as a time index, for example measured in years. For this function holds

Amplitude =  $R$

Period length =  $\frac{2\pi}{\omega}$

Frequency =  $\frac{1}{\text{Period length}} = \frac{\omega}{2\pi}$  ( $\omega$  periods in each  $2\pi$ )

The parameter  $\phi$ , which is called the *phase*, indicates how much the function has been shifted compared to the standard cosine function. Because

$$\cos(\omega t + \phi) = \cos(\omega(t + \frac{\phi}{\omega})) = \cos(\omega(t + d)) \text{ where } d = \frac{\phi}{\omega}\tag{A.1.8}$$

the phase can also be interpreted as a time shift of  $d$  time periods. A positive value of  $\phi$  and  $d$  means the function has been shifted backwards (causing a *lead*) in time while a negative value means the function has been moved forwards (causing a *lag*) in time. The Figures A.4 through A.6 graphically show the effects of respectively the frequency  $\omega$ , the amplitude  $R$  and the phase  $\phi$ .

Figure A.4 Frequency  $\omega$

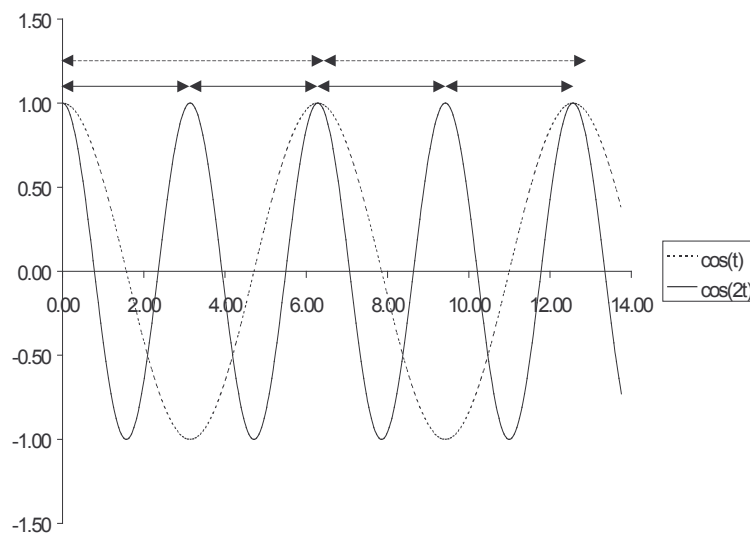
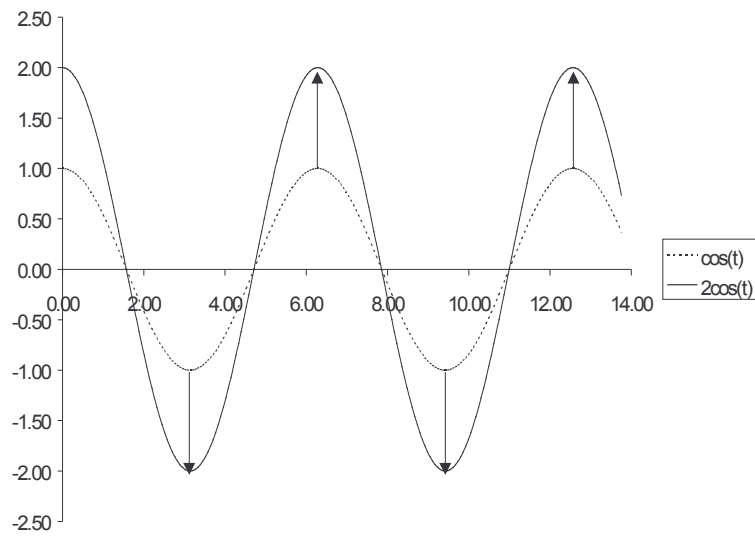
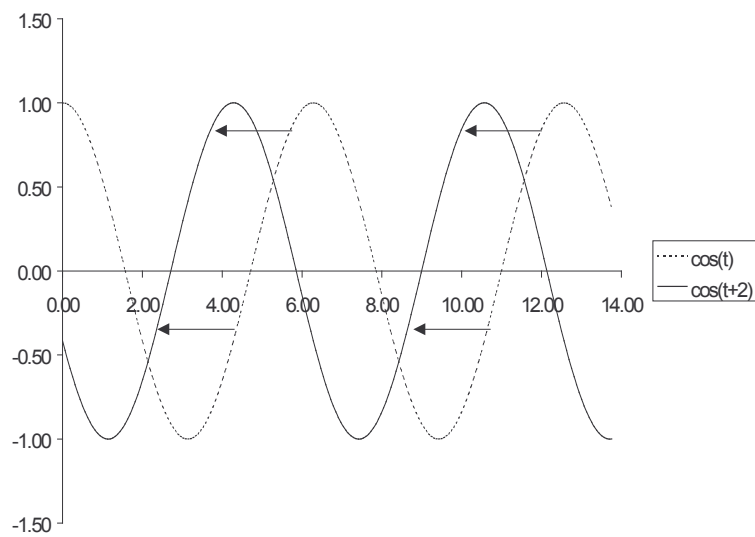


Figure A.5 Amplitude  $R$ Figure A.6 Phase  $\phi$ 

An equivalent representation of (A.1.7) is

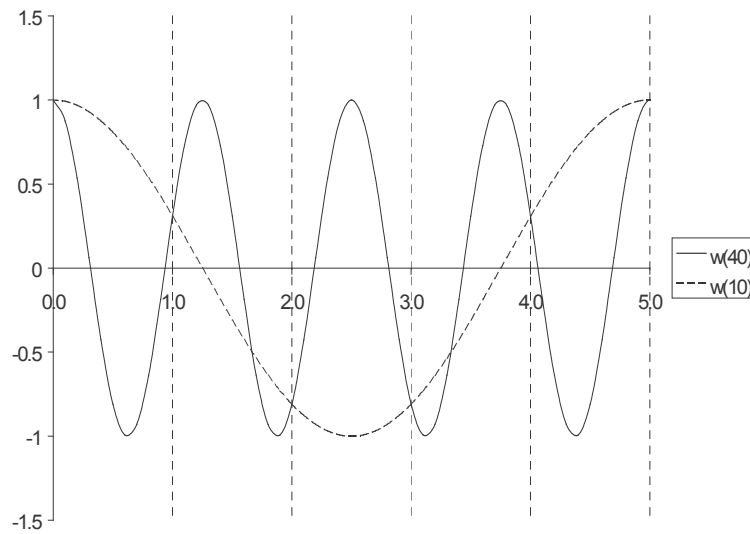
$$f(t) = A \cos(\omega t) + B \sin(\omega t) \quad (\text{A.1.9})$$

for which the following relations hold

$$\begin{aligned} R &= \sqrt{A^2 + B^2} \\ A &= R \cos(\phi) \quad \text{and} \quad B = -R \sin(\phi) \end{aligned} \quad (\text{A.1.10})$$

Note that the second equation follows easily from the second of the doubling formula (A.1.6). These relations can be used to calculate the values of  $A$  en  $B$  from those for  $R$  and  $\phi$  and vice versa. Finally it is important to mention a phenomenon that is particularly important in the case of discrete time (i.e. where  $t=0,1,2,\dots$ ). This phenomenon is called *aliasing* which means that any sine or cosine function with a frequency  $\omega$  outside the interval  $[0,\pi]$  always has an alias frequency  $\omega_a$  within the interval  $[0,\pi]$ . Measured at discrete times the functions  $\cos(\omega t)$  and  $\cos(\omega_a t)$  are identical. As an example look at the cosines with frequencies  $\omega = 2\pi 40/50 \notin [0,\pi]$  and  $\omega_a = 2\pi 10/50 = 2\pi - \omega \in [0,\pi]$  shown in Figure A.7.

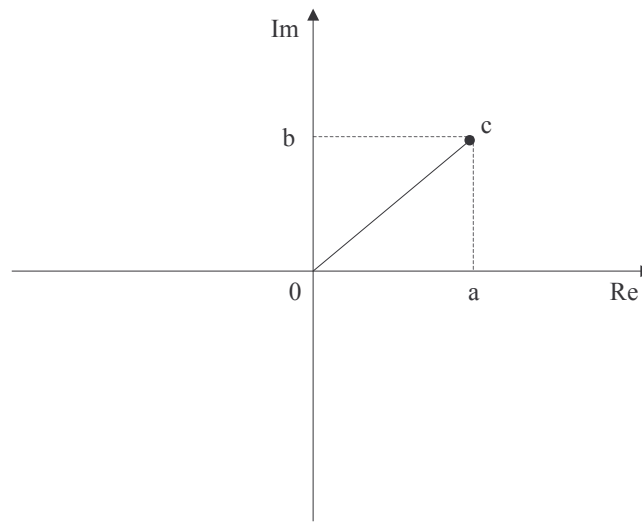
Figure A.7 Aliasing



## A.2 Complex numbers

Based on the *imaginary number*  $i$  for which by definition holds  $i^2 = -1$  and the set of real numbers  $\mathbf{R}$ , the set of *complex numbers*  $\mathbf{C}$  can be defined. Each complex number  $c$  has the format  $c = a + bi$  with  $a \in \mathbf{R}$  and  $b \in \mathbf{R}$ . The numbers  $a$  and  $b$  are called respectively the *real* and the *imaginary* part of the number  $c$ . The notation is  $a = \text{real}(c)$  and  $b = \text{imag}(c)$ . The set of complex numbers can be seen as the set of ordered pairs of real numbers  $(a, b)$  which can be plotted in a two dimensional plain. This plain is also called the complex plain with the real horizontal axis and the complex vertical axis. This is shown in Figure A.8.

Figure A.8 The complex plain



Calculating with complex numbers is basically the same as with real numbers with the addition that  $i^2$  is replaced by  $-1$ . For adding, multiplying and dividing complex numbers hold

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad (\text{A.2.1})$$

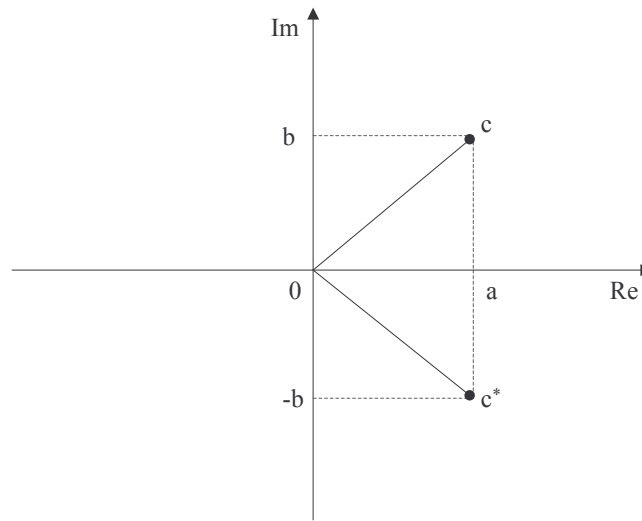
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \quad (\text{A.2.2})$$

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \quad (\text{A.2.3})$$

For each complex number  $c = a + bi$ ,  $c^* = a - bi$  is called the *complex conjugate* of  $c$ . In the complex plain  $c^*$  is the reflection of  $c$  in the horizontal real axis. This is shown in Figure A.9.



Figure A.9 Complex conjugates



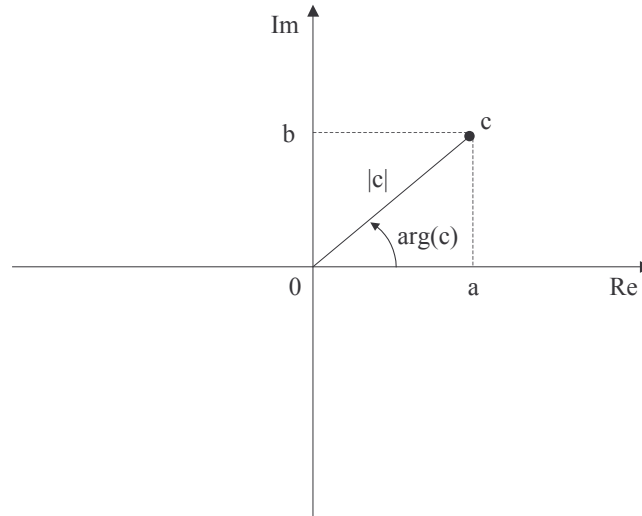
To be able to compare complex numbers for each complex number  $c$  the *modulus* is defined as

$$|c| = \sqrt{cc^*} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2} \quad (\text{A.2.4})$$

The modulus corresponds to the distance between the origin and the complex number in the complex plane as shown in Figure A.10. The modulus of a real number (i.e. with complex part  $b=0$ ) coincides with the absolute value of the number. Besides the modulus, also the *argument* of a complex number can be calculated. The argument is defined as the angle (in radians) between the positive real axis and the line connecting the origin to the complex number as shown in Figure A.10. From the definition of the sine and cosine functions in the previous section of this appendix it is easy to see that the argument of a complex number can be found by solving

$$\cos(\arg(c)) = \frac{a}{|c|} \quad \text{and} \quad \sin(\arg(c)) = \frac{b}{|c|} \quad (\text{A.2.5})$$

Figure A.10 Modulus and argument



The modulus and argument can be used for the following equivalent representation of any complex number

$$c = a + bi = |c|(\cos(\arg(c)) + i \sin(\arg(c))) \quad (\text{A.2.6})$$

The relations between the  $a$  and  $b$  in the original representation and the representation (A.2.6) in terms of its *polar coordinates*  $|c|$  and  $\arg(c)$  are given by (A.2.4) and (A.2.5) as just discussed. Finally the *complex exponential function* is defined as

$$\exp(i\omega) = \cos \omega + i \sin \omega \Leftrightarrow \begin{cases} \cos \omega = \frac{\exp(i\omega) + \exp(-i\omega)}{2} \\ \sin \omega = \frac{\exp(i\omega) - \exp(-i\omega)}{2i} \end{cases} \quad (\text{A.2.7})$$

Given (A.2.6) it is easy to see that for any  $\omega \in \mathbf{R}$ ,  $\exp(i\omega)$  defines a complex number on the unit circle around the origin. The complex exponential function and the sine and cosine function are therefore closely related. The right hand part of the relations (A.2.7) are also called the *Euler* equations.

### A.3 Some basic matrix operators

#### *Vec operator*

The *vec* operator turns an  $m \times n$  matrix  $A$  into an  $(m \times n) \times 1$  column vector by stacking the columns of the matrix.

$$\text{vec}(A) = \text{vec} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,n} \end{bmatrix} \quad (\text{A.3.1})$$

#### *Kronecker product*

The  $\otimes$  operator is the *Kronecker product* which for an  $m \times n$  matrix  $A$  and a  $p \times q$  matrix  $B$  is defined as the  $(m \times p) \times (n \times q)$  matrix

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{bmatrix} \quad (\text{A.3.2})$$

The Kronecker product therefore multiplies each element of  $A$  with the entire matrix  $B$ . Some simple rules for calculating with the Kronecker product are

$$\begin{aligned} (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \\ (A \otimes B)' &= A' \otimes B' \\ (A \otimes B)(C \otimes D) &= AC \otimes BD \\ \text{vec}(ABC) &= (C' \otimes A) \text{vec}(B) \end{aligned}$$

#### *Trace operator*

For an  $n \times n$  matrix  $A$  the *trace* operator is defined as the sum of the diagonal elements.

$$\text{trace}(A) = \text{trace} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = a_{1,1} + a_{2,2} + \cdots + a_{n,n} \quad (\text{A.3.3})$$

## Appendix B: Proofs and derivations

### B.1 Derivation of (3.4.48)

Assume the following specification of a general AR( $p$ ) model without a constant term in which the parameters are given an extra subscript  $p$  to indicate they belong to the AR model of order  $p$ .

$$x_t = \beta_{p,1}x_{t-1} + \beta_{p,2}x_{t-2} + \dots + \beta_{p,p}x_{t-p} + \varepsilon_t \quad \text{where } \varepsilon_t \sim N(0, \sigma_p^2) \quad (\text{B.1.1})$$

Furthermore assume that, given a set of autocovariances  $\gamma_0, \dots, \gamma_p$  to describe, the parameters  $\beta_{2,1}$ ,  $\beta_{2,2}$  and  $\sigma_2^2$  of the AR(2) model that is consistent with the first three autocovariances  $\gamma_0$ ,  $\gamma_1$  and  $\gamma_2$  are known. Following the Yule-Walker equations in (3.2.20) and (3.2.21) these parameters satisfy

$$\sigma_2^2 = \gamma_0 - \gamma_1\beta_{2,1} - \gamma_2\beta_{2,2} \quad (\text{B.1.2})$$

and

$$\begin{bmatrix} \beta_{2,1} \\ \beta_{2,2} \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \quad (\text{B.1.3})$$

Let's now see how the parameters of the AR(3) model that is consistent with the autocovariances  $\gamma_0, \dots, \gamma_3$  can be calculated from the parameters of the known AR(2) model. The Yule-Walker equations for an AR(3) model state

$$\sigma_3^2 = \gamma_0 - \gamma_1\beta_{3,1} - \gamma_2\beta_{3,2} - \gamma_3\beta_{3,3} \quad (\text{B.1.4})$$

and

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \beta_{3,1} \\ \beta_{3,2} \\ \beta_{3,3} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \quad (\text{B.1.5})$$

The first two equations of (B.1.5) can be written as

$$\begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \beta_{3,1} \\ \beta_{3,2} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} - \beta_{3,3} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \quad (\text{B.1.6})$$

or

$$\begin{bmatrix} \beta_{3,1} \\ \beta_{3,2} \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} - \beta_{3,3} \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \quad (\text{B.1.7})$$

By noting the presence of the Yule-Walker equations (B.1.3) of the AR(2) model in (B.1.7) and substituting this gives

$$\begin{bmatrix} \beta_{3,1} \\ \beta_{3,2} \end{bmatrix} = \begin{bmatrix} \beta_{2,1} \\ \beta_{2,2} \end{bmatrix} - \beta_{3,3} \begin{bmatrix} \beta_{2,2} \\ \beta_{2,1} \end{bmatrix} \quad (\text{B.1.8})$$

By filling in these expressions for  $\beta_{3,1}$  and  $\beta_{3,2}$  in the third Yule-Walker equation of the AR(3) model it follows from (B.1.5) that

$$\beta_{3,3} = \frac{\gamma_3 - \gamma_2 \beta_{2,1} - \gamma_1 \beta_{2,2}}{\gamma_0 - \gamma_1 \beta_{2,1} - \gamma_2 \beta_{2,2}} = \frac{\gamma_3 - \gamma_2 \beta_{2,1} - \gamma_1 \beta_{2,2}}{\sigma_2^2} \quad (\text{B.1.9})$$

Doing the same substitution in (B.1.4) gives

$$\begin{aligned} \sigma_3^2 &= \gamma_0 - \gamma_1 \beta_{3,1} - \gamma_2 \beta_{3,2} - \gamma_3 \beta_{3,3} \\ &= (\gamma_0 - \gamma_1 \beta_{2,1} - \gamma_2 \beta_{2,2}) + \beta_{3,3} (\gamma_1 \beta_{2,2} + \gamma_2 \beta_{2,1} - \gamma_3) \\ &= \sigma_2^2 - \frac{(\gamma_1 \beta_{2,2} + \gamma_2 \beta_{2,1} - \gamma_3)}{\sigma_2^2} (\gamma_1 \beta_{2,2} + \gamma_2 \beta_{2,1} - \gamma_3) \\ &= \sigma_2^2 (1 - \beta_{3,3}^2) \end{aligned} \quad (\text{B.1.10})$$

Generalizing (B.1.8), (B.1.9) and (B.1.10) leads to

$$\begin{aligned} \beta_{p,i} &= \beta_{p-1,i} - \beta_{p,p} \beta_{p-1,p-i} \quad \text{for } p > 0 \text{ en } i < p \\ \beta_{p,p} &= \frac{\gamma_p - \sum_{j=1}^{p-1} \gamma_{p-j} \beta_{p-1,j}}{\sigma_{p-1}^2} \\ \sigma_p^2 &= \sigma_{p-1}^2 (1 - \beta_{p,p}^2) \end{aligned} \quad (\text{B.1.11})$$

which is equivalent to (3.4.48).

## B.2 Proof of (4.1.3)

Substitution of (4.1.3) into (4.1.2) gives

$$\begin{aligned} & \sum_{j=0}^{T-1} A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \\ &= \sum_{j=0}^{T-1} \left( \frac{1}{T} \sum_{k=0}^{T-1} x_k \cos(\omega_j k) \right) \cos(\omega_j t) + \sum_{j=0}^{T-1} \left( \frac{1}{T} \sum_{k=0}^{T-1} x_k \sin(\omega_j k) \right) \sin(\omega_j t) \end{aligned} \quad (\text{B.2.1})$$

By changing the order of summation and using  $\omega_j k = 2\pi \frac{j}{T} k = 2\pi \frac{k}{T} j = \omega_k j$  this can be seen to be equal to

$$\frac{1}{T} \sum_{k=0}^{T-1} x_k \sum_{j=0}^{T-1} \cos(\omega_k j) \cos(\omega_t j) + \frac{1}{T} \sum_{k=0}^{T-1} x_k \sum_{j=0}^{T-1} \sin(\omega_k j) \sin(\omega_t j) \quad (\text{B.2.2})$$

Using the following relation for which the proof is not given here

$$\begin{aligned} \sum_{t=0}^{T-1} \cos(\omega_j t) &= \begin{cases} 0 & \text{for } j \neq 0 \\ T & \text{for } j = 0 \end{cases} \\ \sum_{t=0}^{T-1} \sin(\omega_j t) &= 0 \quad \forall j \\ \sum_{t=0}^{T-1} \cos(\omega_j t) \cos(\omega_k t) &= \begin{cases} T/2 & \text{for } j = k \neq 0 \text{ or } T/2 \\ T & \text{for } j = k = 0 \text{ or } T/2 \\ 0 & \text{for } j \neq k \end{cases} \\ \sum_{t=0}^{T-1} \cos(\omega_j t) \sin(\omega_k t) &= 0 \quad \forall j, k \\ \sum_{t=0}^{T-1} \sin(\omega_j t) \sin(\omega_k t) &= \begin{cases} T/2 & \text{for } j = k \neq 0 \text{ or } n/2 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (\text{B.2.3})$$

it now follows that (B.2.2) is equal to

$$\begin{cases} \frac{1}{T} T x_t + \frac{1}{T} 0 = x_t & \text{als } t = 0 \\ \frac{1}{T} \frac{T}{2} x_t + \frac{1}{T} \frac{T}{2} x_t = x_t & \text{als } t \neq 0 \end{cases} \quad (\text{B.2.4})$$

which concludes the proof.

### B.3 Proof of (4.1.9)

In the matrix notation of the Classical Regression Model as described in section 3.4.1 equation (4.1.8) becomes

$$Y = Xb + \varepsilon \quad (\text{B.3.1})$$

where

$$Y = \begin{bmatrix} x_0 \\ \vdots \\ x_{T-1} \end{bmatrix}, \quad X = \begin{bmatrix} \cos(\omega_j 0) & \sin(\omega_j 0) \\ \vdots & \vdots \\ \cos(\omega_j (T-1)) & \sin(\omega_j (T-1)) \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \quad \text{and} \quad \varepsilon = \begin{bmatrix} \varepsilon_0 \\ \vdots \\ \varepsilon_{T-1} \end{bmatrix}$$

The general OLS solution minimizing the sum of squared residuals is as we now

$$\hat{b} = (X'X)^{-1}X'Y \quad (\text{B.3.2})$$

Substituting for the matrices  $X$  and  $Y$  into when assuming  $j \neq 0$  and  $j \neq T/2$  gives

$$\begin{aligned} X'X &= \begin{bmatrix} \cos(\omega_j 0) & \cdots & \cos(\omega_j (T-1)) \\ \sin(\omega_j 0) & \cdots & \sin(\omega_j (T-1)) \end{bmatrix} \begin{bmatrix} \cos(\omega_j 0) & \sin(\omega_j 0) \\ \vdots & \vdots \\ \cos(\omega_j (T-1)) & \sin(\omega_j (T-1)) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{t=0}^{T-1} \cos^2(\omega_j t) & \sum_{t=0}^{T-1} \cos(\omega_j t) \sin(\omega_j t) \\ \sum_{t=0}^{T-1} \cos(\omega_j t) \sin(\omega_j t) & \sum_{t=0}^{T-1} \sin^2(\omega_j t) \end{bmatrix} = \begin{bmatrix} T/2 & 0 \\ 0 & T/2 \end{bmatrix} \end{aligned}$$

and

$$X'Y = \begin{bmatrix} \cos(\omega_j 0) & \cdots & \cos(\omega_j (T-1)) \\ \sin(\omega_j 0) & \cdots & \sin(\omega_j (T-1)) \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \\ \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \end{bmatrix}$$

from which follows

$$\hat{b} = (X'X)^{-1}X'Y = \begin{bmatrix} 2/T & 0 \\ 0 & 2/T \end{bmatrix} \begin{bmatrix} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \\ \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \end{bmatrix} = \begin{bmatrix} \frac{2}{T} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \\ \frac{2}{T} \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \end{bmatrix} = \begin{bmatrix} 2A_j \\ 2B_j \end{bmatrix} \quad (\text{B.3.3})$$



where  $A_j$  and  $B_j$  are defined as in (4.1.4). Since  $\cos(0)=1$  and  $\sin(0)=0$  for  $j=0$  the regression (4.1.8) comes down to only regressing on a constant term. For  $j=0$  therefore simply follows

$$(X'X)^{-1} = \left( \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 \end{bmatrix} \right)^{-1} = \frac{1}{T}$$

and

$$X'Y = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{T-1} \end{bmatrix} = \sum_{t=0}^{T-1} x_t$$

which results in

$$\hat{\alpha}_0 = (X'X)^{-1} X'Y = \frac{1}{T} \sum_{t=0}^{T-1} x_t = A_0 \quad \text{and} \quad \hat{\beta}_0 = 0 = B_0 \quad (\text{B.3.4})$$

For  $j=T/2$  along the same lines follows

$$\hat{\alpha}_{T/2} = -\frac{1}{T} \sum_{t=0}^{T-1} x_t = A_{T/2} \quad \text{and} \quad \hat{\beta}_{T/2} = 0 = B_{T/2} \quad (\text{B.3.5})$$

which concludes the proof.

## B.4 Proof of (4.1.11)

Substituting for  $A_j$  and  $B_j$  gives

$$\begin{aligned}
 \sum_{j=0}^{T-1} A_j^2 + B_j^2 &= \sum_{j=0}^{T-1} \left( \frac{1}{T} \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \right)^2 + \left( \frac{1}{T} \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \right)^2 \\
 &= \frac{1}{T^2} \sum_{j=0}^{T-1} \left( \sum_{t=0}^{T-1} x_t \cos(\omega_j t) \right) \left( \sum_{k=0}^{T-1} x_k \cos(\omega_j k) \right) + \frac{1}{T^2} \sum_{j=0}^{T-1} \left( \sum_{t=0}^{T-1} x_t \sin(\omega_j t) \right) \left( \sum_{k=0}^{T-1} x_k \sin(\omega_j k) \right) \\
 &= \frac{1}{T^2} \sum_{j=0}^{T-1} \sum_{t=0}^{T-1} \sum_{k=0}^{T-1} x_t \cos(\omega_j t) x_k \cos(\omega_j k) + \frac{1}{T^2} \sum_{j=0}^{T-1} \sum_{t=0}^{T-1} \sum_{k=0}^{T-1} x_t \sin(\omega_j t) x_k \sin(\omega_j k)
 \end{aligned} \tag{B.4.1}$$

By changing the order of summation and using  $\omega_j k = 2\pi \frac{j}{T} k = 2\pi \frac{k}{T} j = \omega_k j$  this is equal to

$$\frac{1}{T^2} \sum_{t=0}^{T-1} x_t \sum_{k=0}^{T-1} x_k \sum_{j=0}^{T-1} \cos(\omega_t j) \cos(\omega_k j) + \frac{1}{T^2} \sum_{t=0}^{T-1} x_t \sum_{k=0}^{T-1} x_k \sum_{j=0}^{T-1} \sin(\omega_t j) \sin(\omega_k j) \tag{B.4.2}$$

Using the relations from B.2 this gives

$$\sum_{j=0}^{T-1} A_j^2 + B_j^2 = \begin{cases} \frac{1}{T^2} \sum_{t=0}^{T-1} x_t^2 \frac{T}{2} + \frac{1}{T^2} \sum_{t=0}^{T-1} x_t^2 \frac{T}{2} = \frac{1}{T} \sum_{t=0}^{T-1} x_t^2 & \text{for } t \neq 0 \text{ or } T/2 \\ \frac{1}{T^2} \sum_{t=0}^{T-1} x_t^2 T + \frac{1}{T^2} 0 = \frac{1}{T} \sum_{t=0}^{T-1} x_t^2 & \text{for } t = 0 \text{ or } T/2 \end{cases} \tag{B.4.3}$$

which concludes the proof.

## B.5 Proof of (4.1.21) and (4.1.22)

Start with the following somewhat different, though equivalent, definition of the DDFT and the IDFFT. The only difference compared to (4.1.18) and (4.1.19) is the place of the term  $1/T^*$ .

$$J_j = \sum_{t=0}^{T^*-1} x_t \exp(-i\omega_j t) \quad \text{for } j = 0, \dots, T^* - 1 \quad (\text{B.5.1})$$

and

$$x_t = \frac{1}{T^*} \sum_{j=0}^{T^*-1} J_j \exp(i\omega_j t) \quad \text{for } t = 0, \dots, T - 1 \quad (\text{B.5.2})$$

with

$$\omega_j = 2\pi \frac{j}{T^*} \quad \text{for } j = 0, \dots, T^* - 1 \quad (\text{B.5.3})$$

By taking the limit for  $T^* \rightarrow \infty$  in (B.5.1) and multiplying by  $1/T$  the definition of the DCFT follows directly for all frequencies  $0 \leq \omega \leq 2\pi$ . Taking also the limit  $T^* \rightarrow \infty$  in (B.5.2) and substituting for the frequencies gives

$$\begin{aligned} x_t &= \lim_{T^* \rightarrow \infty} \left( \frac{1}{T^*} \sum_{j=0}^{T^*-1} J_j \exp(i2\pi \frac{j}{T^*} t) \right) \\ &= \lim_{T^* \rightarrow \infty} \left( \frac{T}{T^*} \sum_{j=0}^{T^*-1} \frac{1}{T} J_j \exp(i2\pi \frac{j}{T^*} t) \frac{2\pi T^*}{T^* 2\pi} \right) \\ &= \frac{T}{2\pi} \lim_{T^* \rightarrow \infty} \left( \sum_{j=0}^{T^*-1} \frac{1}{T} J_j \exp(i2\pi \frac{j}{T^*} t) \frac{2\pi}{T^*} \right) \\ &= \frac{T}{2\pi} \int_0^{2\pi} J(\omega) \exp(i\omega t) d\omega \end{aligned} \quad (\text{B.5.4})$$

where  $J(\omega)$  is as in (4.1.21). We chose to include the factors  $1/T$  for the analogy with the discrete case. An alternative proof follows by substituting the DCFT into the IDCFT.

$$\begin{aligned}
 & \frac{T}{2\pi} \int_0^{2\pi} \frac{1}{T} \sum_{k=0}^{T-1} x_k \exp(-i\omega k) \exp(i\omega t) d\omega \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{T-1} x_k \exp(-i\omega(k-t)) d\omega \\
 &= \frac{1}{2\pi} \sum_{k=0}^{T-1} x_k \left\{ \int_0^{2\pi} \cos(\omega(k-t)) d\omega - i \int_0^{2\pi} \sin(\omega(k-t)) d\omega \right\} \\
 &= \frac{1}{2\pi} x_t 2\pi \\
 &= x_t
 \end{aligned} \tag{B.5.5}$$

This uses the simple relations

$$\begin{aligned}
 \int_0^{2\pi} \cos(\omega k) d\omega &= \begin{cases} 2\pi & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \\
 \int_0^{2\pi} \sin(\omega k) d\omega &= 0 \quad \forall k
 \end{aligned} \tag{B.5.6}$$

## B.6 Proof of (4.1.23)

Again by expanding a time series with zeros (4.1.16) becomes

$$\begin{aligned}
 \frac{1}{T^*} \sum_{t=0}^{T-1} x_t^2 &= \sum_{j=0}^{T^*-1} |J_j|^2 \\
 &= \sum_{j=0}^{T^*-1} \left| \frac{1}{T^*} \sum_{t=0}^{T-1} x_t \exp(-i\omega_j t) \right|^2 \\
 &= \frac{T^2}{T^{*2}} \sum_{j=0}^{T^*-1} \left| \frac{1}{T} \sum_{t=0}^{T-1} x_t \exp(-i\omega_j t) \right|^2 \\
 &= \frac{T^2}{T^{*2}} \sum_{j=0}^{T^*-1} |J(\omega_j)|^2
 \end{aligned} \tag{B.6.1}$$

This can be written as

$$\begin{aligned}
 \frac{1}{T} \sum_{t=0}^{T-1} x_t^2 &= \frac{T^*}{T} \frac{T^2}{T^{*2}} \sum_{j=0}^{T^*-1} |J(\omega_j)|^2 \frac{2\pi}{T^*} \frac{T^*}{2\pi} \\
 &= \frac{T}{2\pi} \sum_{j=0}^{T^*-1} |J(\omega_j)|^2 \frac{2\pi}{T^*}
 \end{aligned} \tag{B.6.2}$$

Taking the limit in this last expression for  $T^* \rightarrow \infty$  and substituting for  $\omega_j$  gives

$$\begin{aligned}
 \frac{1}{T} \sum_{t=0}^{T-1} x_t^2 &= \lim_{T^* \rightarrow \infty} \frac{T}{2\pi} \sum_{j=0}^{T^*-1} \left| J\left(2\pi \frac{j}{T^*}\right) \right|^2 \frac{2\pi}{T^*} \\
 &= \frac{T}{2\pi} \int_0^{2\pi} |J(\omega)|^2 d\omega
 \end{aligned} \tag{B.6.3}$$

which concludes the proof.

## B.7 Proof of (4.2.8)

The proof starts with the original definition (4.2.1) of the periodogram.

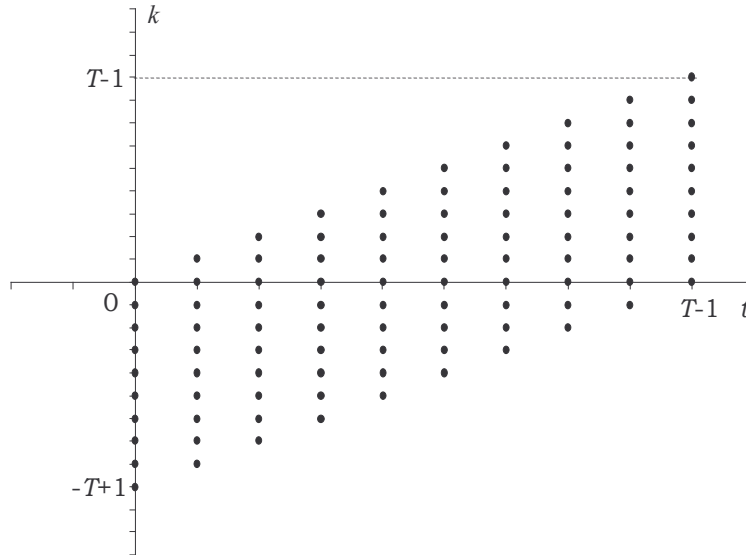
$$\begin{aligned}
 P(\omega) &= \frac{T}{2\pi} |J(\omega)|^2 \\
 &= \frac{T}{2\pi} J(\omega) J(\omega)^* \\
 &= \frac{T}{2\pi} \frac{1}{T^2} \left( \sum_{t=0}^{T-1} x_t \exp(-i\omega t) \right) \left( \sum_{s=0}^{T-1} x_s \exp(i\omega s) \right) \\
 &= \frac{1}{2\pi T} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t x_s \exp(-i\omega(t-s))
 \end{aligned} \tag{B.7.1}$$

By substituting  $k = t-s$  this is equal to

$$\frac{1}{2\pi T} \sum_{t=0}^{T-1} \sum_{k=t}^{t-T+1} x_t x_{t-k} \exp(-i\omega k) \tag{B.7.2}$$

In a two dimensional plain the values of  $t$  and  $k$  in the summation are shown in Figure B.1.

Figure B.1 Summation over  $t$  and  $k$  in (B.7.2)



From this it can be seen that (B.7.2) is equal to

$$\begin{aligned}
 & \frac{1}{2\pi T} \left( \sum_{k=-T+1}^0 \sum_{t=0}^{k+T-1} x_t x_{t-k} \exp(-i\omega k) + \sum_{k=1}^{T-1} \sum_{t=k}^{T-1} x_t x_{t-k} \exp(-i\omega k) \right) \\
 &= \frac{1}{2\pi T} \left( \sum_{k=-T+1}^0 \sum_{t=-k}^{T-1} x_{t+k} x_t \exp(-i\omega k) + \sum_{k=1}^{T-1} \sum_{t=k}^{T-1} x_t x_{t-k} \exp(-i\omega k) \right) \quad (\text{B.7.3}) \\
 &= \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \left( \frac{1}{T} \sum_{t=|k|}^{T-1} x_t x_{t-|k|} \right) \exp(-i\omega k)
 \end{aligned}$$

Dividing this by  $\sigma^2$  and multiplying by two concludes the proof.



## B.8 Proof of (4.4.5)

First note that

$$\begin{aligned}
 P_{xy}(\omega) &= \frac{T}{2\pi} J_x(\omega) J_y(\omega)^* \\
 &= \frac{T}{2\pi} \left( \frac{1}{T} \sum_{t=0}^{T-1} x_t \exp(-i\omega t) \right) \left( \frac{1}{T} \sum_{s=0}^{T-1} y_s \exp(i\omega s) \right) \\
 &= \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t y_s \exp(-i\omega(t-s))
 \end{aligned} \tag{B.8.1}$$

Using this and the relations

$$\begin{aligned}
 \int_0^{2\pi} \cos(\omega - k) d\omega &= \begin{cases} 2\pi & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \\
 \int_0^{2\pi} \sin(\omega - k) d\omega &= 0 \quad \forall k
 \end{aligned} \tag{B.8.2}$$

gives

$$\begin{aligned}
 \int_0^{2\pi} P_{xy}(\omega) d\omega &= \int_0^{2\pi} \frac{1}{T^2} J_x(\omega) J_y(\omega)^* d\omega \\
 &= \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t y_s \int_0^{2\pi} \exp(-i\omega(t-s)) d\omega \\
 &= \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t y_s \int_0^{2\pi} \exp(-i\omega(t-s)) d\omega \\
 &= \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t y_s \int_0^{2\pi} \cos(\omega(t-s)) - i \sin(\omega(t-s)) d\omega \\
 &= \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} x_t y_s \left( \int_0^{2\pi} \cos(\omega(t-s)) d\omega - i \int_0^{2\pi} \sin(\omega(t-s)) d\omega \right) \\
 &= \frac{1}{T} \sum_{t=0}^{T-1} x_t y_t \\
 &= \sigma_{xy}^2
 \end{aligned} \tag{B.8.3}$$

which concludes the proof.

## B.9 Proof of (4.4.7)

By substituting  $k=t-s$  in (B.8.1) we obtain

$$\begin{aligned}
 P_{xy}(\omega) &= \frac{1}{T2\pi} \sum_{t=0}^{T-1} \sum_{k=t}^{t-T+1} x_t y_{t-k} \exp(-i\omega k) \\
 &= \frac{1}{T2\pi} \left( \sum_{k=0}^{T-1} \sum_{t=k}^{T-1} x_t y_{t-k} \exp(-i\omega k) + \sum_{k=-n+1}^{-1} \sum_{t=0}^{T-1+k} x_t y_{t-k} \exp(-i\omega k) \right) \\
 &= \frac{1}{2\pi} \left( \sum_{k=0}^{T-1} \frac{1}{T} \sum_{t=k}^{T-1} x_t y_{t-k} \exp(-i\omega k) + \sum_{k=-n+1}^{-1} \frac{1}{T} \sum_{t=0}^{T-1+k} x_t y_{t-k} \exp(-i\omega k) \right) \\
 &= \frac{1}{2\pi} \sum_{k=-(T-1)}^{T-1} \hat{\gamma}_{xy,K} \exp(-i\omega k)
 \end{aligned} \tag{B.9.1}$$

which concludes the proof.

## B.10 Derivation of (4.4.27)

For convenience we left out the  $(\omega)$  notation for the various variables. The expectation of the variables is then obtained as

$$E(y_t) = \cos(\omega t + \phi)Ea_y + \sin(\omega t + \phi)Eb_y = 0 \quad (\text{B.10.1})$$

The auto covariances are obtained as

$$\begin{aligned} E(y_t y_{t-k}) &= E(a_y \cos(\omega t + \phi) + b_y \sin(\omega t + \phi)) \times \\ &\quad (a_y \cos(\omega(t-k) + \phi) + b_y \sin(\omega(t-k) + \phi)) \\ &= Ea_y^2 (\cos(\omega t + \phi) \cos(\omega(t-k) + \phi)) + Eb_y^2 (\sin(\omega t + \phi) \sin(\omega(t-k) + \phi)) \\ &= \sigma_y^2 (\cos(\omega t + \phi) (\cos(\omega t + \phi) \cos(\omega k) + \sin(\omega t + \phi) \sin(\omega k))) \\ &\quad + \sigma_y^2 (\sin(\omega t + \phi) (\sin(\omega t + \phi) \cos(\omega k) - \cos(\omega t + \phi) \sin(\omega k))) \\ &= \sigma_y^2 (\cos(\omega t + \phi)^2 + \sin(\omega t + \phi)^2) \cos(\omega k) \\ &= \sigma_y^2 \cos(\omega k) \end{aligned} \quad (\text{B.10.2})$$

in which both  $E(a_y b_y) = 0$  and the relations (A.1.6) are used. The cross covariances are obtained in a similar way as

$$\begin{aligned} E(x_t y_{t-k}) &= E(a_x \cos(\omega t) + b_x \sin(\omega t)) \times \\ &\quad (a_y \cos(\omega(t-k) + \phi) + b_y \sin(\omega(t-k) + \phi)) \\ &= Ea_x a_y (\cos(\omega t) \cos(\omega(t-k) + \phi)) \\ &\quad + Eb_x b_y (\sin(\omega t) \sin(\omega(t-k) + \phi)) \\ &= \sigma_{xy}^2 \cos(\omega t) (\cos(\omega t) \cos(\phi - \omega k) - \sin(\omega t) \sin(\phi - \omega k)) \\ &\quad + \sigma_{xy}^2 \sin(\omega t) (\sin(\omega t) \cos(\phi - \omega k) + \cos(\omega t) \sin(\phi - \omega k)) \\ &= \sigma_{xy}^2 \cos(\phi - \omega k) (\cos(\omega t)^2 + \sin(\omega t)^2) \\ &= \sigma_{xy}^2 \cos(\phi - \omega k) \end{aligned} \quad (\text{B.10.3})$$

and

$$\begin{aligned} E(x_{t-k} y_t) &= E(a_x \cos(\omega(t-k)) + b_x \sin(\omega(t-k))) \times \\ &\quad (a_y \cos(\omega t + \phi) + b_y \sin(\omega t + \phi)) \\ &= Ea_x a_y (\cos(\omega(t-k)) \cos(\omega t + \phi)) + Eb_x b_y (\sin(\omega(t-k)) \sin(\omega t + \phi)) \\ &= \sigma_{xy}^2 ((\cos(\omega t) \cos(\omega k) + \sin(\omega t) \sin(\omega k)) (\cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi))) \\ &\quad + \sigma_{xy}^2 ((\sin(\omega t) \cos(\omega k) - \cos(\omega t) \sin(\omega k)) (\sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi))) \\ &= \sigma_{xy}^2 (\cos(\omega t)^2 + \sin(\omega t)^2) (\cos(\omega k) \cos(\phi) - \sin(\omega k) \sin(\phi)) \\ &= \sigma_{xy}^2 (\omega) \cos(\omega k - \phi) \end{aligned} \quad (\text{B.10.4})$$

which uses  $E(a_x b_y) = E(a_y b_x) = 0$ .

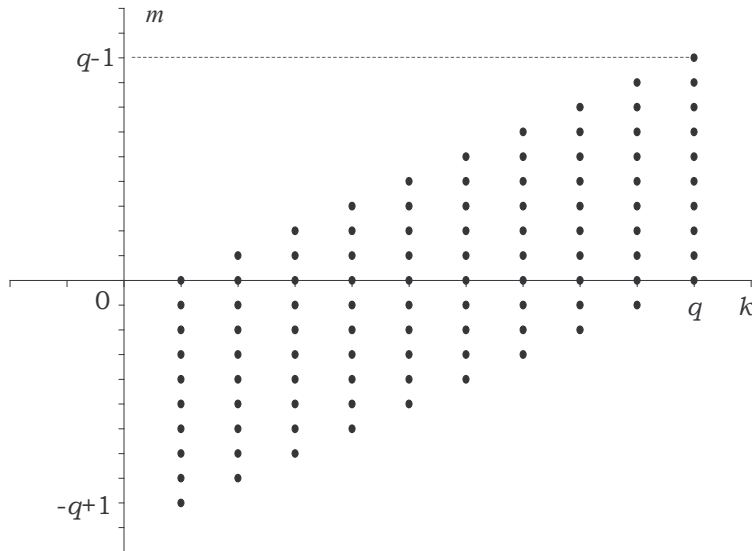
### B.11 Proof of (4.5.7)

The second part of (4.5.7) can be written as

$$\begin{aligned}
 \left| 1 + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \right|^2 &= \left( 1 + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \right) \left( 1 + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \right)^* \\
 &= \left( 1 + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \right) \left( 1 + \sum_{l=1}^q \alpha_l \exp(i\omega l) \right) \\
 &= 1 + \sum_{l=1}^q \alpha_l \exp(i\omega l) + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \\
 &\quad + \sum_{k=1}^q \alpha_k \exp(-i\omega k) \sum_{l=1}^q \alpha_l \exp(i\omega l) \\
 &= 1 + \sum_{l=1}^q \alpha_l (\cos(\omega l) + i \sin(\omega l)) + \sum_{k=1}^q \alpha_k (\cos(\omega k) - i \sin(\omega k)) \\
 &\quad + \sum_{k=1}^q \sum_{l=1}^q \alpha_k \alpha_l \exp(-i\omega(k-l)) \\
 &= 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \sum_{m=k-1}^{k-q} \alpha_k \alpha_{k-m} \exp(-i\omega m)
 \end{aligned} \tag{B.11.1}$$

In a two dimensional plain the values of  $k$  and  $m$  in the second summation are shown in Figure B.2.

Figure B.2 Summation over  $k$  and  $m$  in (B.11.1).



This shows that (B.11.1) is equal to

$$\begin{aligned}
 & 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \alpha_k^2 + \sum_{m=1}^{q-1} \sum_{k=m+1}^q \alpha_k \alpha_{k-m} \exp(-i\omega m) \\
 & + \sum_{m=-q+1}^{-1} \sum_{k=1}^{q+m} \alpha_k \alpha_{k-m} \exp(-i\omega m) \\
 & = 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \alpha_k^2 + \sum_{m=1}^{q-1} \sum_{k=m+1}^q \alpha_k \alpha_{k-m} (\cos(\omega m) - i \sin(\omega m)) \\
 & + \sum_{m=1}^{q-1} \sum_{k=1}^{q-m} \alpha_k \alpha_{k+m} (\cos(\omega m) + i \sin(\omega m)) \\
 & = 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \alpha_k^2 + \sum_{m=1}^{q-1} \sum_{k=m+1}^q \alpha_k \alpha_{k-m} (\cos(\omega m) - i \sin(\omega m)) \\
 & + \sum_{m=1}^{q-1} \sum_{k=m+1}^q \alpha_{k-m} \alpha_k (\cos(\omega m) + i \sin(\omega m)) \\
 & = 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \alpha_k^2 + 2 \sum_{m=1}^{q-1} \sum_{k=m+1}^q \alpha_k \alpha_{k-m} \cos(\omega m) \\
 & = 1 + 2 \sum_{k=1}^q \alpha_k \cos(\omega k) + \sum_{k=1}^q \alpha_k^2 + 2 \sum_{m=1}^{q-1} \sum_{k=1}^{q-m} \alpha_k \alpha_{m+k} \cos(\omega m)
 \end{aligned} \tag{B.11.2}$$

which concludes the proof.

## B.12 Proof of (5.1.18) and (5.1.19)

Suppose that first a linear filter with FRF  $G_1(\exp(-i\omega))$  is applied and that second a filter with FRF  $G_2(\exp(-i\omega))$  is applied on the time series after having filtered with the first filter. Applying (5.1.6) this means that in terms of a complex representation the resulting time series is given by

$$y_t^* = x_t^* G_1(\exp(-i\omega)) G_2(\exp(-i\omega)) \quad (\text{B.12.1})$$

Assuming the polar formats

$$G_1(\exp(-i\omega)) = R_1(\omega) \exp(i\phi_1(\omega)) \text{ and } G_2(\exp(-i\omega)) = R_2(\omega) \exp(i\phi_2(\omega)) \quad (\text{B.12.2})$$

the combined filter therefore has a FRF equal to

$$\begin{aligned} G_{1+2}(\exp(-i\omega)) &= G_1(\exp(-i\omega)) G_2(\exp(-i\omega)) \\ &= R_1(\omega) \exp(i\phi_1(\omega)) R_2(\omega) \exp(i\phi_2(\omega)) \\ &= R_1(\omega) R_2(\omega) \exp(i(\phi_1(\omega) + \phi_2(\omega))) \end{aligned} \quad (\text{B.12.3})$$

and thus

$$\begin{aligned} |G_{1+2}(\exp(-i\omega))| &= R_1(\omega) R_2(\omega) = |G_1(\exp(-i\omega))| |G_2(\exp(-i\omega))| \\ \arg(G_{1+2}(\exp(-i\omega))) &= \phi_1(\omega) + \phi_2(\omega) = \arg(G_1(\exp(-i\omega))) + \arg(G_2(\exp(-i\omega))) \end{aligned} \quad (\text{B.12.4})$$

which concludes the proof.

### B.13 Proof of (5.3.15)

Start by writing the FRF  $G(\omega)$  and the weights  $g_l$  of the IRF in the general complex formats

$$G(\omega) = A(\omega) + iB(\omega) \quad (\text{B.13.1})$$

and

$$g_l = a_l + ib_l \quad (\text{B.13.2})$$

Substituting these into (5.3.14) yields

$$\begin{aligned} & \int_0^{2\pi} \left| G(\omega) - \sum_{l=-a}^a g_l \exp(-il\omega) \right|^2 d\omega \quad (\text{B.13.3}) \\ &= \int_0^{2\pi} \left| A(\omega) + iB(\omega) - \sum_{l=-a}^a (a_l + ib_l)(\cos(l\omega) - i\sin(l\omega)) \right|^2 d\omega \\ &= \int_0^{2\pi} \left| \left( A(\omega) - \sum_{l=-a}^a a_l \cos(l\omega) - \sum_{l=-a}^a b_l \sin(l\omega) \right) + i \left( B(\omega) + \sum_{l=-a}^a a_l \sin(l\omega) - \sum_{l=-a}^a b_l \cos(l\omega) \right) \right|^2 d\omega \\ &= \int_0^{2\pi} \left( \left( A(\omega) - \sum_{l=-a}^a a_l \cos(l\omega) - \sum_{l=-a}^a b_l \sin(l\omega) \right)^2 + \left( B(\omega) + \sum_{l=-a}^a a_l \sin(l\omega) - \sum_{l=-a}^a b_l \cos(l\omega) \right)^2 \right) d\omega \\ &= \int_0^{2\pi} \left\{ A(\omega)^2 + \left( \sum_{l=-a}^a a_l \cos(l\omega) \right)^2 + \left( \sum_{l=-a}^a b_l \sin(l\omega) \right)^2 - 2A(\omega) \sum_{l=-a}^a a_l \cos(l\omega) - 2A(\omega) \sum_{l=-a}^a b_l \sin(l\omega) \right. \\ &\quad \left. + B(\omega)^2 + \left( \sum_{l=-a}^a a_l \sin(l\omega) \right)^2 + \left( \sum_{l=-a}^a b_l \cos(l\omega) \right)^2 + 2B(\omega) \sum_{l=-a}^a a_l \sin(l\omega) - 2B(\omega) \sum_{l=-a}^a b_l \cos(l\omega) \right\} d\omega \\ &= \int_0^{2\pi} \left\{ A(\omega)^2 + B(\omega)^2 + \sum_{l=-a}^a a_l^2 + \sum_{l=-a}^a b_l^2 + 2 \sum_{l=-a}^a \sum_{j=l+1}^a a_l \cos(l\omega) a_j \cos(j\omega) \right. \\ &\quad \left. + 2 \sum_{l=-a}^a \sum_{j=l+1}^a b_l \sin(l\omega) b_j \sin(j\omega) - 2A(\omega) \sum_{l=-a}^a a_l \cos(l\omega) - 2A(\omega) \sum_{l=-a}^a b_l \sin(l\omega) \right. \\ &\quad \left. + 2 \sum_{l=-a}^a \sum_{j=l+1}^a a_l \sin(l\omega) a_j \sin(j\omega) + 2 \sum_{l=-a}^a \sum_{j=l+1}^a b_l \cos(l\omega) b_j \cos(j\omega) \right. \\ &\quad \left. + 2B(\omega) \sum_{l=-a}^a a_l \sin(l\omega) - 2B(\omega) \sum_{l=-a}^a b_l \cos(l\omega) \right\} d\omega \end{aligned}$$



The first order conditions with respect to  $a_l$  and  $b_l$  of (B.13.3) are

$$\begin{aligned}
 \frac{\partial}{\partial a_l} &= \int_0^{2\pi} \left\{ 2a_l + 2 \sum_{j=l+1}^a a_j \cos(l\omega) \cos(j\omega) - 2A(\omega) \cos(l\omega) \right. \\
 &\quad \left. + 2 \sum_{j=l+1}^a a_j \sin(l\omega) \sin(j\omega) + 2B(\omega) \sin(l\omega) \right\} d\omega = 0 \\
 &\Leftrightarrow \int_0^{2\pi} a_l d\omega + \sum_{j=l+1}^a a_j \left( \int_0^{2\pi} \cos(l\omega) \cos(j\omega) d\omega + \int_0^{2\pi} \sin(l\omega) \sin(j\omega) d\omega \right) \\
 &\quad - \int_0^{2\pi} A(\omega) \cos(l\omega) + \int_0^{2\pi} B(\omega) \sin(l\omega) d\omega = 0 \tag{B.13.4} \\
 &\Leftrightarrow 2\pi a_l + \sum_{j=l+1}^a \frac{a_j}{l-j} \int_0^{2\pi} \sin((l-j)\omega) d\omega - \int_0^{2\pi} A(\omega) \cos(l\omega) + \int_0^{2\pi} A(\omega) \cos(l\omega) d\omega = 0 \\
 &\Leftrightarrow a_l^* = \frac{1}{2\pi} \int_0^{2\pi} A(\omega) \cos(l\omega) - \frac{1}{2\pi} \int_0^{2\pi} B(\omega) \sin(l\omega) d\omega = 0
 \end{aligned}$$

and

$$\frac{\partial}{\partial b_l} = 0 \Leftrightarrow b_l^* = \frac{1}{2\pi} \int_0^{2\pi} A(\omega) \sin(l\omega) + \frac{1}{2\pi} \int_0^{2\pi} B(\omega) \cos(l\omega) d\omega = 0 \tag{B.13.5}$$

This uses the relations

$$\begin{aligned}
 \int \sin(mx) \sin(nx) dx &= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \text{ for } m^2 \neq n^2 \\
 \int \cos(mx) \cos(nx) dx &= \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \text{ for } m^2 \neq n^2
 \end{aligned} \tag{B.13.6}$$

and

$$\int_0^{2\pi} \sin mx dx = 0 \quad \forall m \tag{B.13.7}$$

Rewriting (5.3.15) as

$$\begin{aligned}
 g_l &= \frac{1}{2\pi} \int_0^{2\pi} G(\omega) \exp(il\omega) d\omega \\
 &= \frac{1}{2\pi} \int_0^{2\pi} (A(\omega) + iB(\omega)) (\cos(l\omega) + i \sin(l\omega)) d\omega \\
 &= \frac{1}{2\pi} \int_0^{2\pi} (A(\omega) \cos(l\omega) - B(\omega) \sin(l\omega)) + i(A(\omega) \sin(l\omega) + B(\omega) \cos(l\omega)) d\omega \\
 &= a_l^* + ib_l^*
 \end{aligned} \tag{B.13.8}$$

where  $a^*$  and  $b^*$  are as defined in (B.13.4) and (B.13.5), concludes the proof.

## B.14 Proof of (5.3.22) and (5.3.23)

The first of the two relations follows by seeing the FRF

$$G(\omega) = \sum_{l=a}^a g_l \exp(-i\omega l) \quad (\text{B.14.1})$$

as a polynomial in terms of  $\exp(-i\omega l)$  with coefficients  $g_l$ . If  $G(\omega)=0$  for  $\omega=0$  then this means that a root of the polynomial must be  $\exp(-i\omega l)=1$  or  $\exp(i\omega l)=1$ . In terms of the factorisation proposition from the theory of linear algebra this means that the polynomial can be divided by either  $(1-\exp(-i\omega l))$  or  $(1-\exp(i\omega l))$ . So

$$G(\omega) = (1 - \exp(-i\omega l))H(\omega) \vee G(\omega) = (1 - \exp(i\omega l))I(\omega) \quad (\text{B.14.2})$$

where  $H(\omega)$  and  $I(\omega)$  are the appropriate remainder polynomials. Setting  $\exp(i\omega l)=L$  gives (5.3.22). In case  $g_l = g_{-l}$  from (B.14.1) it is easily seen that

$$G(\omega) = G(-\omega) \quad (\text{B.14.3})$$

Together with the previous restriction  $G(0)=0$  this implies that the polynomial can both be divided by  $(1-\exp(-i\omega l))$  and  $(1-\exp(i\omega l))$ . That is

$$G(\omega) = (1 - \exp(-i\omega l))(1 - \exp(i\omega l))J(\omega) \quad (\text{B.14.4})$$

where again  $J(\omega)$  is the appropriate remainder polynomial. Substituting again  $\exp(i\omega l)=L$  gives (5.2.23) which concludes the proof.

## B.15 Proof of (5.3.34)

Substituting the IDCFT (4.1.22) expression for  $x_t$  in (5.3.33) gives

$$\begin{aligned}
 |y_t - \hat{y}_t|^2 &= \left| \sum_{l \in Q} g_l x_{t-l} \right|^2 \\
 &= \left| \sum_{l \in Q} g_l \frac{T}{2\pi} \int_0^{2\pi} J(\omega) \exp(i\omega(t-l)) d\omega \right|^2 \\
 &= \left| \frac{T}{2\pi} \int_0^{2\pi} \sum_{l \in Q} g_l J(\omega) \exp(i\omega(t-l)) d\omega \right|^2 \\
 &\leq \left\{ \frac{T}{2\pi} \int_0^{2\pi} \left| \sum_{l \in Q} g_l J(\omega) \exp(i\omega(t-l)) \right| d\omega \right\}^2 \\
 &\leq \left\{ \frac{T}{2\pi} \int_0^{2\pi} |J(\omega)| \cdot \sum_{l \in Q} |g_l| \cdot |\exp(i\omega(t-l))| d\omega \right\}^2 \\
 &= \left\{ \frac{T}{2\pi} \int_0^{2\pi} |J(\omega)| \cdot \sum_{l \in Q} |g_l| d\omega \right\}^2
 \end{aligned} \tag{B.15.1}$$

The set  $Q$  is as defined in (5.3.33). The first inequality follows from

$$\left| \int f(x) dx \right| \leq \int |f(x)| dx \tag{B.15.2}$$

which hold for any function  $f(x)$ . The second inequality follows from the relations

$$\begin{aligned}
 |a \cdot b| &= |a| \cdot |b| \\
 |a \pm b| &\leq |a| + |b|
 \end{aligned} \tag{B.15.3}$$

for any (complex) numbers  $a$  and  $b$ . The final equality simply follows from

$$|\exp(-i\omega(t-l))| = \sqrt{\cos^2(\omega(t-l)) + \sin^2(\omega(t-l))} = 1 \tag{B.15.4}$$

If we now define

$$M = \max_{0 \leq \omega < 2\pi} |T \cdot J(\omega)| \quad (\text{B.15.5})$$

then (B.15.1) can further be bounded by

$$\begin{aligned} |y_t - \hat{y}_t|^2 &\leq \left\{ \frac{T}{2\pi} \int_0^{2\pi} |J(\omega)| \cdot \sum_{l \in Q} |g_l| d\omega \right\}^2 \\ &\leq \left\{ \frac{1}{2\pi} \sum_{l \in Q} |g_l| \int_0^{2\pi} M d\omega \right\}^2 = M^2 \left\{ \sum_{l \in Q} |g_l| \right\}^2 \end{aligned} \quad (\text{B.15.6})$$

which concludes the proof.

## B.16 Derivation of spectrum shown in Figure 5.28

From (5.3.22) we know that for a filter with FRF  $G(\omega)$  which is zero for  $\omega=0$  it holds that

$$G(L) = (1 - L)H(L) \quad (\text{B.16.1})$$

for some remainder polynomial  $H(L)$ . Furthermore we know from section 5.2.2 that a linear zero phase filter has symmetrical weights  $g_l$  around  $l=0$ . For example for  $a=3$  such a filter looks like

$$G(L) = \sum_{l=-a}^a g_l L^l = g_3 L^{-3} + g_2 L^{-2} + g_1 L^{-1} + g_0 + g_1 L^1 + g_2 L^2 + g_3 L^3 \quad (\text{B.16.2})$$

(5.3.22) also states that if  $G(0)=0$  this means that the filter weights must sum to zero. That is

$$\sum_{l=-a}^a g_l = 0 \Leftrightarrow g_0 = -2 \sum_{l=1}^K g_l \quad (\text{B.16.3})$$

Substituting this into (B.16.2), for  $a=3$  we can write

$$G(L) = g_3 L^{-3} + g_2 L^{-2} + g_1 L^{-1} - 2(g_1 + g_2 + g_3) + g_1 L^1 + g_2 L^2 + g_3 L^3 \quad (\text{B.16.4})$$

Combining this with (B.16.1) shows that the remainder polynomial  $H(L)$  can be solved from

$$\begin{aligned} G(L) &= (1 - L)(g_3 L^{-3} + (g_2 + g_3)L^{-2} \\ &\quad + (g_1 + g_2 + g_3)L^{-1} - (g_1 + g_2 + g_3) \\ &\quad - (g_2 + g_3)L^1 - g_3 L^2) \end{aligned} \quad (\text{B.16.5})$$

If  $x_t$  is the random walk process (5.6.1) then the filtered time series  $y_t = G(L)x_t$  is equal to

$$\begin{aligned} y_t &= g_3 \varepsilon_{t+3} + (g_2 + g_3) \varepsilon_{t+2} \\ &\quad + (g_1 + g_2 + g_3) \varepsilon_{t+1} - (g_1 + g_2 + g_3) \varepsilon_t \\ &\quad - (g_2 + g_3) \varepsilon_{t-1} - g_3 \varepsilon_{t-2} \end{aligned} \quad (\text{B.16.6})$$

Because this is a stationary MA process  $y_t^* = y_{t-3}$  follows the same stochastic process as  $y_t$  and we get the MA(5) process

$$\begin{aligned} y_t^* &= g_3 \varepsilon_t + (g_2 + g_3) \varepsilon_{t-1} \\ &\quad + (g_1 + g_2 + g_3) \varepsilon_{t-2} - (g_1 + g_2 + g_3) \varepsilon_{t-3} \\ &\quad - (g_2 + g_3) \varepsilon_{t-4} - g_3 \varepsilon_{t-5} \end{aligned} \quad (\text{B.16.7})$$

For an arbitrary value of  $a$  this generalizes to the MA(2a-1) process

$$y_t^* = \sum_{i=0}^{2a-1} \alpha_i \varepsilon_{t-i} \quad (\text{B.16.8})$$

with the parameters

$$\alpha_i = \begin{cases} \sum_{j=a-i}^a g_j & \text{if } 0 \leq i \leq a-1 \\ -\sum_{j=1}^a g_j & \text{if } i = a \\ -\sum_{j=i-a+1}^K g_j & \text{if } a+1 \leq i \leq 2a-1 \end{cases} \quad (\text{B.16.9})$$

This shows that if the  $2a-1$  weights  $g_l$  of a linear zero phase filter with  $G(0)=0$  are known, the spectrum of a random walk filtered by this filter can be calculated from the spectral density (4.5.8) of the MA process (B.16.8). In principle the weights of a filter having an ideal FRF, as is the case for the zero phase frequency filter in the text, can be calculated as the Fourier transform of the FRF resulting in (5.3.6). However because of the discontinuous edges in an ideal band-pass filter, an infinite number of weights are needed to describe the filter in the time domain. Thereby also the MA process will be of infinite order. Also if a very large, though finite, number of weights are calculated, disturbing effects due to the leakage effect remain. Note that we know that as the number of weights increases the leakage effect does not disappear but is merely confined to smaller frequency intervals. See for example Figure 5.7 in the text. An alternative approach to calculate the appropriate filter weights is by using the Baxter-King approximation to an ideal pass-band filter as described in section 5.3.2. By allowing for a large number of weights, the ideal FRF can be approximated with sufficient accuracy. The spectrum in 5.22 is calculated using 5000 weights in the Baxter-King filter, filling in these weights into the MA process (B.16.8) and calculating the corresponding spectrum.



## B.17 Derivation of relation used in Table 17.2

Consider a time series of a perfect sinusoid of some frequency  $\omega$  and amplitude  $R$ .

$$x_t = R \cos(\omega t) \text{ for } t = 0, \dots, T-1 \quad (\text{B.17.1})$$

Because its (long term) average value is zero, the standard deviation of this time series is

$$\sigma = \sqrt{\frac{\sum_{t=0}^{T-1} R^2 (\cos(\omega t))^2}{T}} \quad (\text{B.17.2})$$

Because of (B.2.3) this is equal to

$$\sigma = \sqrt{\frac{R^2 T/2}{T}} = \frac{R}{\sqrt{2}} \quad (\text{B.17.3})$$

This shows that the amplitude  $R$  of the sinusoid is approximately  $\sqrt{2}$  times its standard deviation.

## Appendix C: Models used in section 4.7

This appendix contains the parameters of the VAR models used in the Monte Carlo experiment described in section 4.7. The parameters of the univariate AR( $p$ ) models

$$x_t = \beta_1 x_{t-1} + \dots + \beta_p x_{t-p} + \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim N(0, \sigma^2) \quad (\text{C.1})$$

are given in Table C.1 for the various orders  $p$ .

Table C.1 Parameters of univariate AR( $p$ ) models

$p$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	$\beta_6$	$\beta_7$	$\beta_8$	$\sigma^2$
1	0.40								0.40
2	0.40	-0.30							1.20
3	0.05	0.05	0.30						1.70
4	-0.70	-0.35	-0.45	-0.35					1.80
5	0.50	0.10	-0.20	-0.50	0.50				1.00
6	0.60	-0.40	0.10	-0.20	0.10	-0.40			1.10
7	0.05	-0.25	0.20	0.00	-0.10	-0.10	-0.60		1.50
8	-0.20	-0.80	-0.35	-0.13	0.10	0.20	0.35	0.30	0.30

The parameters of the multivariate VAR( $p$ ) models

$$x_t = A_1 x_{t-1} + A_2 x_{t-2} + \dots + A_p x_{t-p} + \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim N(0, \Sigma) \quad (\text{C.2})$$

are given in the Tables C.2 through to C.5 for respectively the dimensions  $n=2, \dots, 5$  and for the various orders  $p$ .

Table C.2 Parameters of VAR( $p$ ) models of dimension  $n=2$

$p$	$A_1$		$A_2$		$A_3$		$A_4$		$\Sigma$	
1	0.77	-0.93							0.83	0.08
	0.93	-0.55							0.08	0.49
2	0.52	-0.69	0.23	0.24					0.43	-0.17
	0.78	-0.65	-0.04	0.16					-0.17	0.77
3	0.52	0.24	0.38	-0.25	-0.79	0.12			1.37	-0.73
	-0.30	-0.43	0.33	-0.89	0.10	-0.23			-0.73	1.39
4	-0.02	-0.87	0.41	-0.24	0.84	-0.88	0.11	-0.32	1.31	-0.92
	0.71	0.74	0.22	-0.19	-0.54	-0.40	0.35	0.57	-0.92	1.05

Table C.3 Parameters of VAR( $p$ ) models of dimension  $n=3$

$p$	$A_1$			$A_2$			$A_3$			$\Sigma$		
1	-0.53	0.18	0.43							1.70	1.22	0.27
	-0.63	0.59	-0.21							1.22	1.46	-0.12
	-0.32	0.87	0.62							0.27	-0.12	0.29
2	0.23	0.91	0.04	-0.64	0.09	0.81				1.16	-0.07	0.21
	0.22	0.42	0.03	0.29	-0.24	-0.75				-0.07	1.74	-0.68
	0.53	0.14	-0.73	-0.72	-0.45	0.37				0.21	-0.68	1.85
3	-0.29	-0.42	-0.60	0.88	-0.74	0.70	0.09	-0.73	-0.10	0.96	0.45	-0.60
	-0.75	-0.10	-0.43	0.94	-0.91	0.13	0.08	-0.77	-0.24	0.45	0.70	-0.86
	-0.74	0.74	0.64	-0.16	-0.12	-0.35	0.53	0.60	0.71	-0.60	-0.86	1.95

Table C.4 Parameters of VAR( $p$ ) models of dimension  $n=4$

$p$	$A_1$				$A_2$				$\Sigma$			
1	0.31	0.61	-0.03	-0.09					1.16	0.07	-0.43	-0.75
	-0.66	1.00	-0.37	0.13					0.07	1.27	0.54	-0.63
	-0.82	0.54	-0.81	-0.24					-0.43	0.54	1.31	-0.06
	0.30	0.80	0.78	-0.22					-0.75	-0.63	-0.06	1.96
2	0.78	-0.69	-0.17	-0.01	-0.27	0.46	-0.17	-0.93	0.47	0.09	-0.12	0.10
	-0.32	0.35	0.05	-0.42	0.87	-0.07	-0.95	0.30	0.09	1.34	-0.90	1.38
	0.45	-0.83	-0.05	-0.55	-0.73	0.31	-0.13	-0.87	-0.12	-0.90	1.69	-0.40
	0.65	0.44	-0.53	-0.25	0.50	0.44	-0.40	-0.82	0.10	1.38	-0.40	1.84

Table C.5 Parameters of VAR( $p$ ) models of dimension  $n=5$

$p$	$A_1$						$\Sigma$					
1	-0.19	0.93	0.09	-0.22	0.37		1.14	0.25	-0.80	0.21	-0.62	
	-0.44	-0.23	-0.94	-0.78	-0.49		0.25	0.63	-0.24	0.12	0.40	
	-0.44	0.21	0.21	-0.70	0.55		-0.80	-0.24	1.08	0.16	0.73	
	0.12	-0.89	-0.52	-0.07	-0.78		0.21	0.12	0.16	0.61	0.08	
	0.09	0.94	0.36	-0.01	0.10		-0.62	0.40	0.73	0.08	1.27	

## Appendix D: Results United Kingdom

This appendix contains the results of the filtering and spectral analyses of the time series for the United Kingdom. The results are obtained by means of the specific methodology described in Chapter 6. The numerical parts of the results are contained in the tables in the chapters of Part III.

### D.1 National Product

Figure D.1 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a National Product index for the UK. For comparison, the pictures also show the long wave and stochastic component of a NL National Product index.

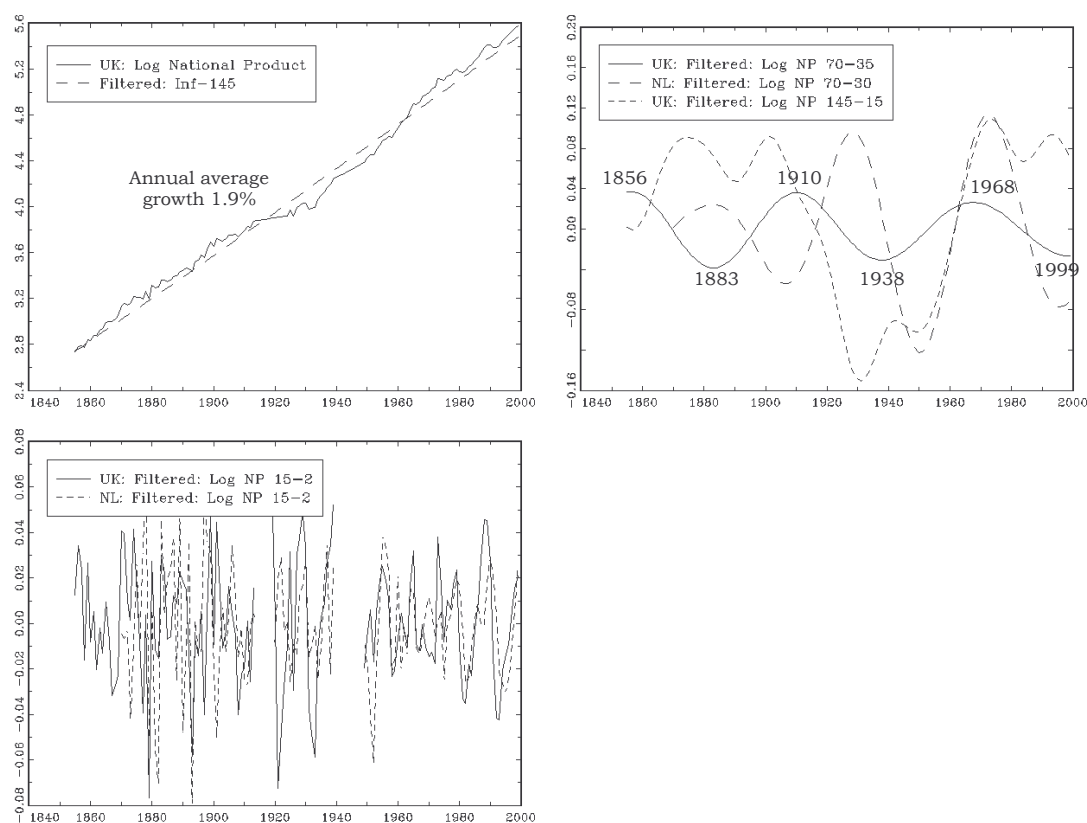
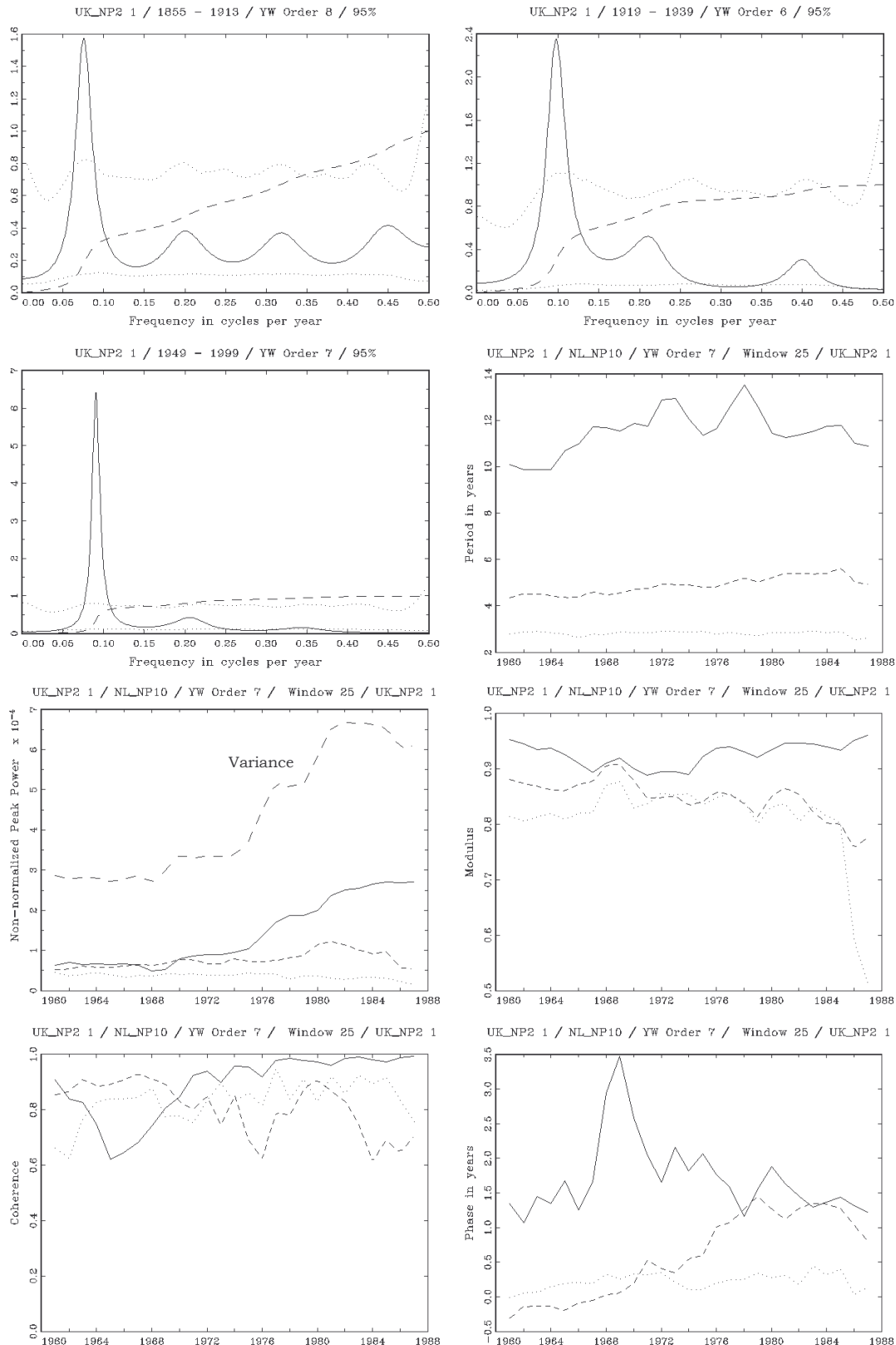


Figure D.2 Static and dynamic estimates of the spectral density of the stochastic component of a National Product index for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the NL. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.2 Industrial Production

Figure D.3 Trend (top left), periodic component (top right) and stochastic component (bottom left) of an Industrial Production index for the UK. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the UK.

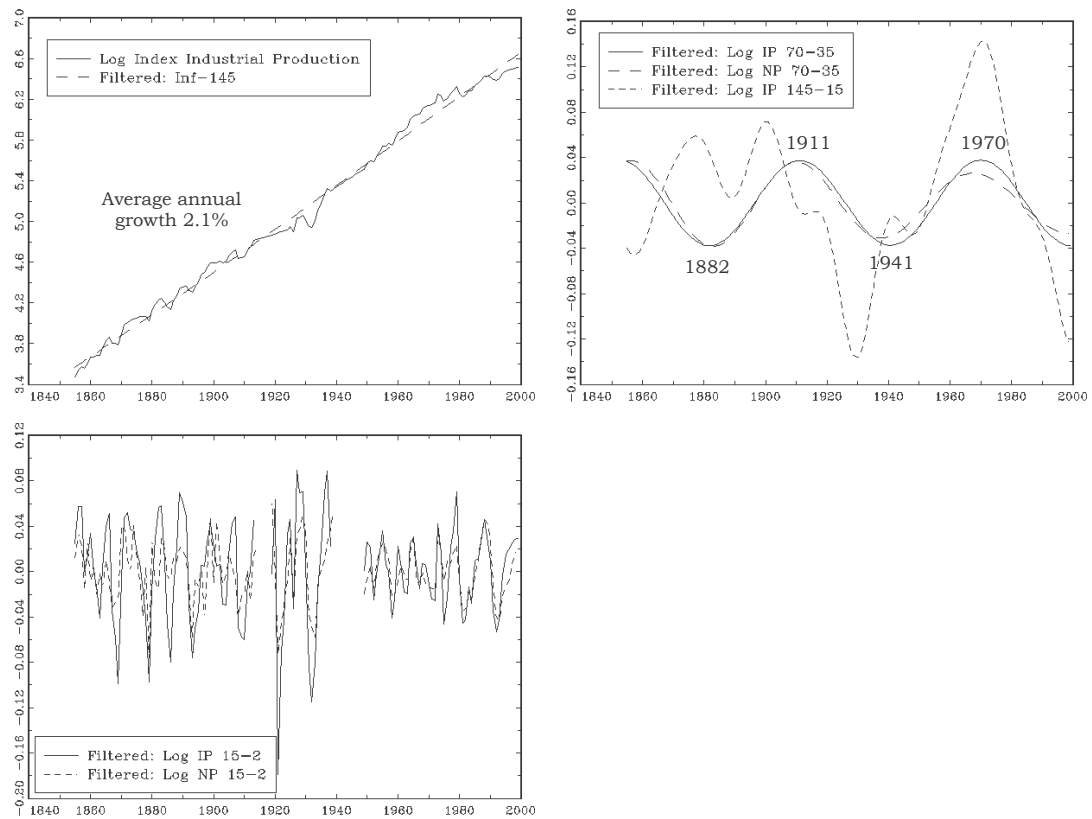
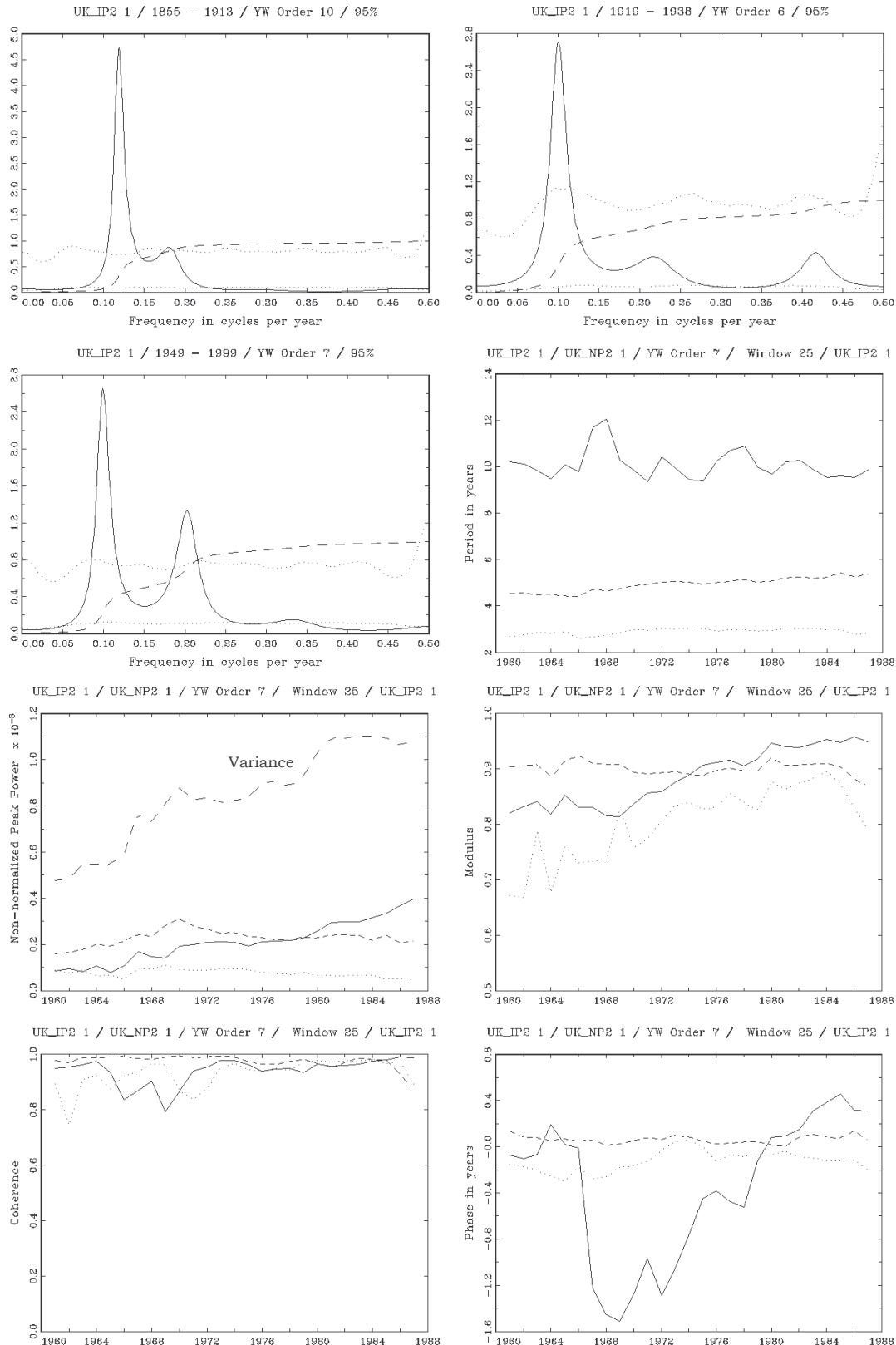


Figure D.4 Static and dynamic estimates of the spectral density of the stochastic component of an Industrial Production index for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.





## D.3 Employment rate

Figure D.5 Trend (top left), periodic component (top right) and stochastic component (bottom left) of the Employment rate for the UK. For comparison, the pictures also show the long wave and stochastic component of a UK National Product index.

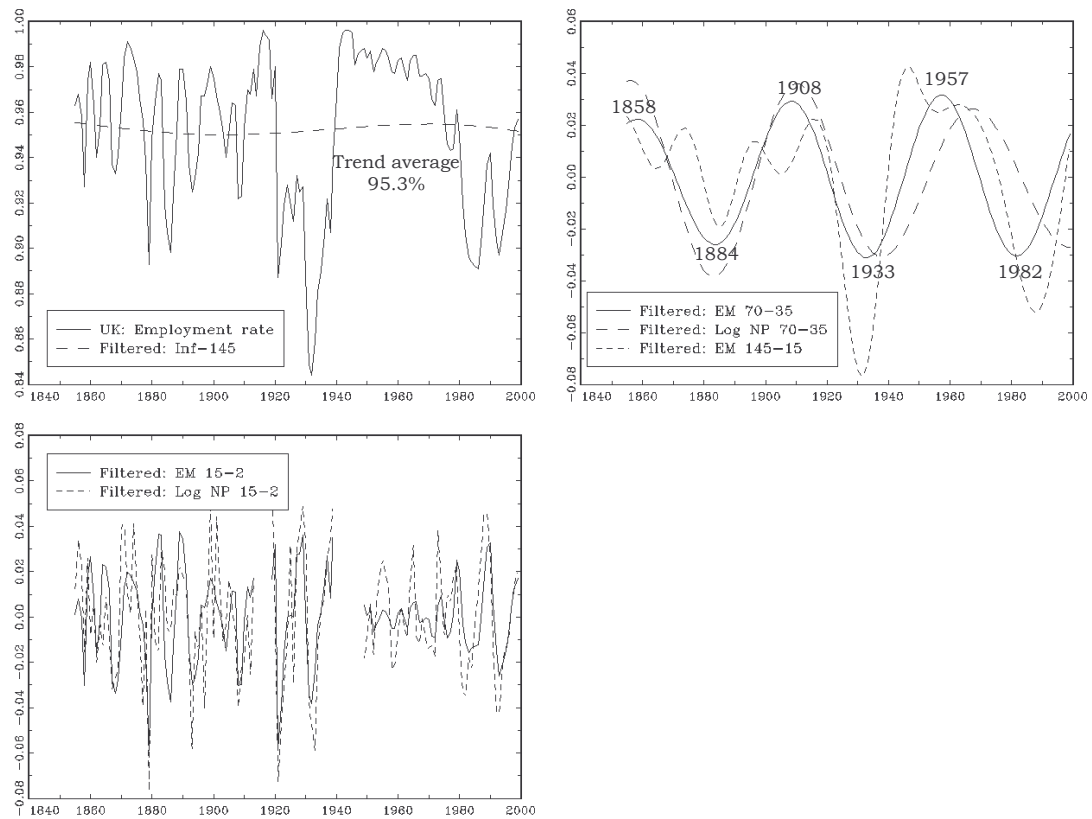
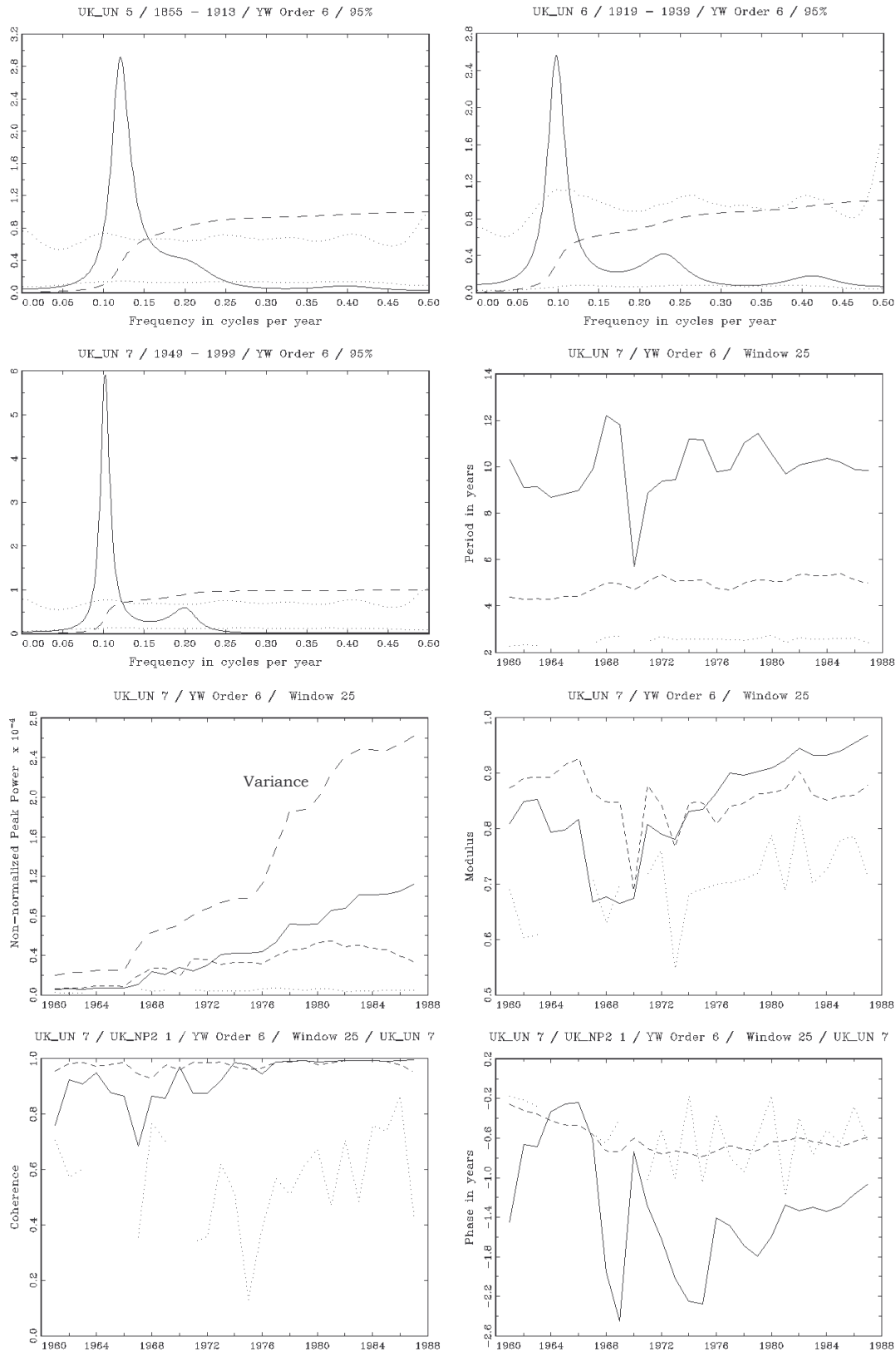


Figure D.6 Static and dynamic estimates of the spectral density of the stochastic component of the Employment rate for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.4 Consumer Prices

Figure D.7 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Consumer Price index for the UK. For comparison, the pictures also show the long wave and stochastic component of a UK National Product index.

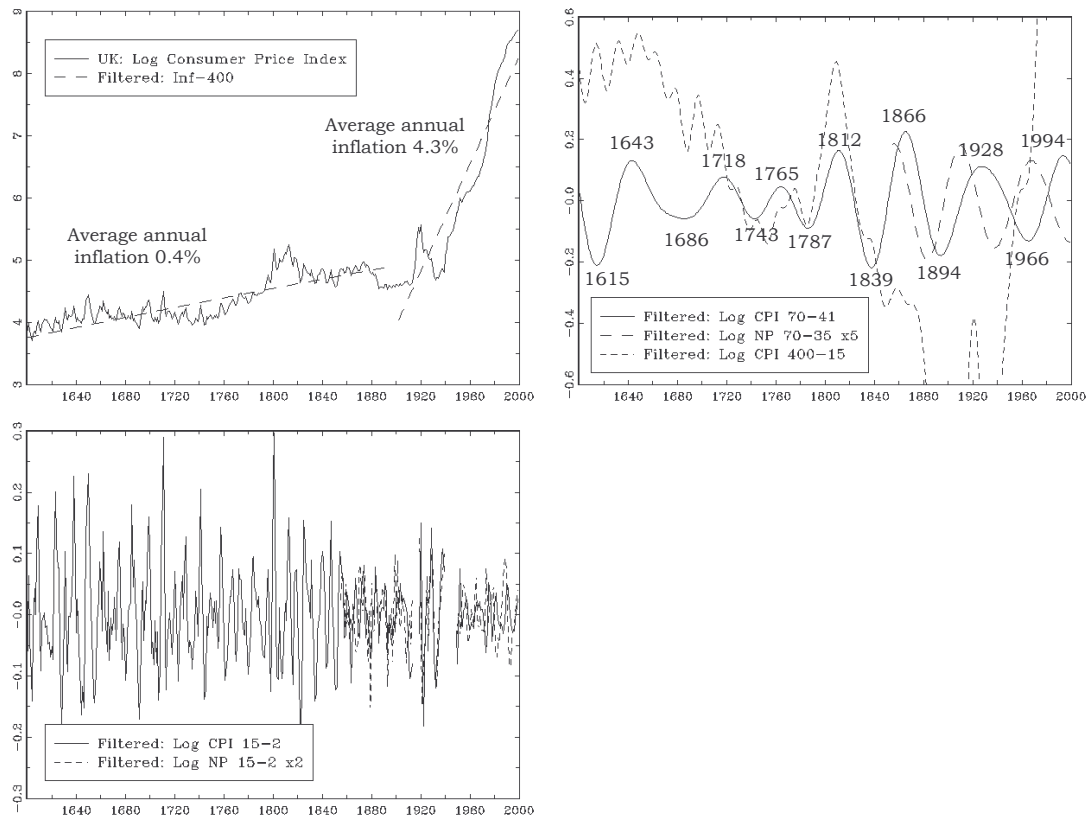
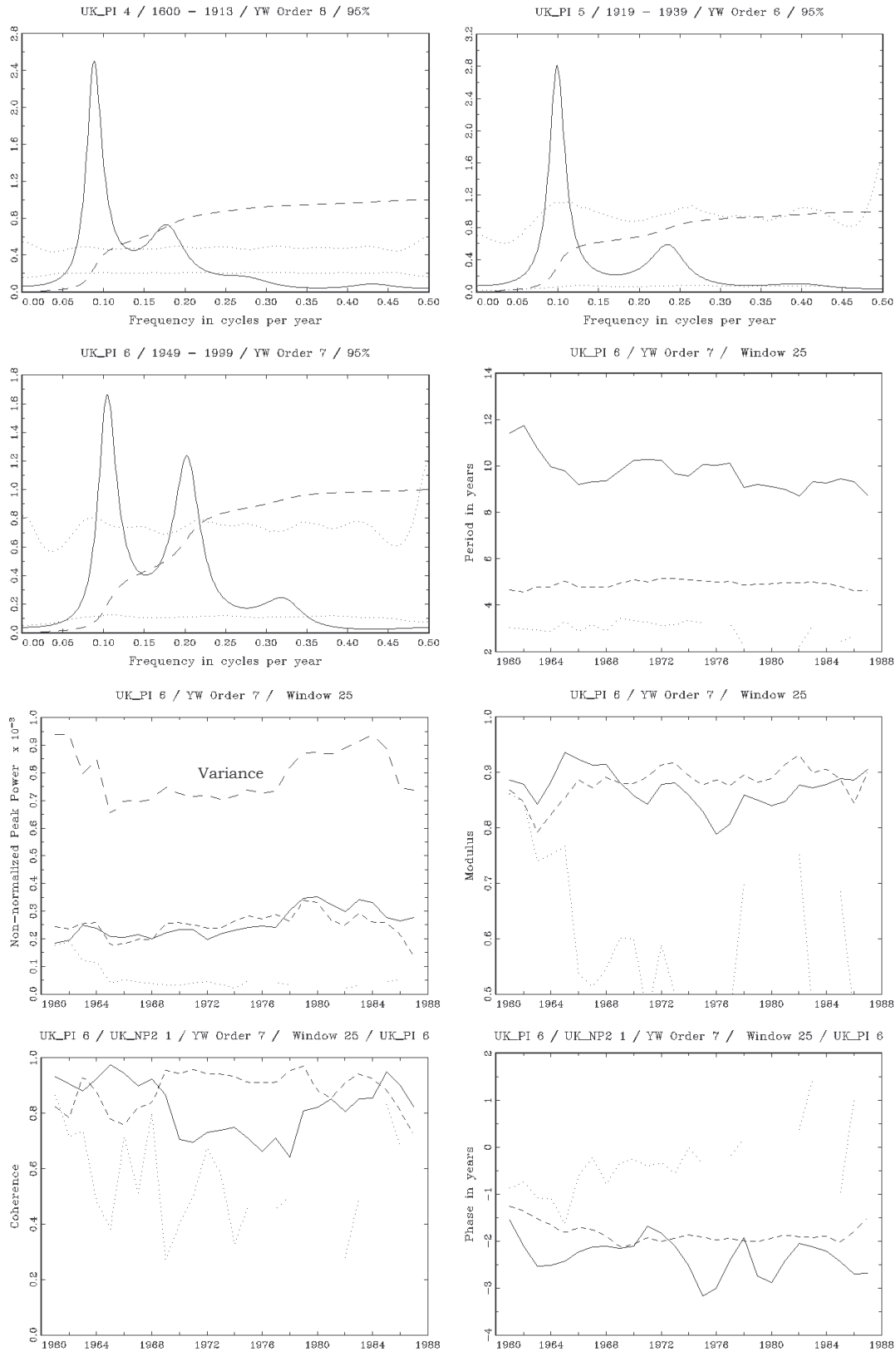


Figure D.8 Static and dynamic estimates of the spectral density of the stochastic component of a Consumer Price index for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.5 Industry Wages

Figure D.9 Trend (top left), periodic component (top right) and stochastic component (bottom left) of an Industry Wage index for the UK. For comparison, the pictures also show the long wave and stochastic component of a UK National Product index.

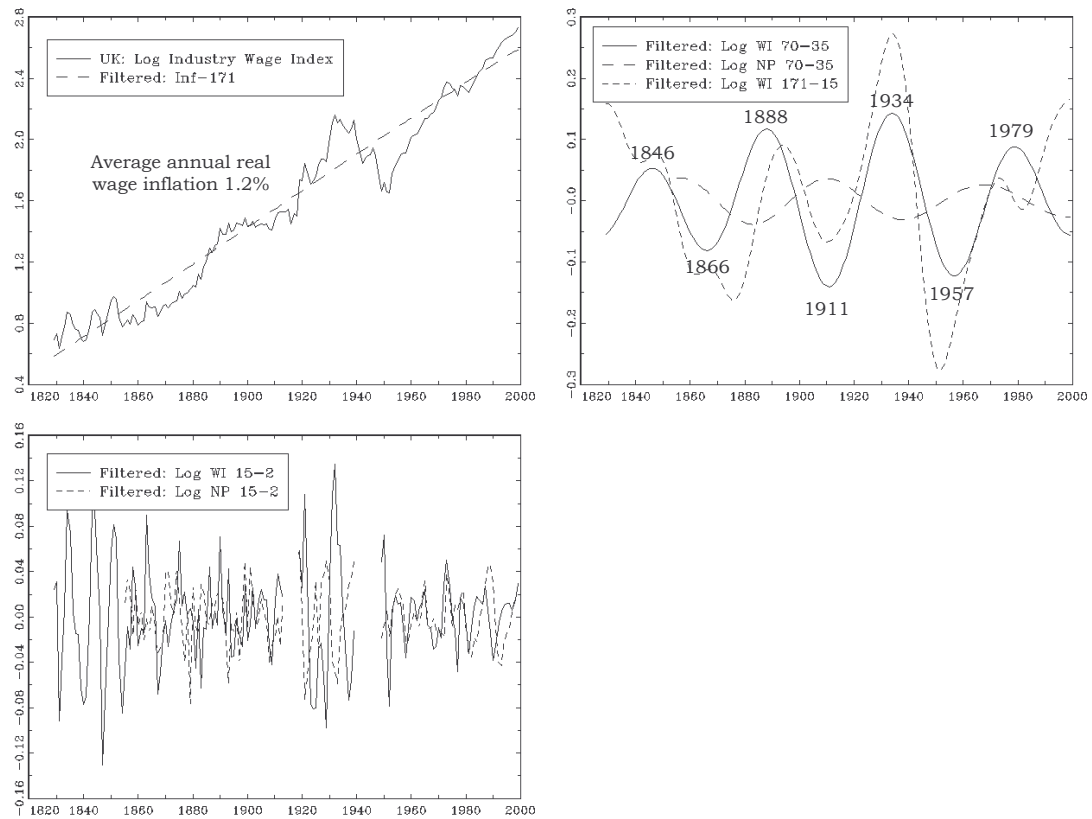
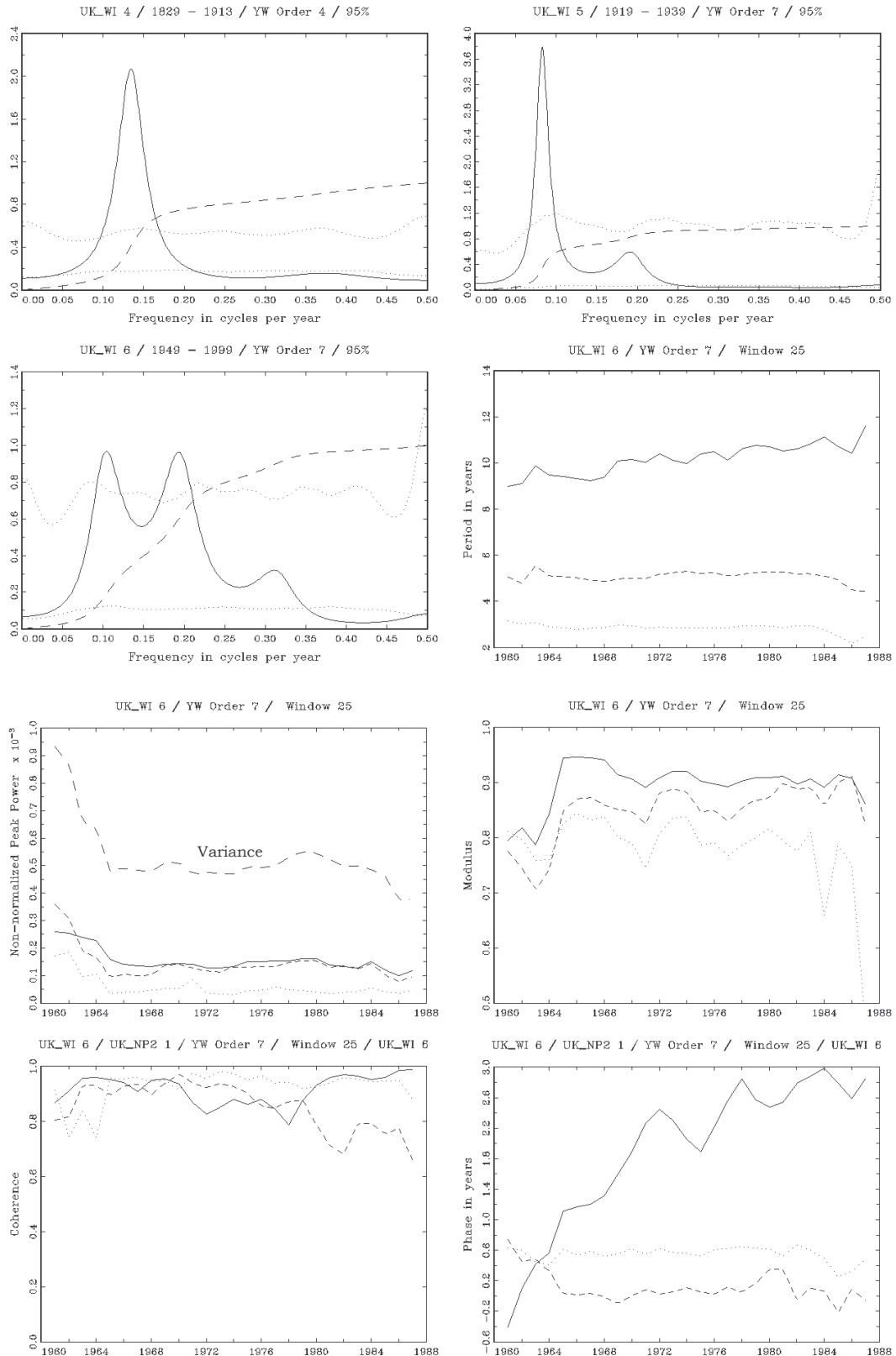


Figure D.10 Static and dynamic estimates of the spectral density of the stochastic component of an Industry Wage index for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures.



## D.6 Short Interest Rate

Figure D.11 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Short Interest Rate for the UK. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the UK.

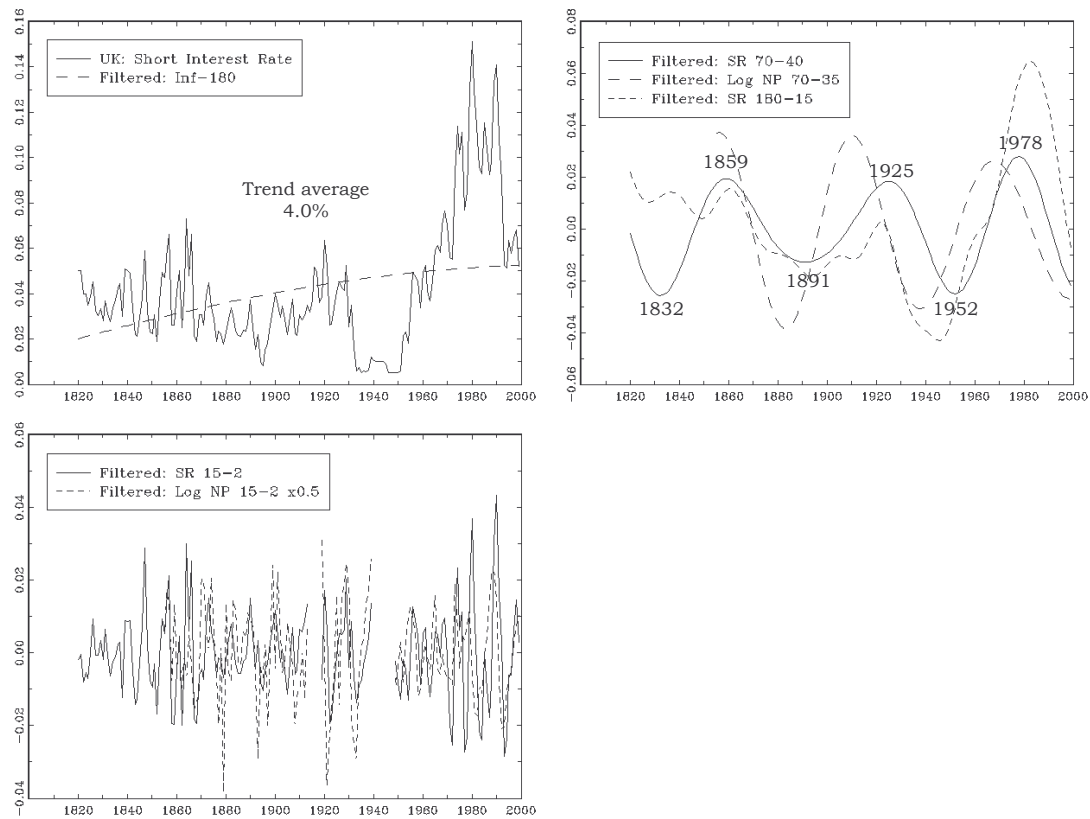


Figure D.12 For a (nominal) Short Interest Rate for the UK, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).

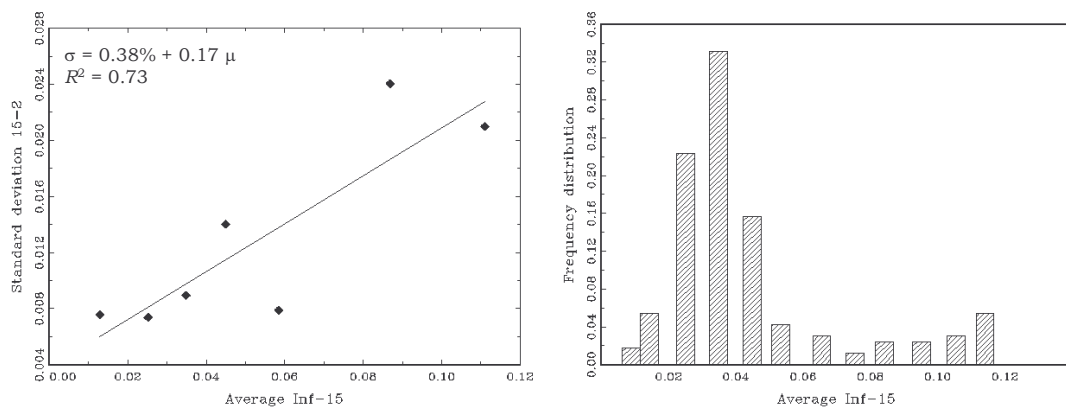
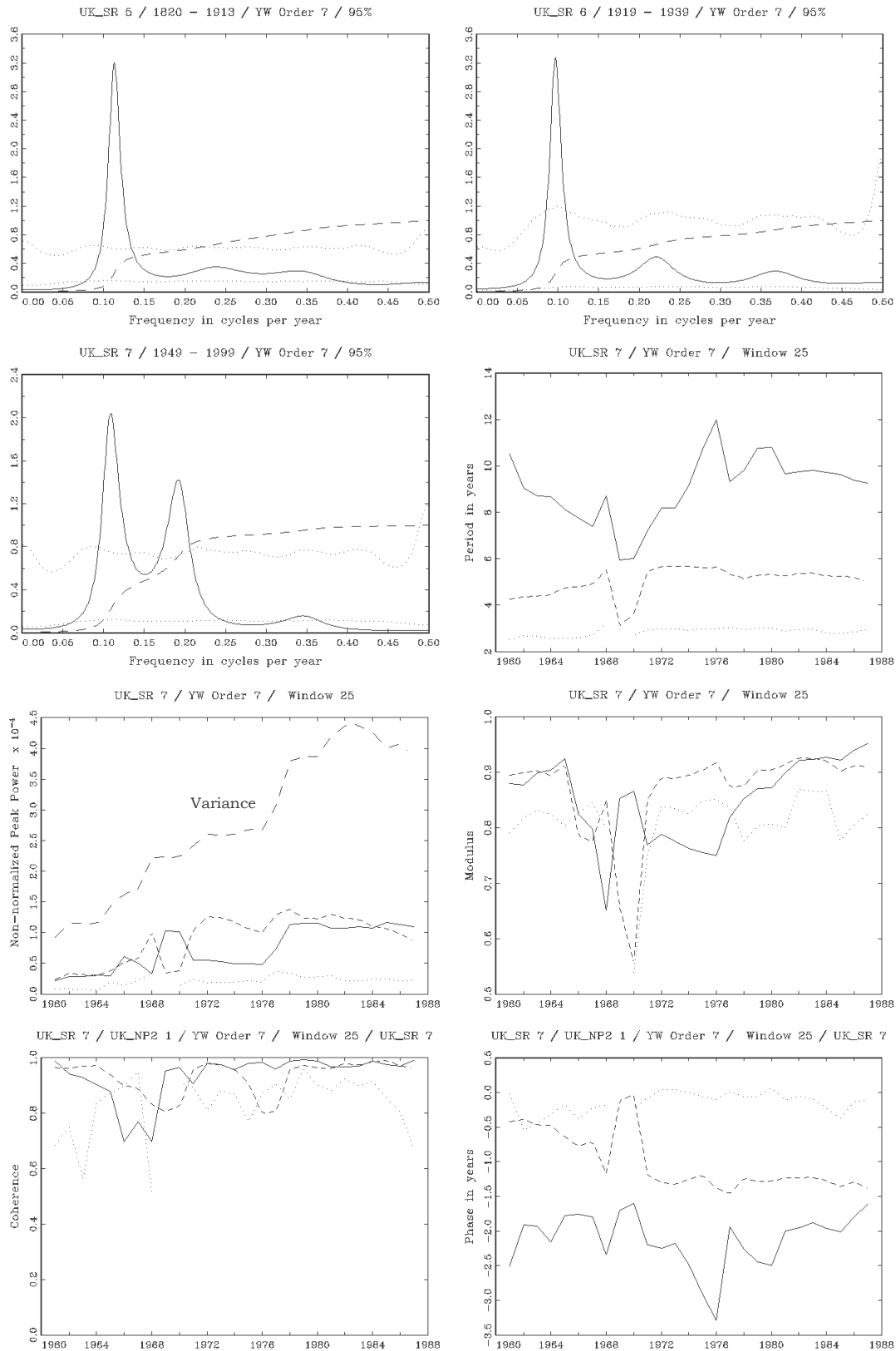




Figure D.13 Static and dynamic estimates of the spectral density of the stochastic component of (nominal) Short Interest Rate for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.7 Long Interest Rate

Figure D.14 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Long Interest Rate for the UK. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the UK. Also the long wave component in (ex post) real long interest rate versus long wave in National Product (bottom right).

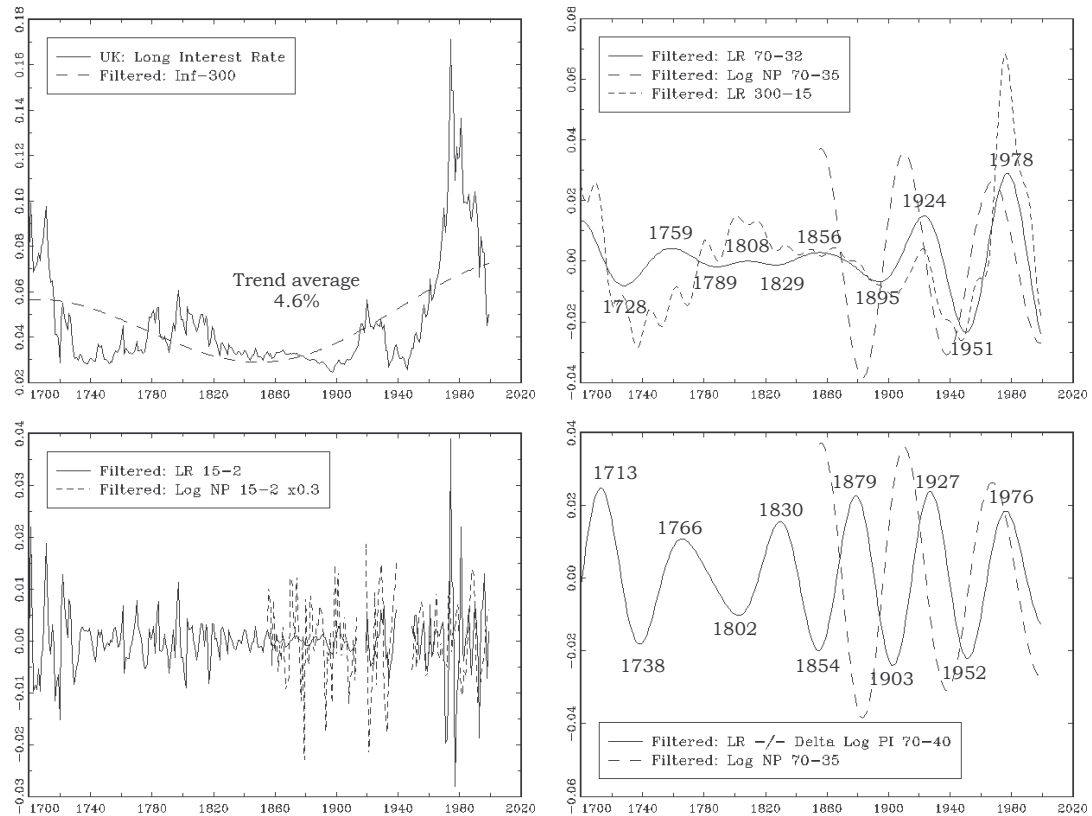


Figure D.15 For a (nominal) Long Interest Rate for the UK, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).

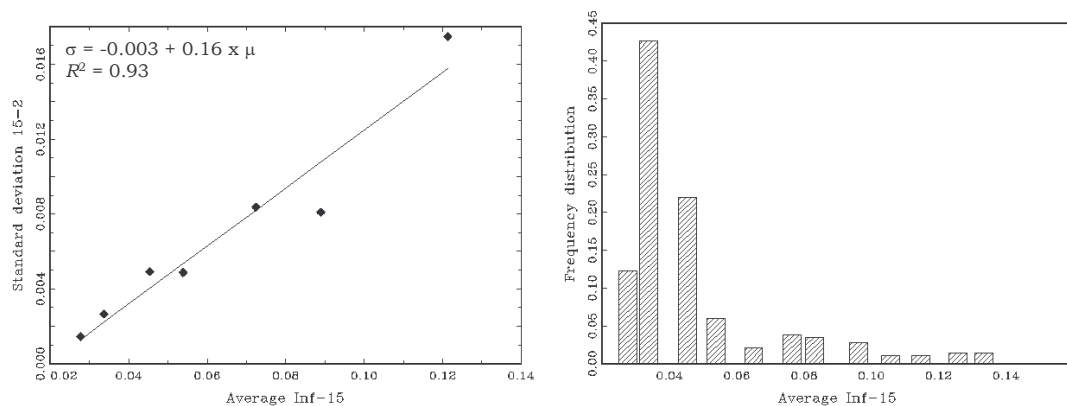
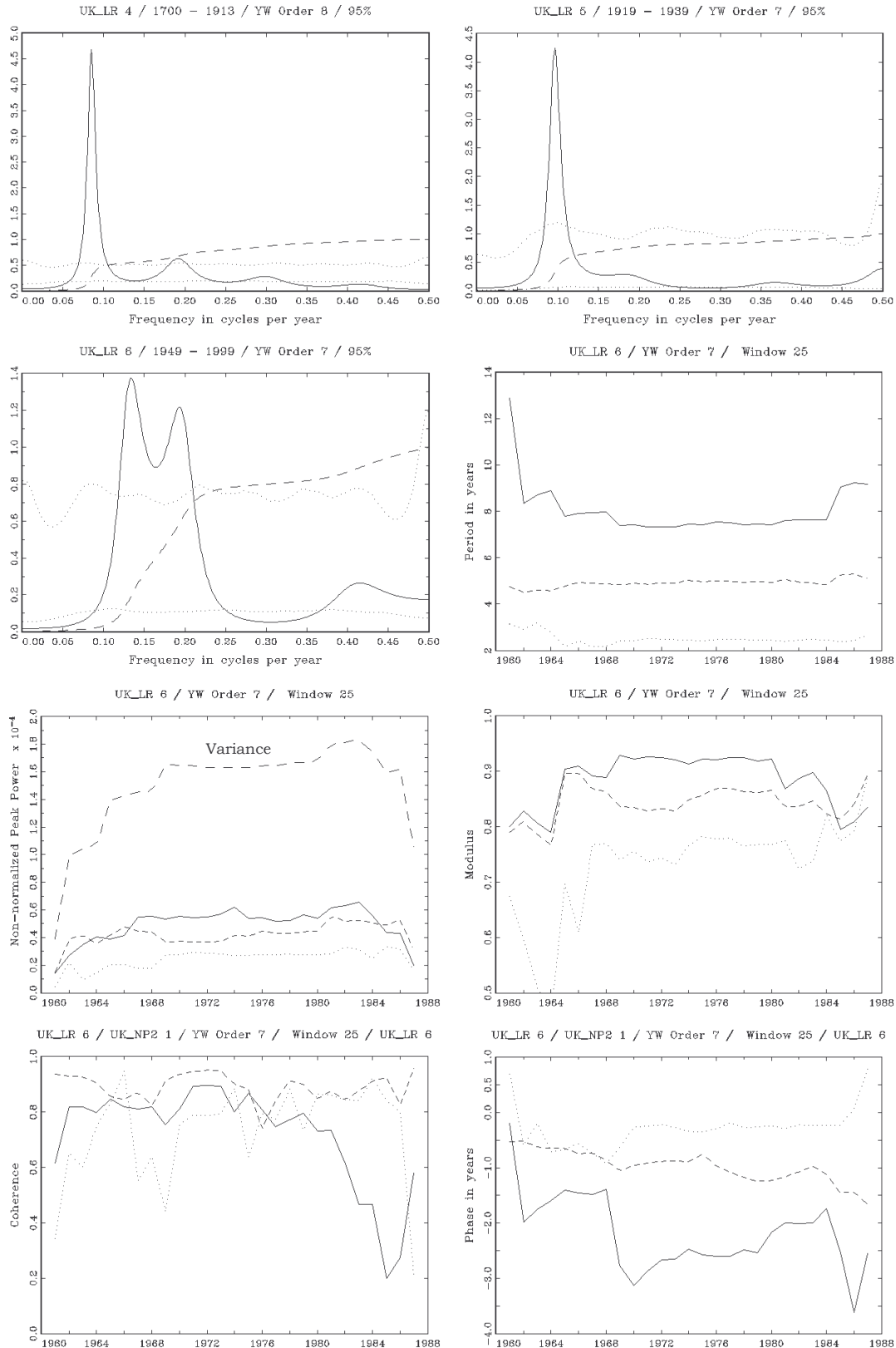


Figure D.16 Static and dynamic estimates of the spectral density of the stochastic component of (nominal) Long Interest Rate for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.8 Term Spread

Figure D.17 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Term Spread for the UK. For comparison, the pictures also show the long wave and stochastic component of a UK National Product index.

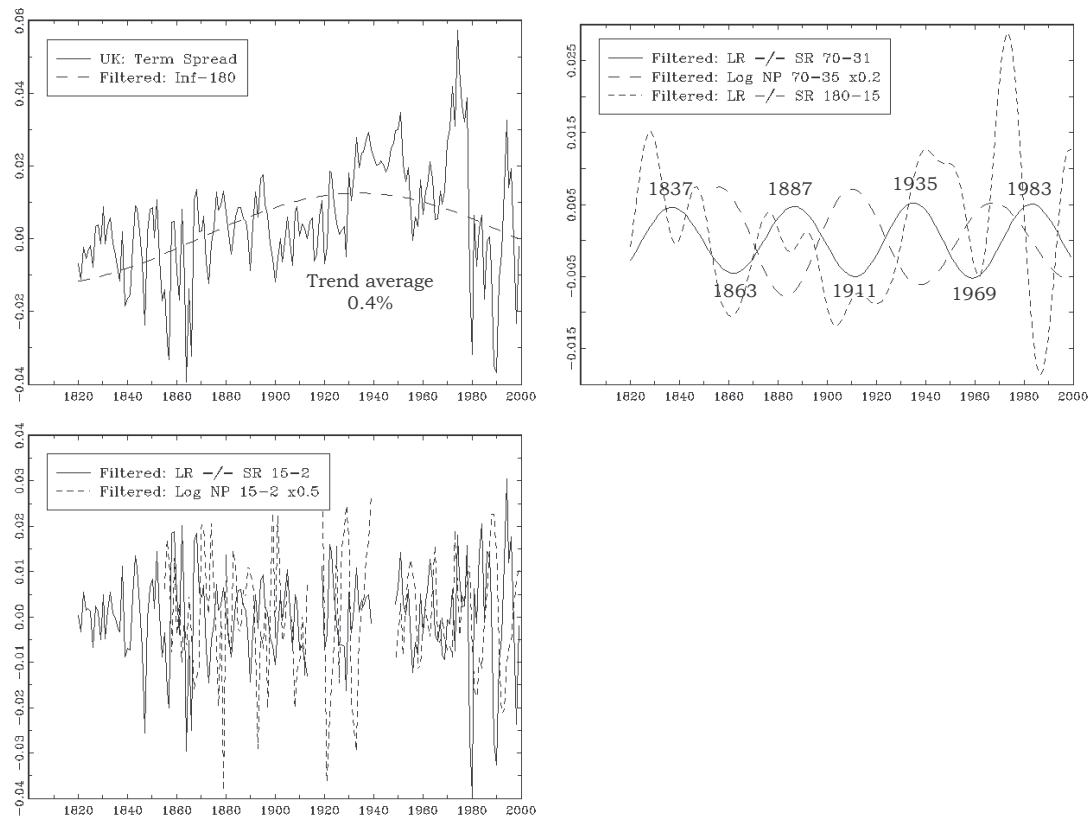
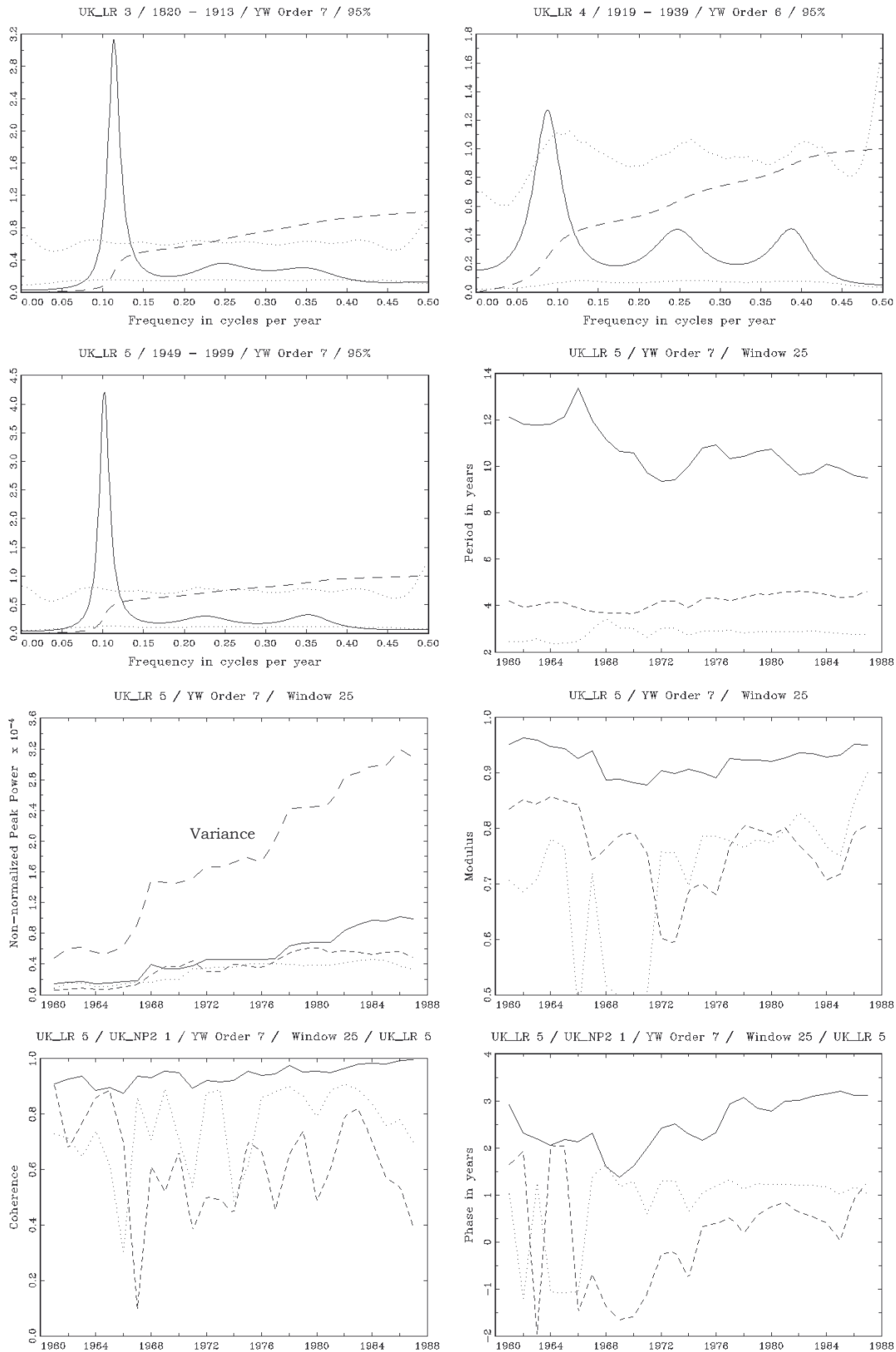


Figure D.18 Static and dynamic estimates of the spectral density of the stochastic component of a Term Spread for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## D.9 Default Spread

Figure D.19 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Default Spread for the UK. For comparison, the pictures also show the long wave and stochastic component of a UK National Product index.

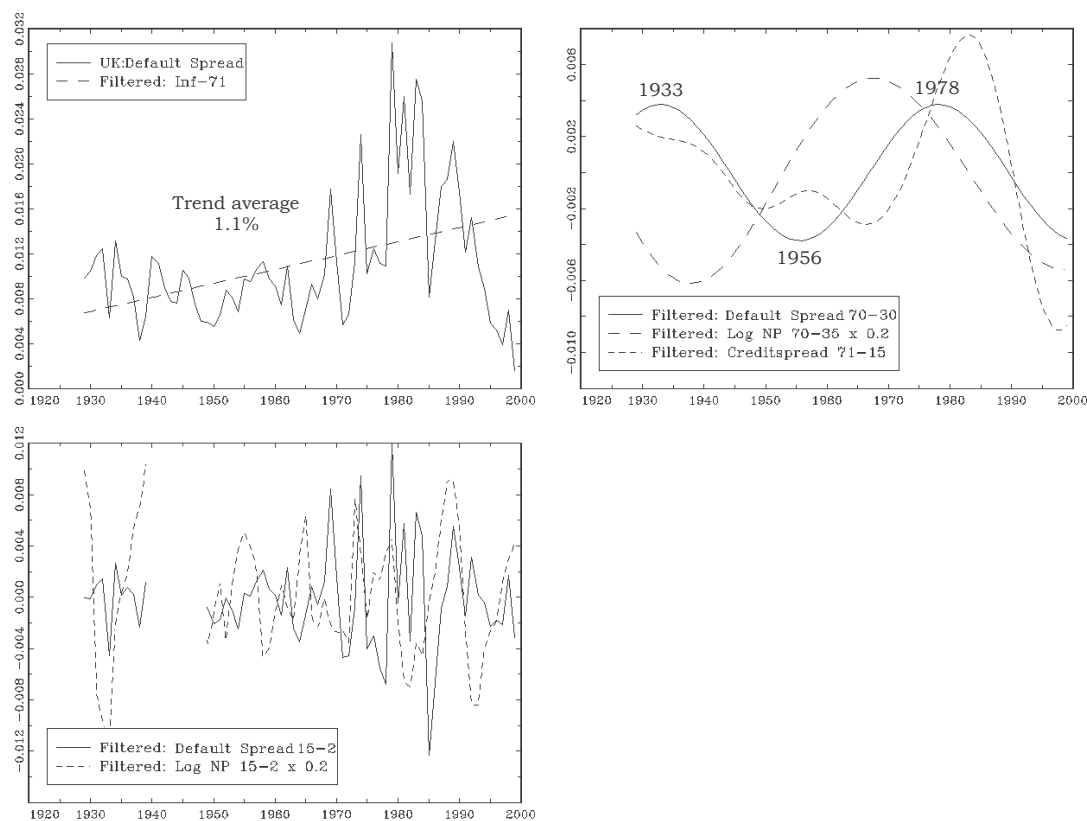


Figure D.20 For a Default Spread for the UK, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the spread and its “volatility” represented by the standard deviation of the stochastic component (left).

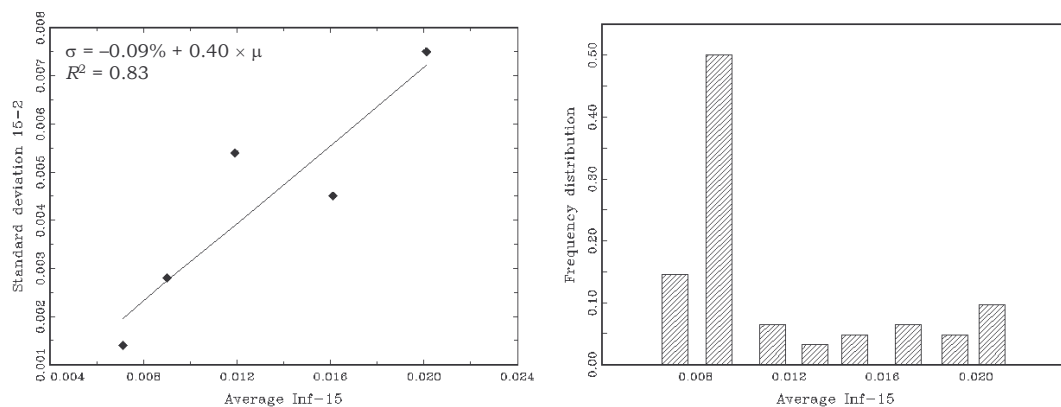
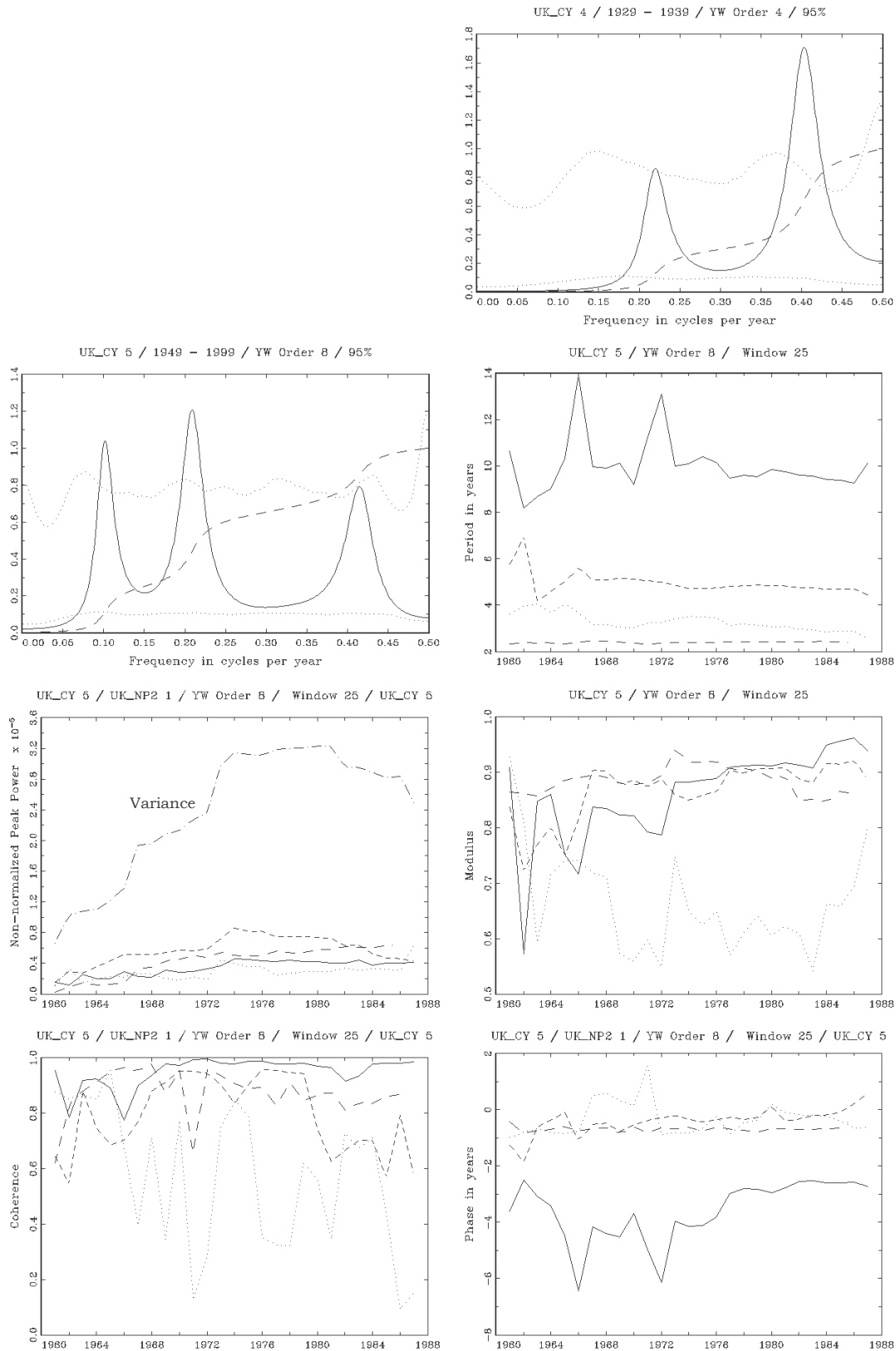


Figure D.21 Static and dynamic estimates of the spectral density of the stochastic component of a Default Spread for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.





## D.10 Equities

Figure D.22 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a real Total Return Equity index for the UK. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the UK. Also the long wave component in the equity risk premium versus long wave in National Product (bottom right).

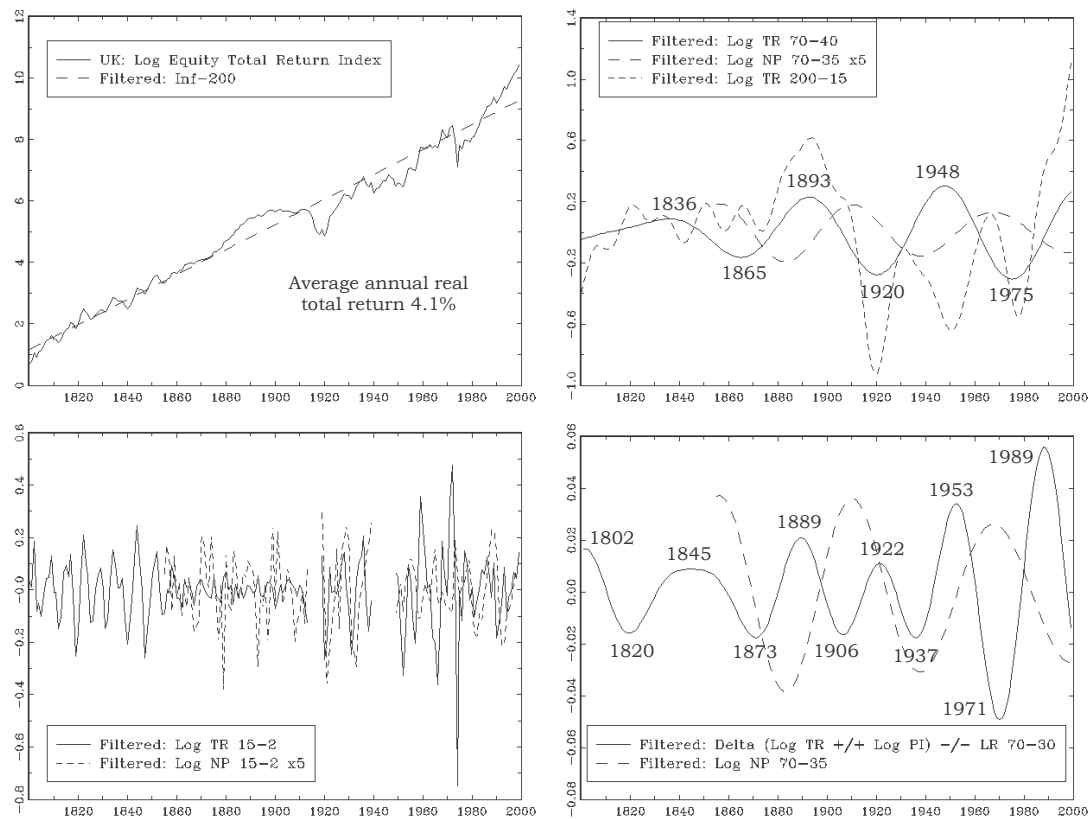
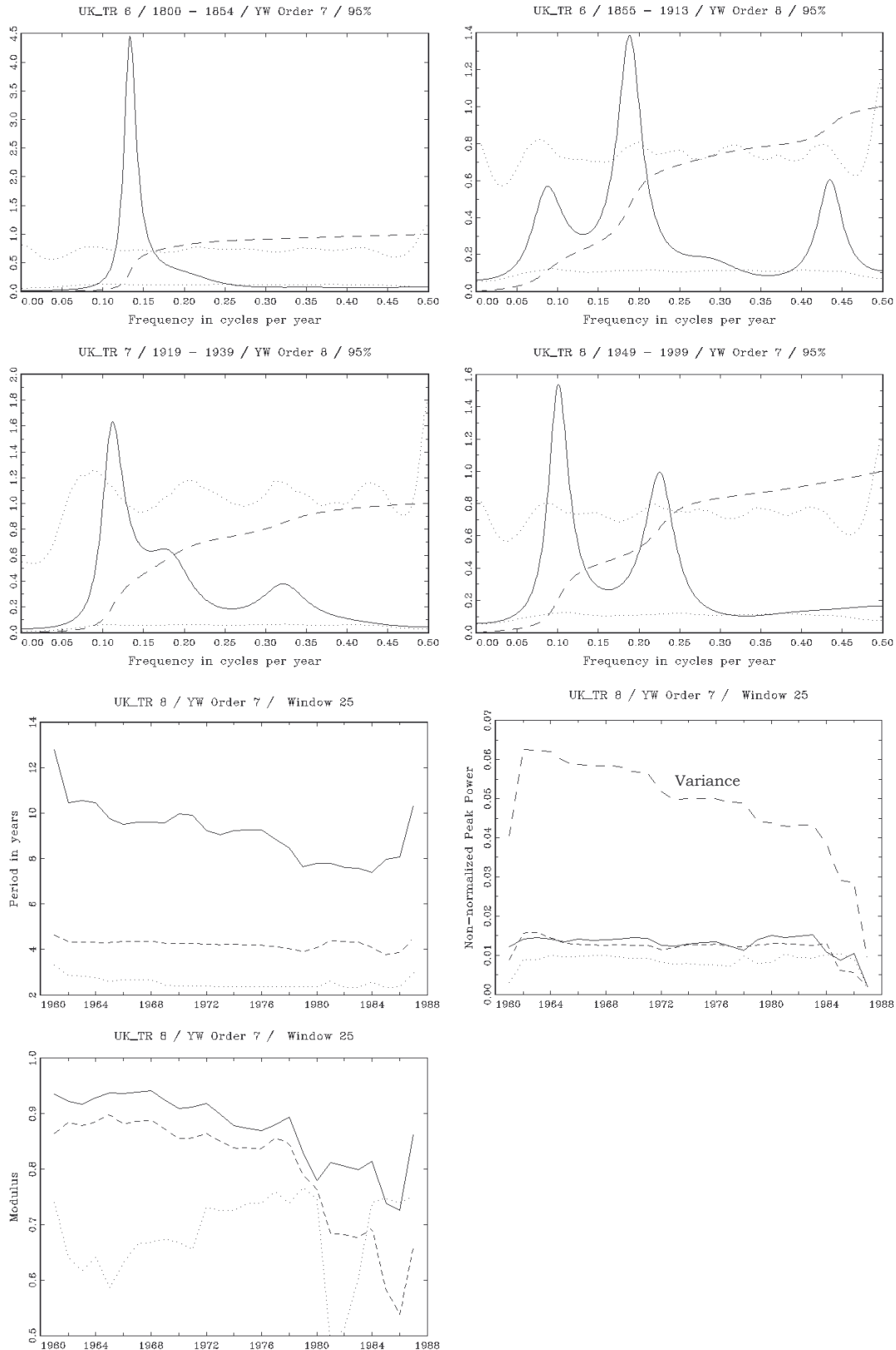
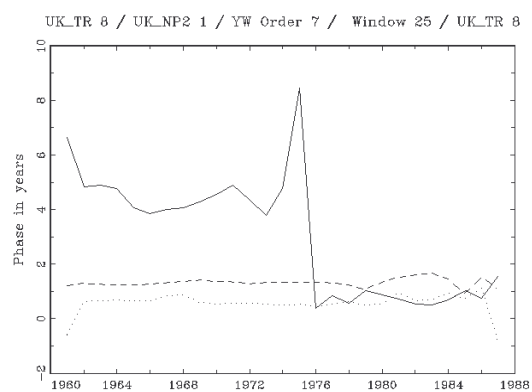
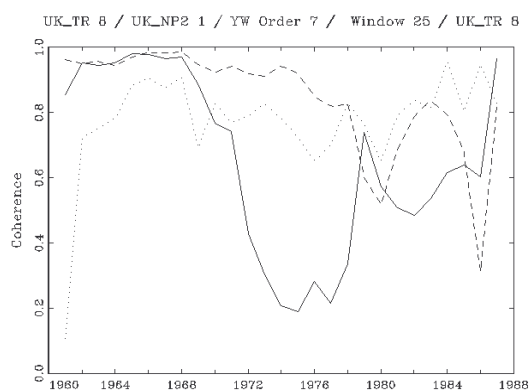


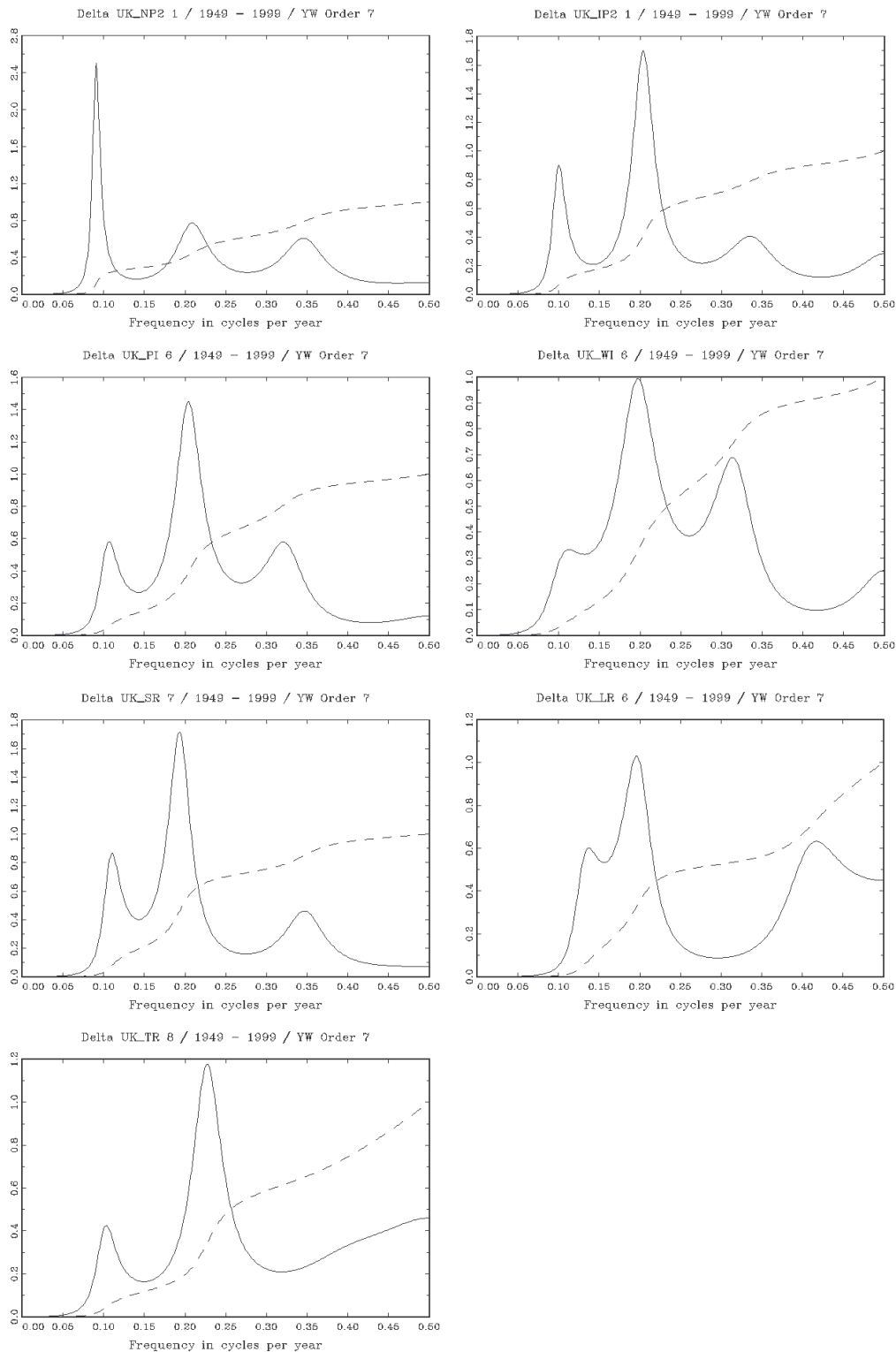
Figure D.23 Static and dynamic estimates of the spectral density of the stochastic component of a real Total Return Equity index for the UK for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.





## D.11 First order differences

Figure D.24 Normalized spectral densities of the first order differences of the stochastic components of the original (level) variables for the UK for the postwar period, possibly after a logarithmic transformation. These are obtained by multiplying the estimated (non-normalized) spectral densities of the original (level) series by the PTF (squared gain) of the first order differencing operator.



## Appendix E: Results United States

This appendix contains the results of the filtering and spectral analyses of the time series for the United States. The results are obtained by means of the specific methodology described in Chapter 6. The numerical parts of the results are contained in the tables in the chapters of Part III.

### E.1 National Product

Figure E.1 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a National Product index for the US. For comparison, the pictures also show the long wave and stochastic component of a NL National Product index.

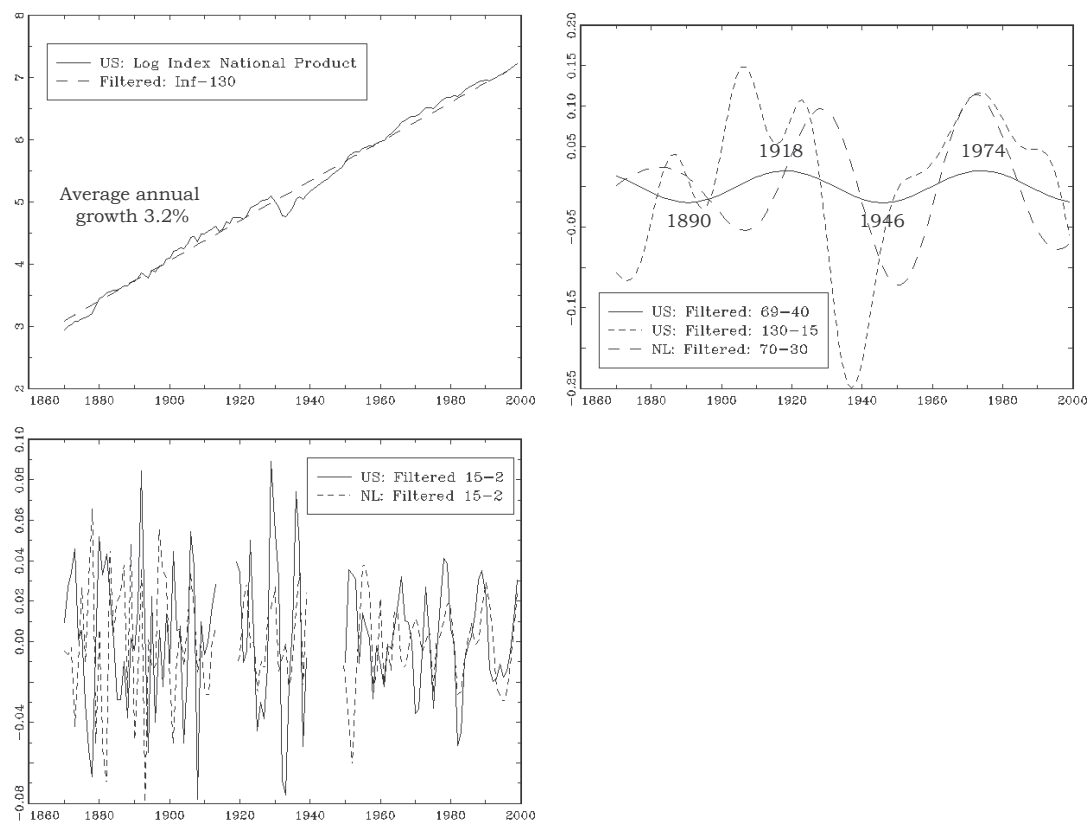
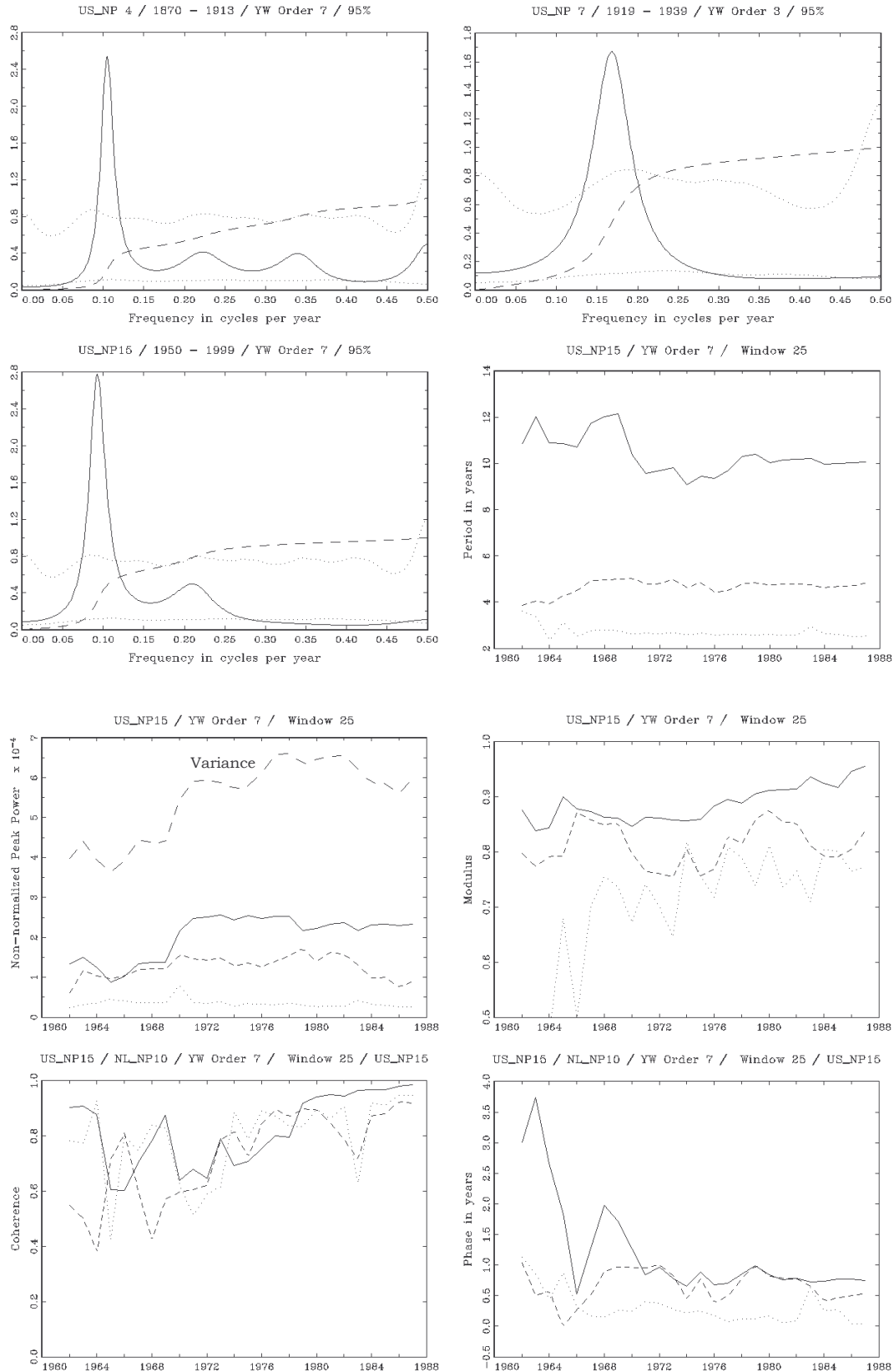


Figure E.2 Static and dynamic estimates of the spectral density of the stochastic component of a National Product index for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the NL. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.2 Industrial Production

Figure E.3 Trend (top left), periodic component (top right) and stochastic component (bottom left) of an Industrial Production index for the US. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the US.

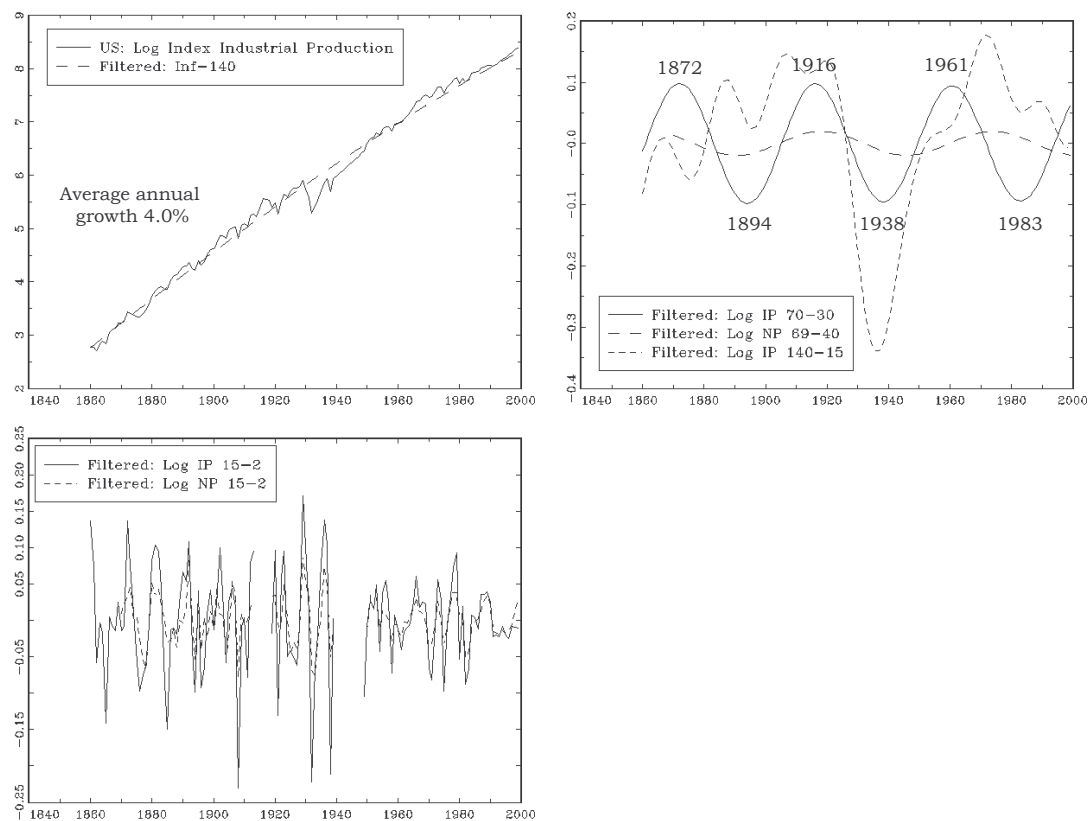
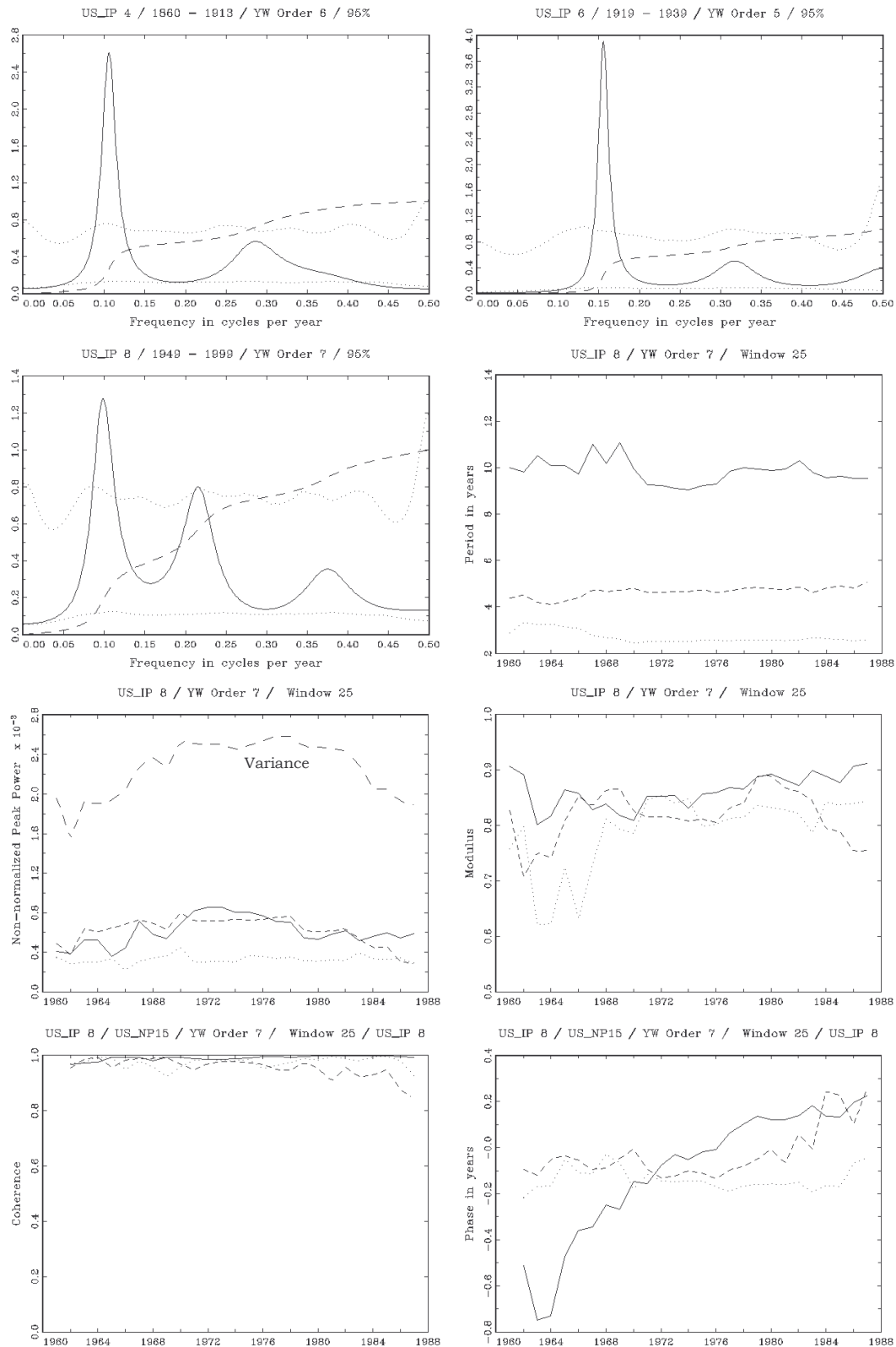




Figure E.4 Static and dynamic estimates of the spectral density of the stochastic component of a National Product index for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures.



### E.3 Employment rate

Figure E.5 Trend (top left), periodic component (top right) and stochastic component (bottom left) of the Employment rate for the US. For comparison, the pictures also show the long wave and stochastic component of a US National Product index.

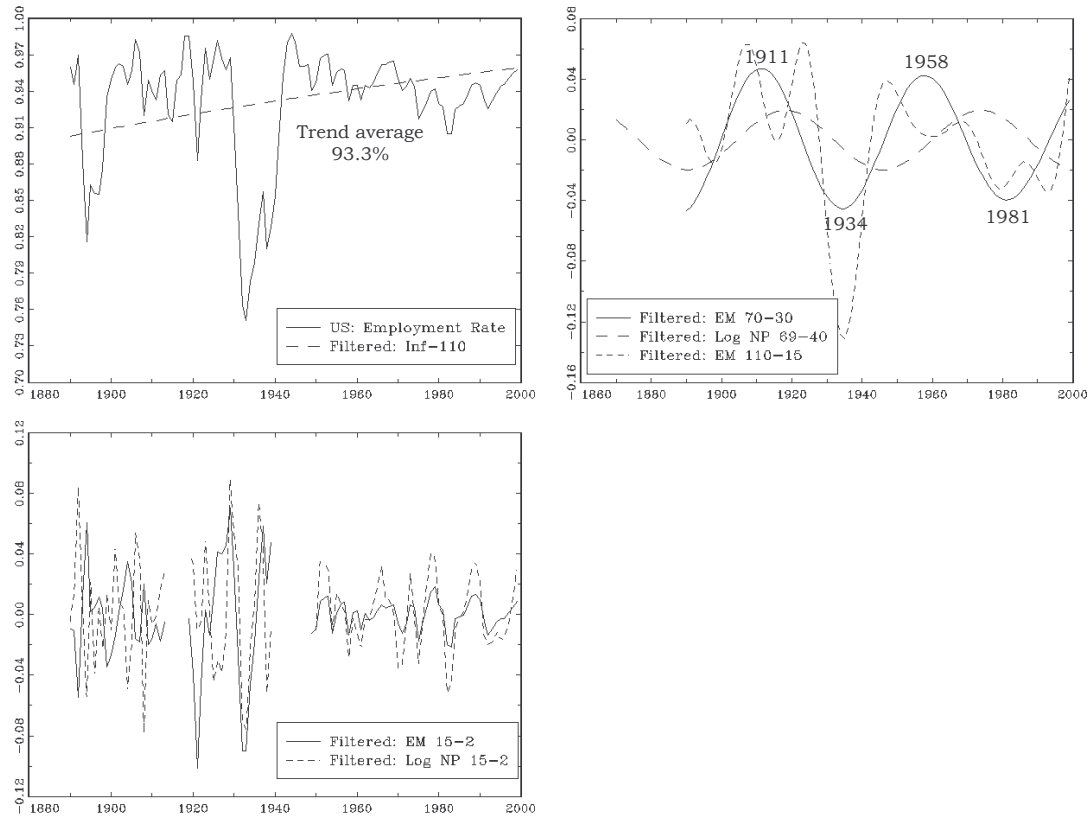
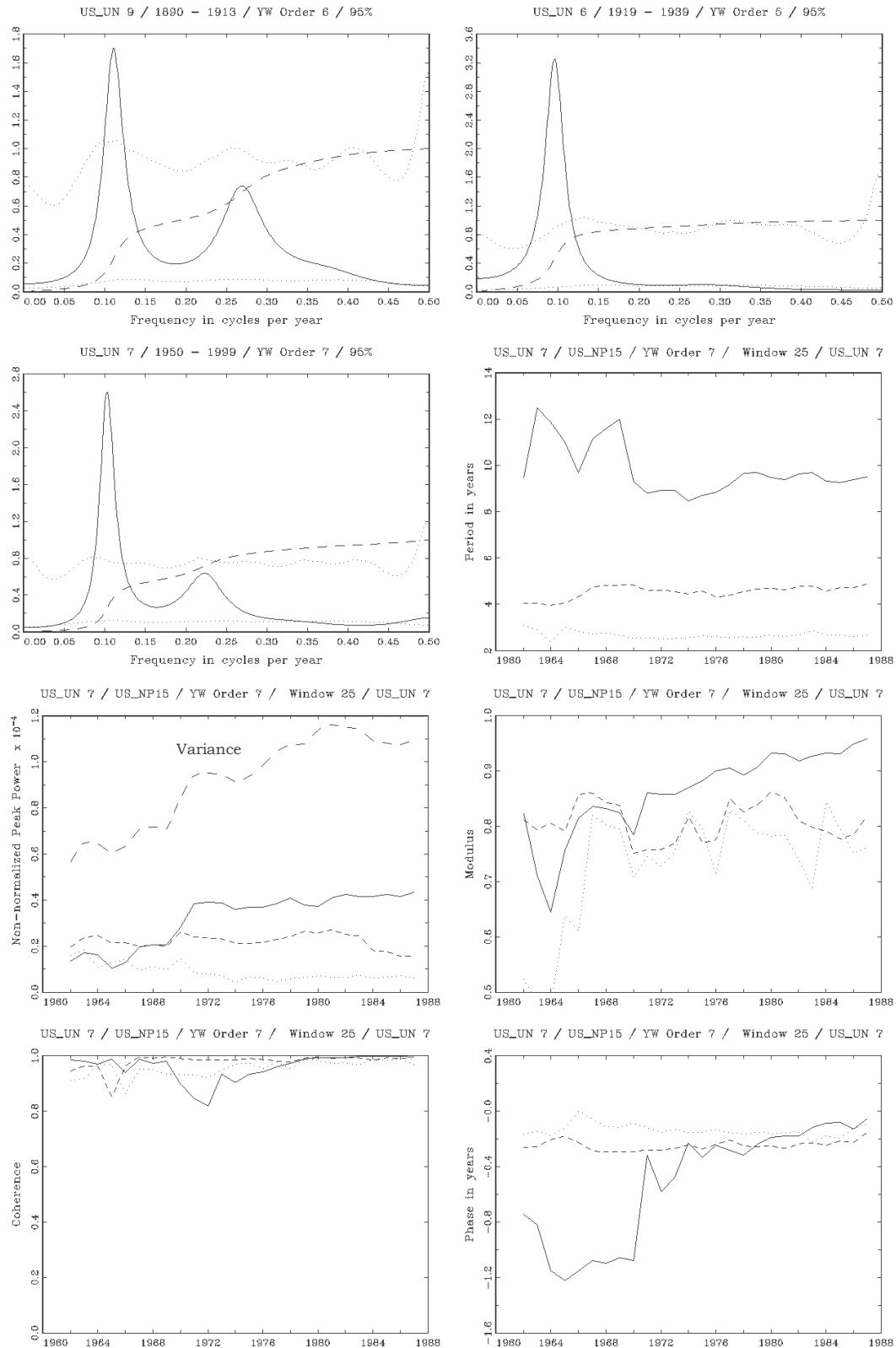


Figure E.6 Static and dynamic estimates of the spectral density of the stochastic component of the Employment rate for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.4 Consumer Prices

Figure D.7 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Consumer Price index for the US. For comparison, the pictures also show the long wave and stochastic component of a US National Product index.

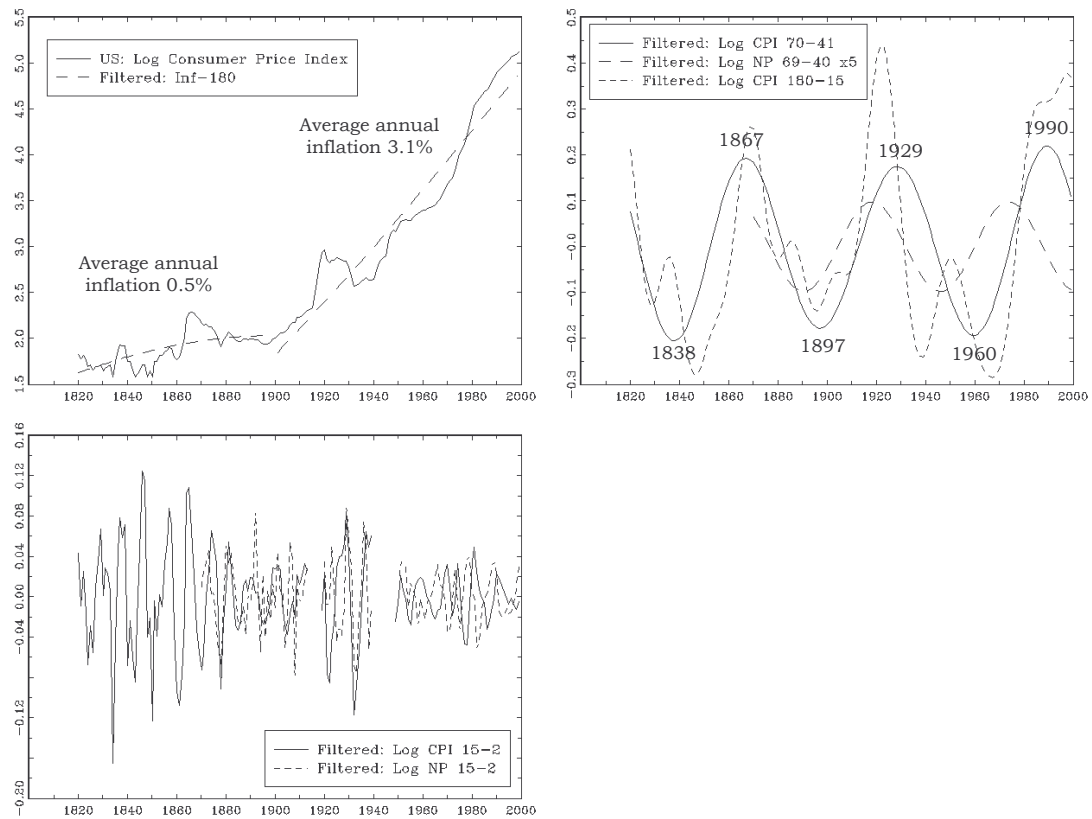
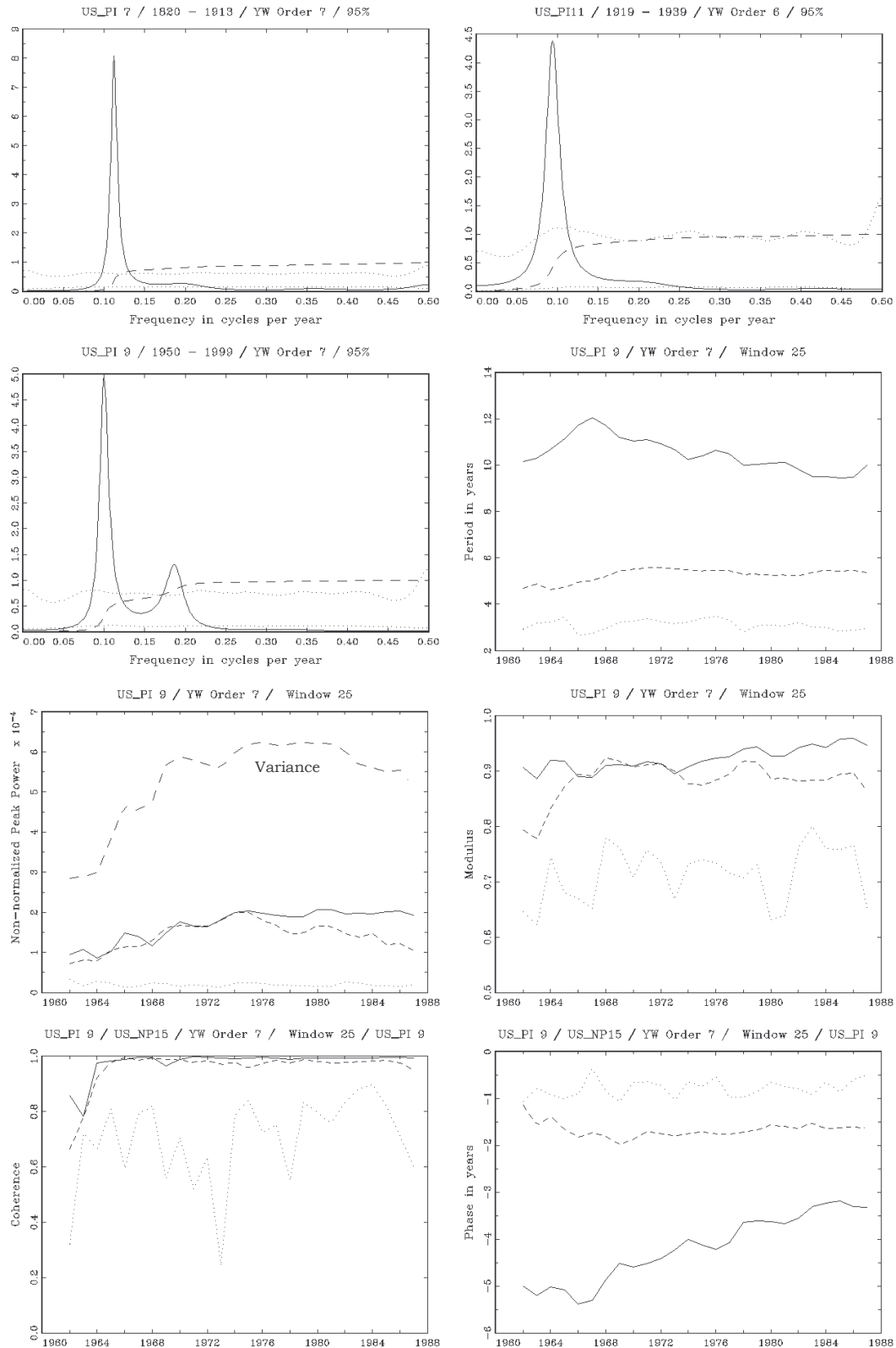


Figure E.8 Static and dynamic estimates of the spectral density of the stochastic component of a Consumer Price index for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.5 Industry Wages

Figure E.9 Trend (top left), periodic component (top right) and stochastic component (bottom left) of an Industry Wage index for the US. For comparison, the pictures also show the long wave and stochastic component of a US National Product index.

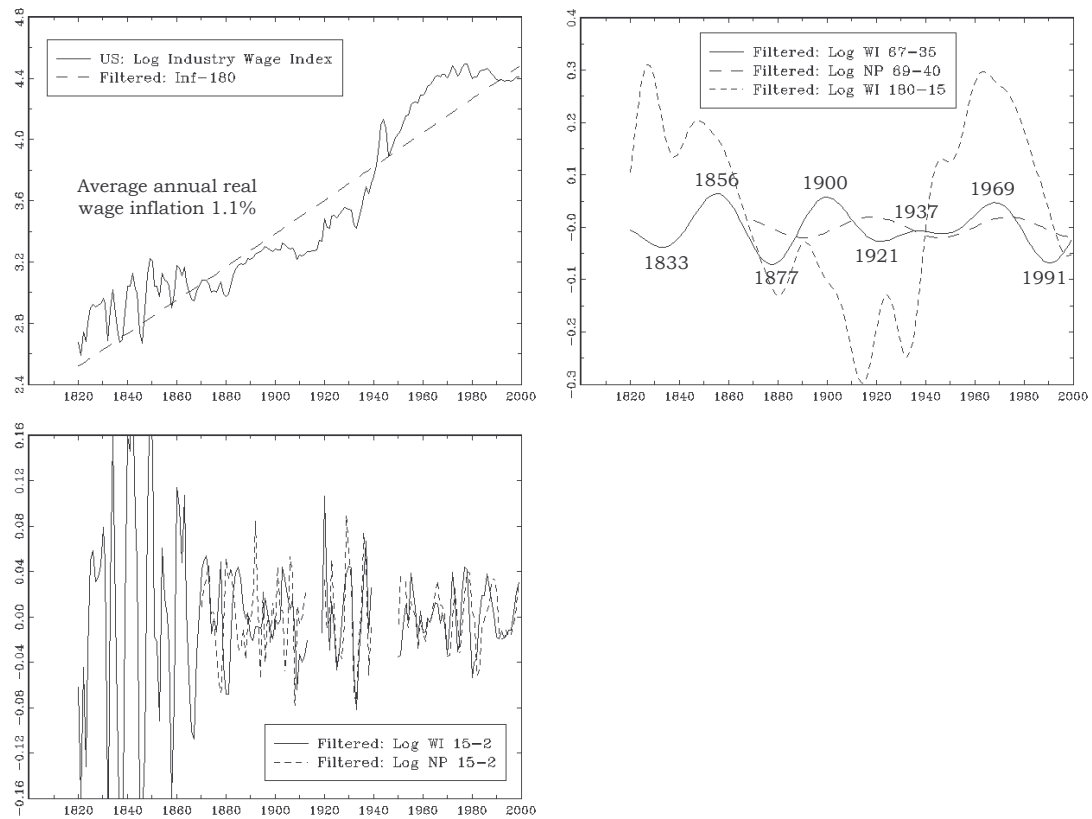
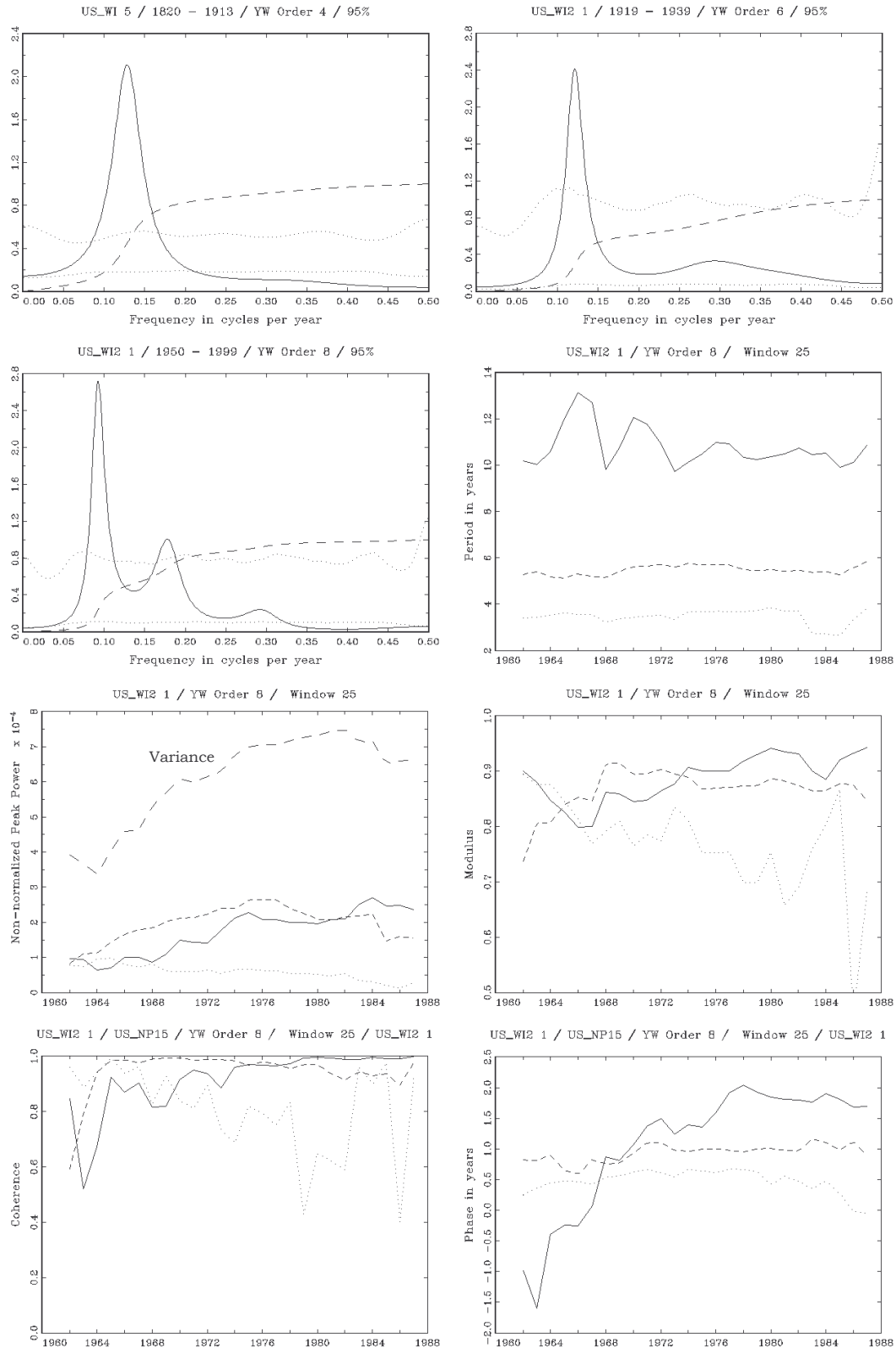


Figure E.10 Static and dynamic estimates of the spectral density of the stochastic component of a Industry Wage index for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.6 Short Interest Rate

Figure E.11 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Short Interest Rate for the US. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the US.

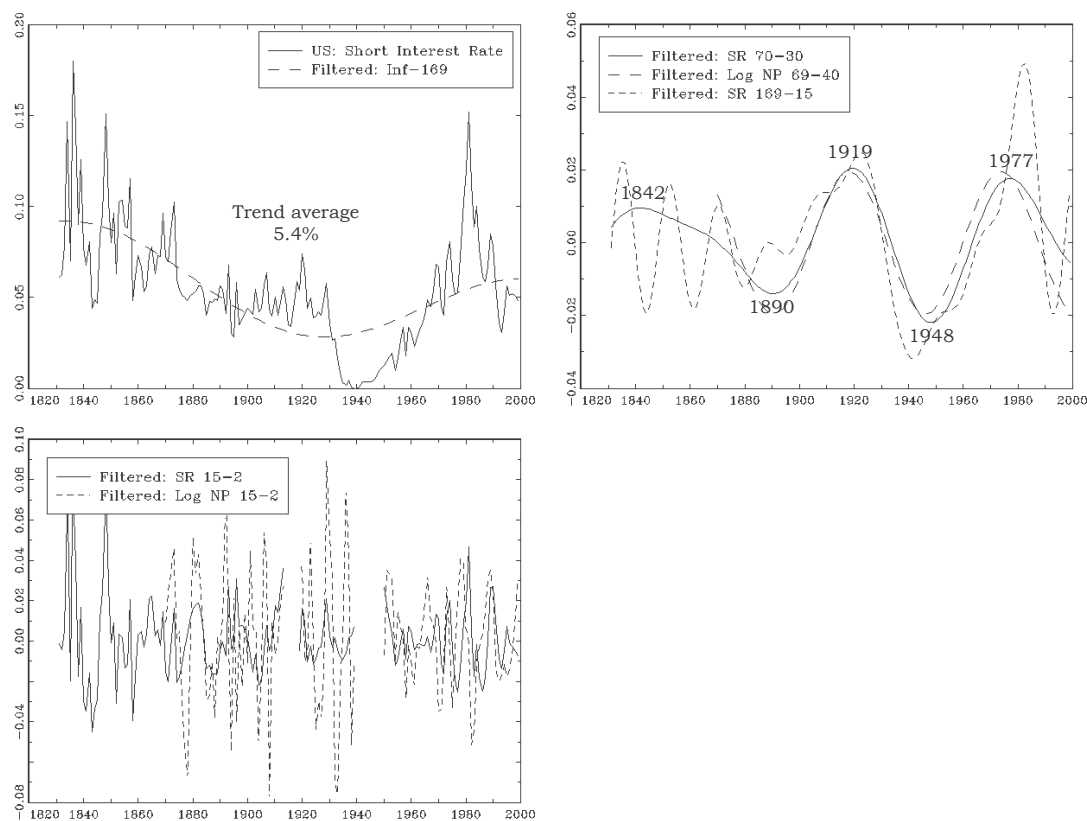


Figure E.12 For a (nominal) Short Interest Rate for the US, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).

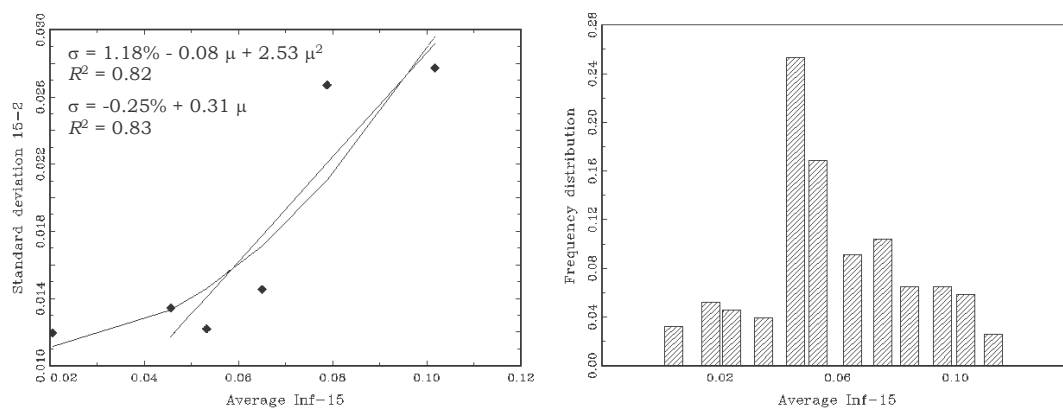
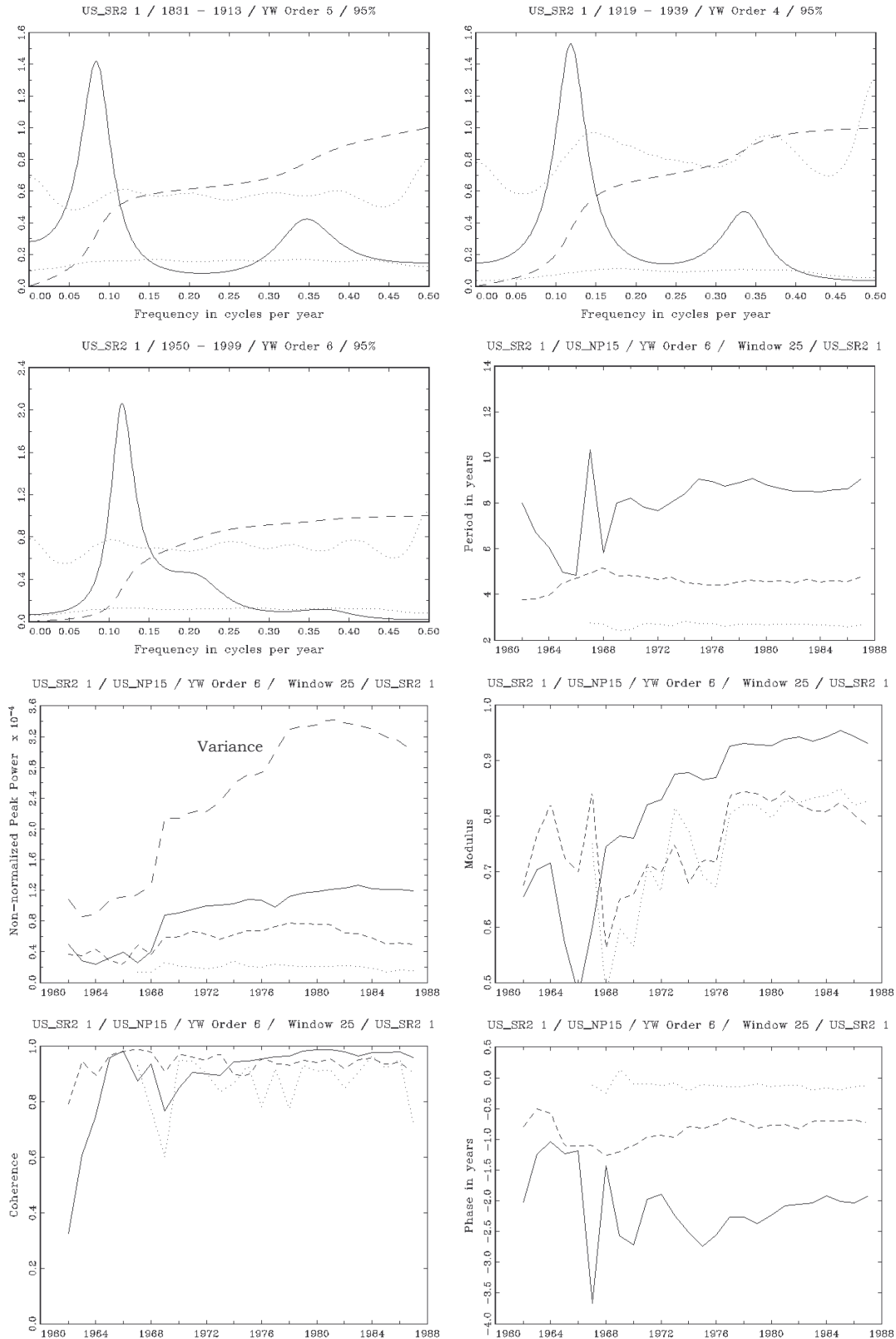




Figure D.13 Static and dynamic estimates of the spectral density of the stochastic component of a (nominal) Short Interest Rate for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.7 Long Interest Rate

Figure E.14 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a (nominal) Long Interest Rate for the US. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the US. Also the long wave component in (ex post) real long interest rate versus long wave in National Product (bottom right).

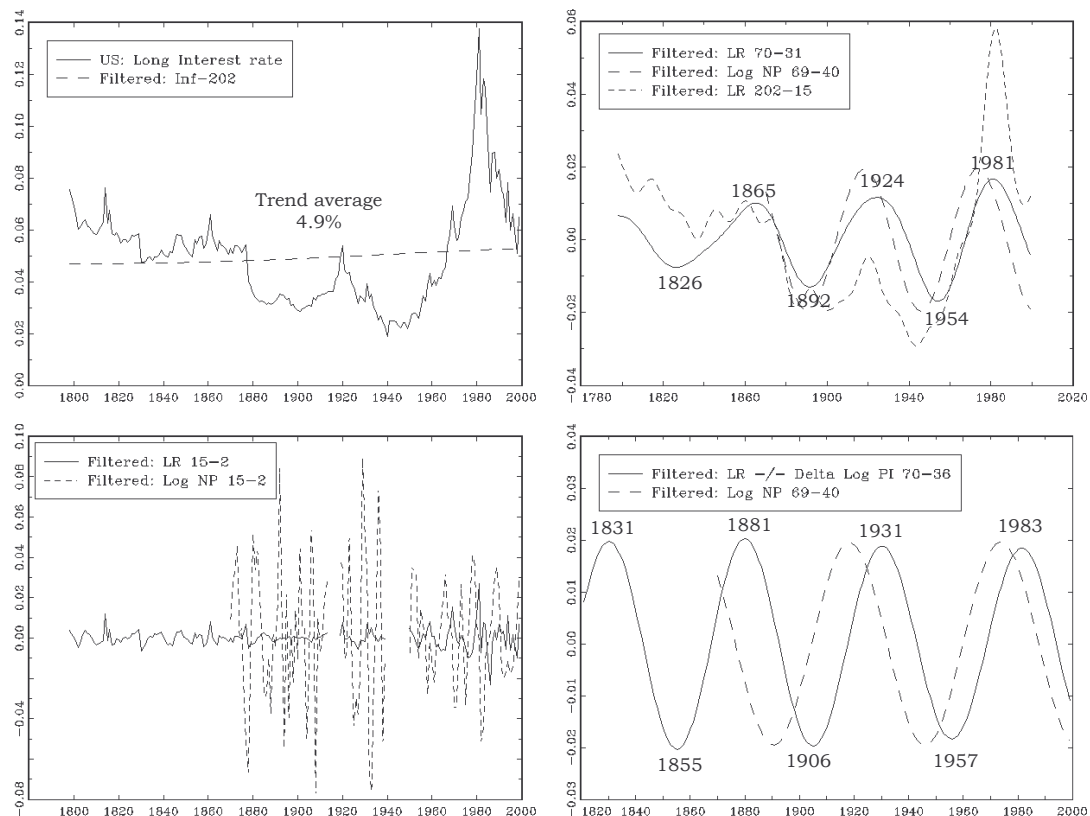


Figure E.15 For a (nominal) Long Interest Rate for the US, the frequency distribution of the sum of the filtered trend and periodic component (right) and estimated relations between this “level” of the interest rate and its “volatility” represented by the standard deviation of the stochastic component (left).

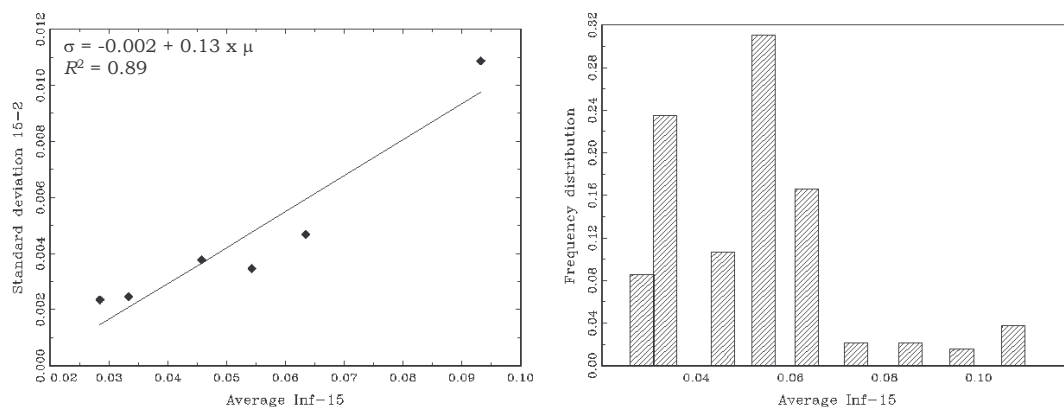
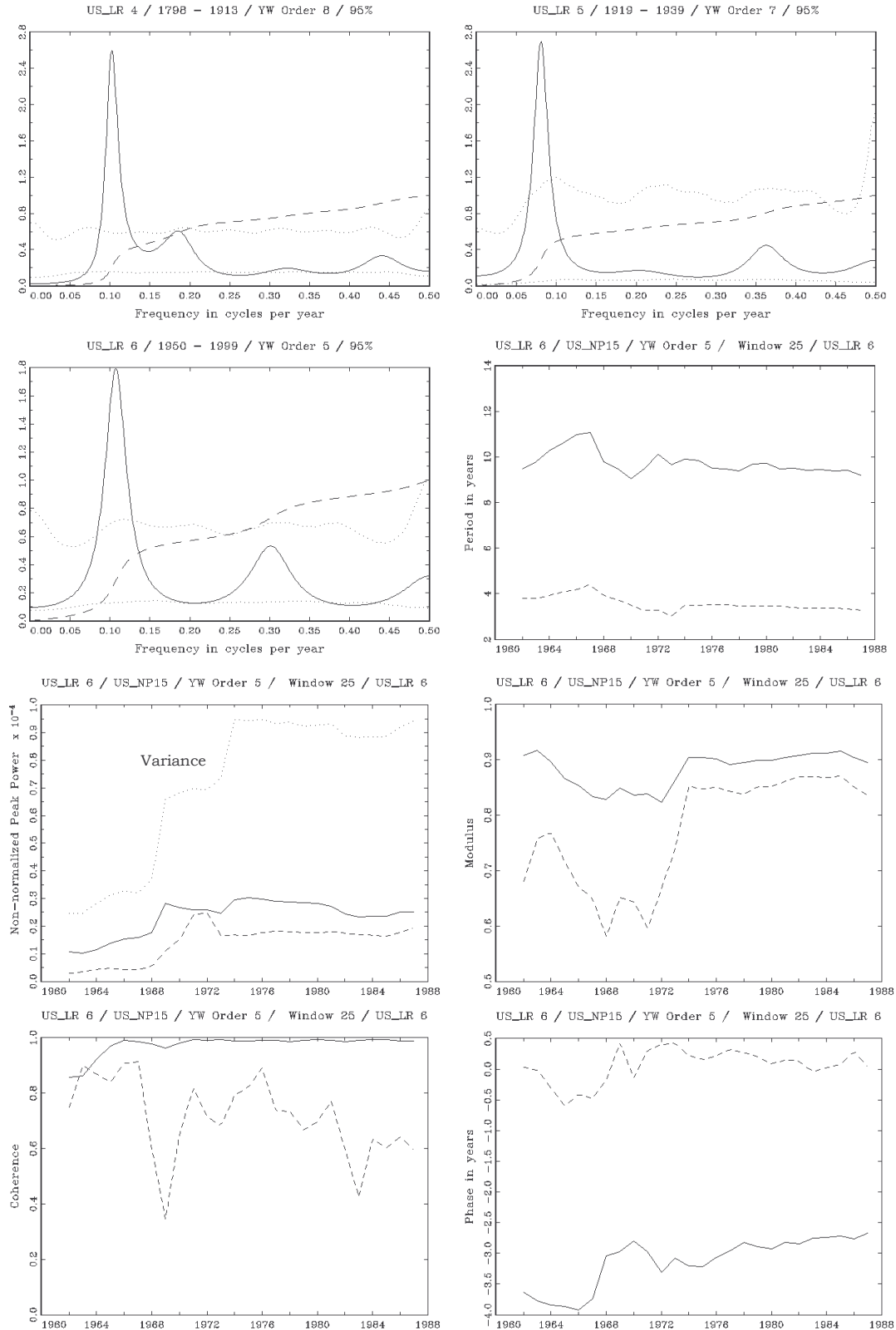


Figure E.16 Static and dynamic estimates of the spectral density of the stochastic component of (nominal) Long Interest Rate for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.8 Term Spread

Figure E.17 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a Term Spread for the US. For comparison, the pictures also show the long wave and stochastic component of a US National Product.

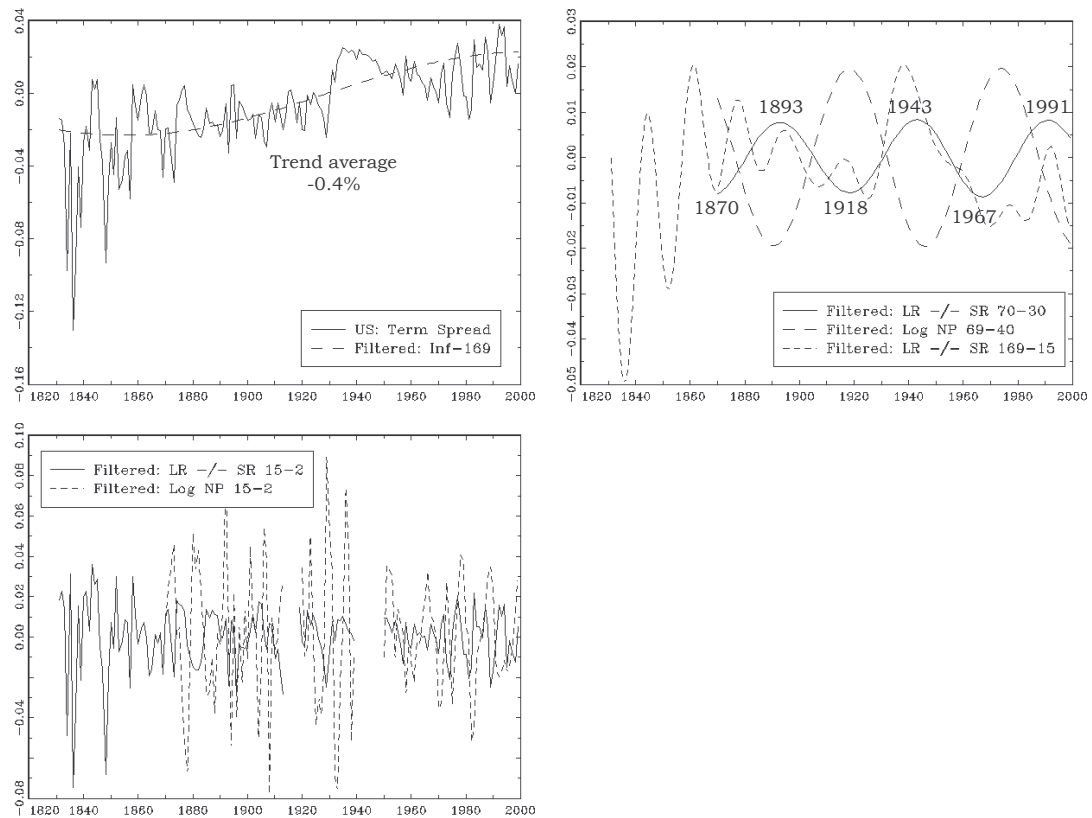
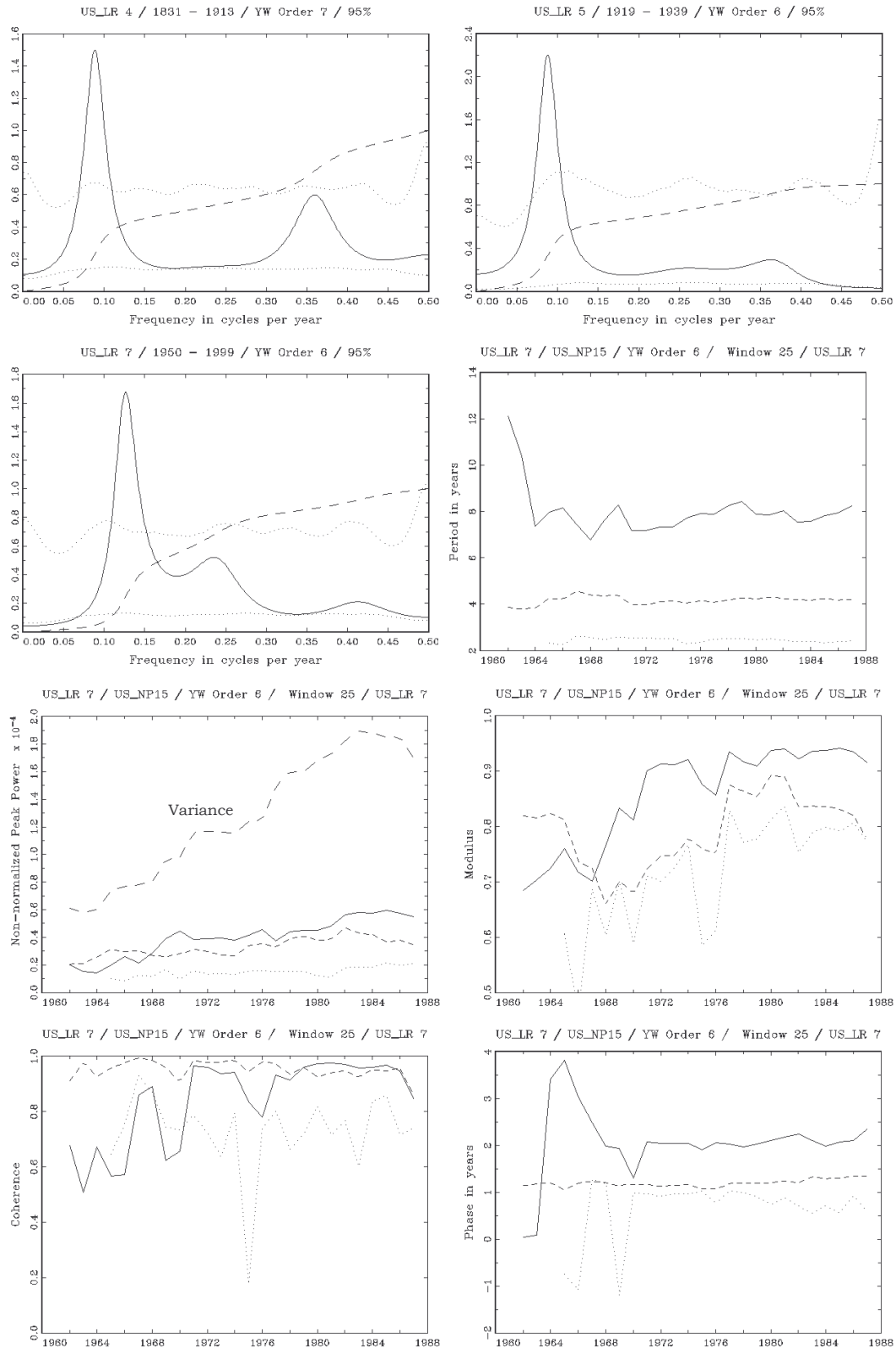


Figure E.18 Static and dynamic estimates of the spectral density of the stochastic component of a Term Spread for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.9 Equities

Figure E.19 Trend (top left), periodic component (top right) and stochastic component (bottom left) of a real Total Return Equity index for the US. For comparison, the pictures also show the long wave and stochastic component of a National Product index for the US. Also the long wave component in the equity risk premium versus long wave in National Product (bottom right).

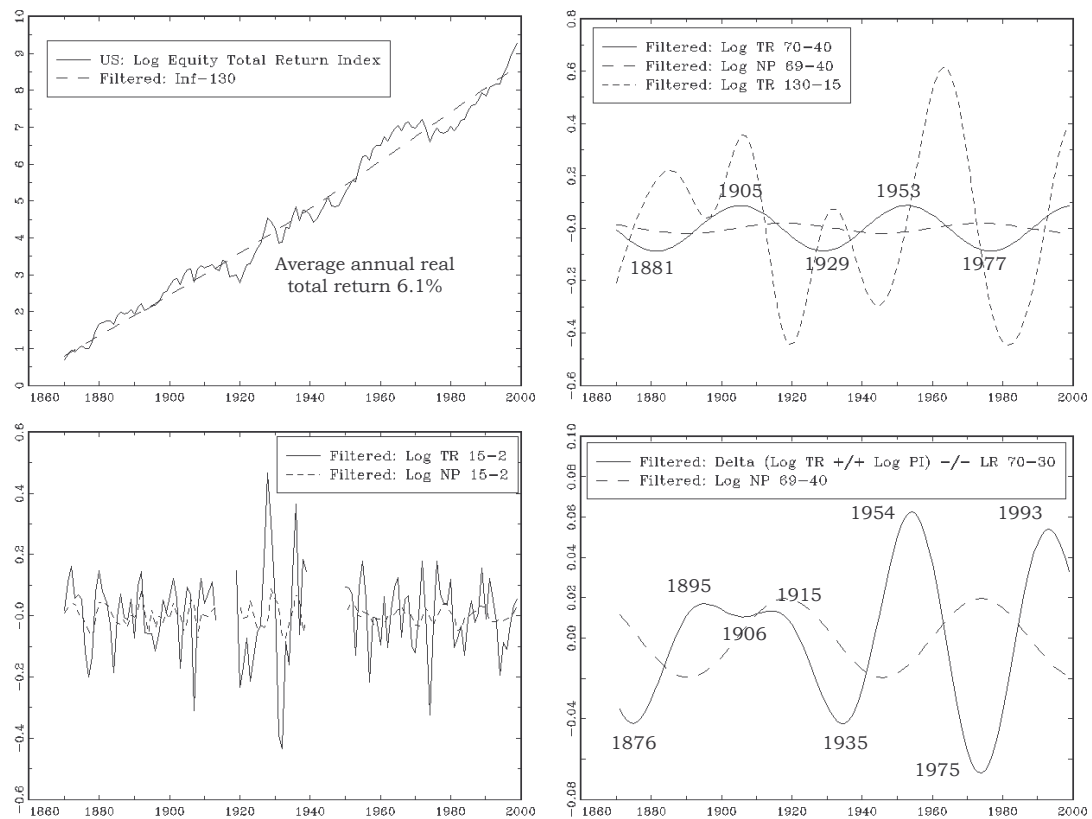
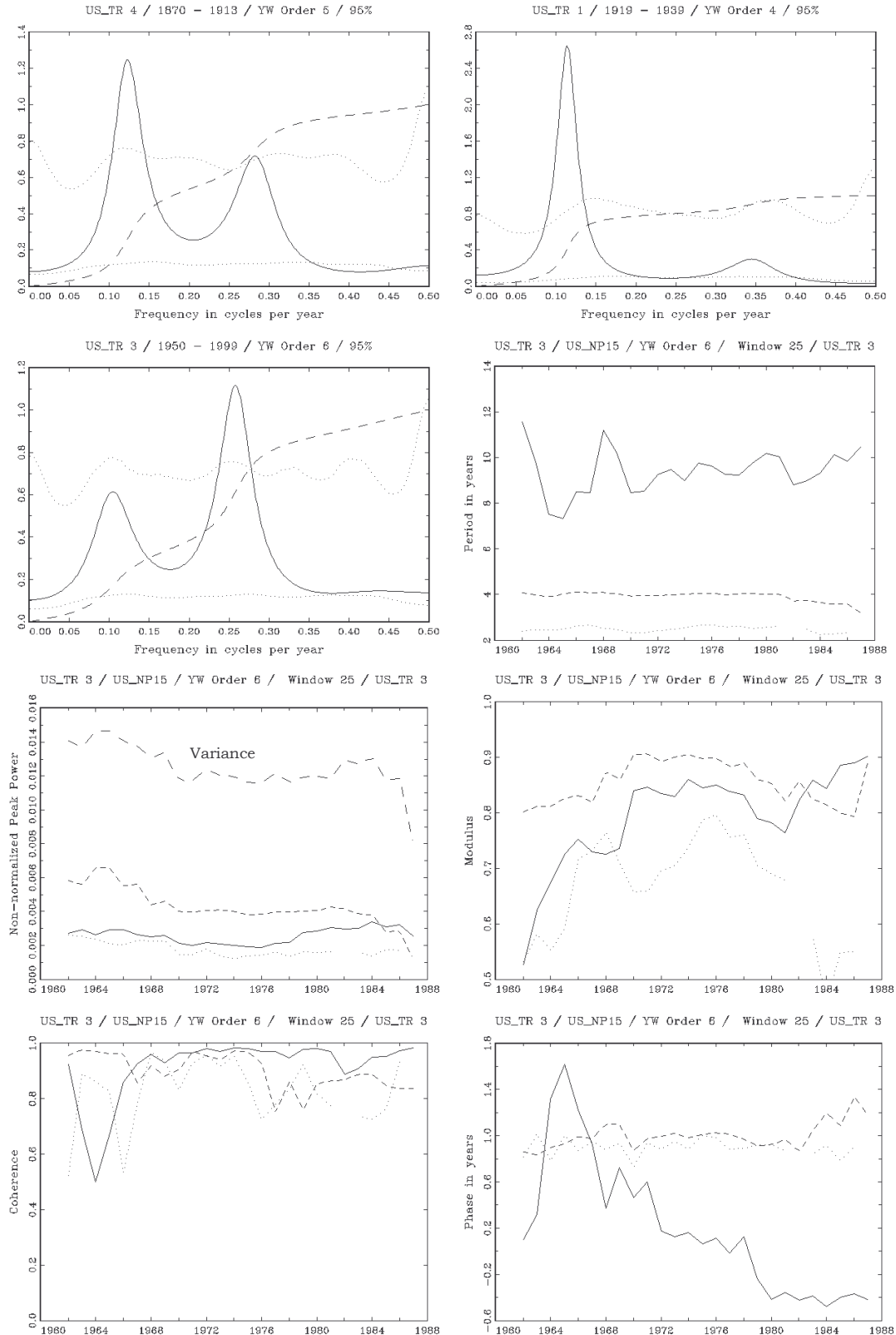
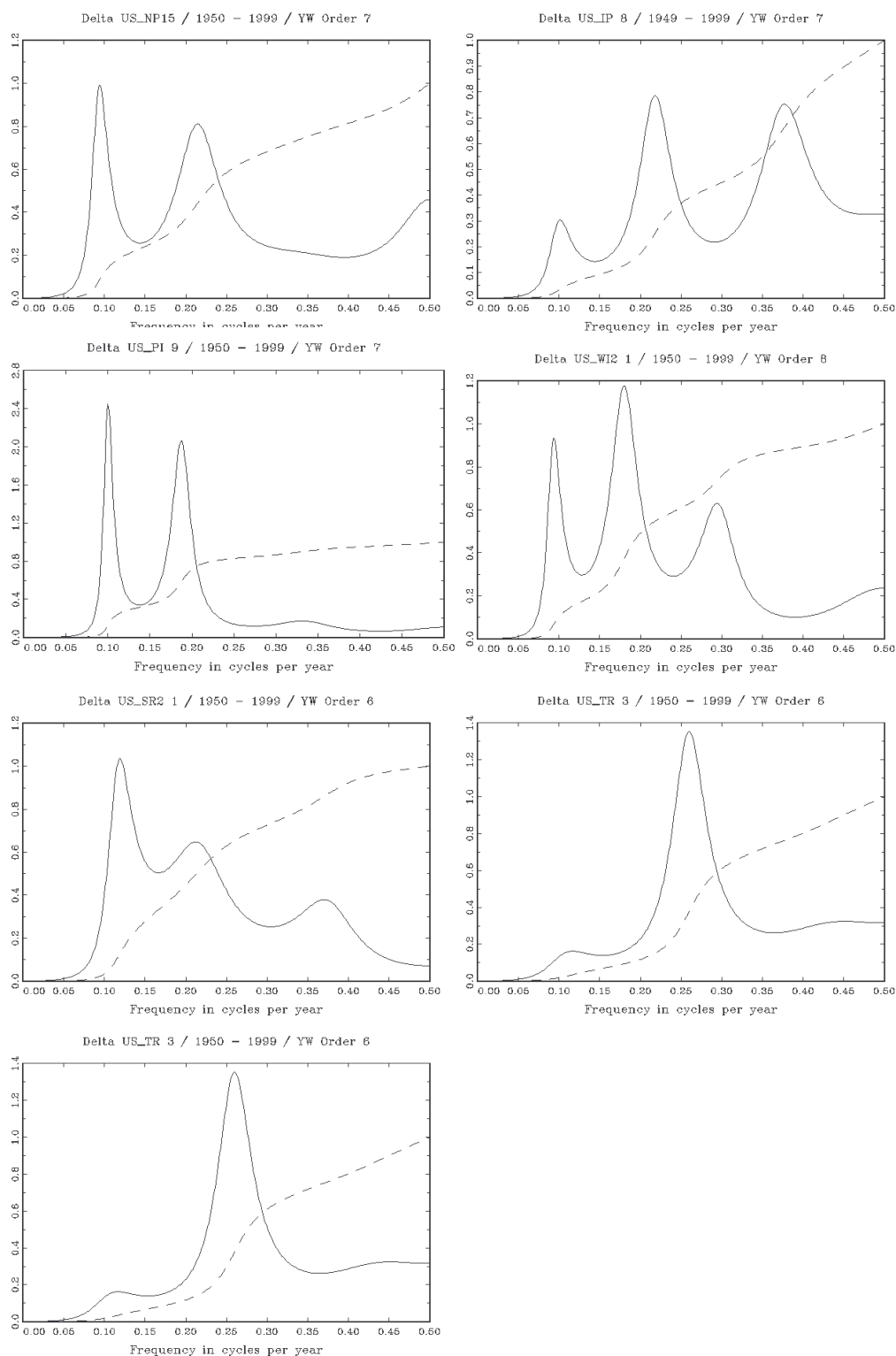


Figure E.20 Static and dynamic estimates of the spectral density of the stochastic component of a real Total Return Equity index for the US for various sub-periods. The static estimates also show the 95% white noise confidence interval. The phase is to be interpreted as the expected lead over the National Product index. See section 6.4 and the header of Figure 7.3 for more information on the pictures shown here.



## E.10 First order differences

Figure E.21 Normalized spectral densities of the first order differences of the stochastic components of the original (level) variables for the US for the postwar period, possibly after a logarithmic transformation. These are obtained by multiplying the estimated (non-normalized) spectral densities of the original (level) series by the PTF (squared gain) of the first order differencing operator.





## Appendix F: Time series and their sources

This appendix contains the time series that are used for the analyses in Part III and IV as well as a description of their sources. Please note that:

- All indices (NP, IP, PI, WI, EP and ET) in this appendix start at a reference value of 100. The NP and IP indices are volume indices and therefore already hold in real terms. The other indices hold in nominal terms in the currency of the relevant country. The real indices used in Part III are constructed by dividing these nominal indices by the Consumer Price Index (PI).
- The Real Total Return equity indices (TR) used in Chapter 16 are constructed as follows. First calculate the nominal returns from the Equity Price index (EP). Then add the Dividend Yield (DY) to obtain the nominal total rate of return. From these returns, construct a nominal total return index and finally divide this index by the Consumer Price Index (PI) to obtain a Real Total Return equity index.
- All employment rates (EM), interest rates (SR and LR) and dividend yields (DY) are denoted in percentage points.
- All population numbers (POP) are denoted in millions.

Table F.1 Historical time series for the Netherlands. Descriptions of the series and their sources are given at the end of the table.

<i>Year</i>	<i>1</i> <i>NP</i>	<i>2</i> <i>IP</i>	<i>3</i> <i>EM</i>	<i>4</i> <i>PI</i>	<i>5</i> <i>WI</i>	<i>6</i> <i>SR</i>	<i>7</i> <i>LR</i>	<i>8</i> <i>EP</i>	<i>9</i> <i>DY</i>	<i>10</i> <i>POP</i>
1800										
1801										
1802										
1803										
1804										
1805										
1806										
1807										
1808										
1809										
1810										
1811										
1812										
1813				100.0						
1814				88.7			7.43			
1815				90.1			6.83			
1816				107.7			5.92	100.0		
1817				113.4			5.80	112.7		
1818				102.5			5.90	126.3		
1819				93.1			5.78	108.2		
1820				79.3			5.63	101.6		
1821				68.7			5.45	98.8		
1822				62.1			5.34	97.3		
1823				62.5			5.30	100.7		
1824				60.6			4.76	132.5	3.2	
1825				64.3			4.66	127.6	3.2	

1826		66.1		5.00	120.6	3.2	
1827		75.8		4.85	117.4	3.2	
1828		74.4	2.90	4.59	119.9	3.2	
1829		75.9	3.02	4.19	117.1	3.2	
1830		80.3	4.48	6.73	107.2	3.2	
1831		89.4	5.05	6.69	102.6	3.2	
1832		85.0	3.83	6.24	113.5	3.2	
1833		80.7	2.75	5.56	120.4	3.2	
1834		74.0	2.73	4.84	127.3	3.2	
1835		66.3	3.60	4.60	134.7	6.4	
1836		69.0	4.17	4.84	163.7	8.1	
1837		67.4	3.75	4.81	175.8	8.8	
1838		79.6	3.21	4.67	187.6	9.2	
1839		82.6	4.56	4.87	206.3	9.2	2.64
1840		83.7	3.04	5.21	198.5	5.2	2.88
1841		86.9	3.21	4.94	177.3	3.4	2.91
1842		86.5	3.90	4.84	146.3	2.4	2.94
1843		82.5	3.13	4.66	130.4	2.4	2.97
1844		76.4	3.88	4.22	172.2	3.8	3.00
1845		84.6	4.38	4.16	232.4	5.0	3.04
1846		95.2	4.67	4.15	210.4	4.8	3.06
1847		114.6	4.27	4.40	200.0	3.4	3.05
1848		77.5	3.56	5.60	147.0	3.0	3.05
1849		75.5	2.67	4.88	137.7	2.0	3.06
1850		64.9	2.63	4.46	132.5	2.6	3.00
1851		75.2	2.54	4.33	126.4	3.2	3.05
1852		74.8	2.50	4.02	144.4	3.7	3.14
1853		74.4	2.79	3.95	162.2	3.3	3.15
1854		90.8	2.60	4.29	140.1	3.2	3.18
1855		93.7	3.38	4.00	177.6	3.5	3.21
1856		88.6	4.63	3.95	184.6	4.6	3.23
1857		74.2	4.60	3.95	158.9	4.3	3.27
1858		72.6	3.19	3.89	173.1	3.9	3.29
1859		72.0	3.21	3.98	157.1	5.0	3.31
1860		77.3	2.67	3.93	165.4	4.9	3.32
1861		79.5	2.81	3.94	190.0	4.9	3.35
1862		80.6	3.52	3.93	207.3	6.4	3.39
1863		71.5	4.17	3.93	247.0	6.7	3.43
1864		65.6	5.35	4.03	237.0	7.2	3.47
1865		65.6	4.96	4.05	265.7	7.3	3.51
1866		75.3	5.05	4.39	253.8	5.0	3.54
1867		77.1	3.05	4.64	217.8	4.6	3.57
1868		79.6	2.66	4.52	219.6	4.8	3.61
1869		74.4	3.50	4.54	224.9	6.6	3.61
1870	100.0	68.5	4.30	4.67	240.5	6.4	3.60
1871	101.5	74.2	3.28	4.69	260.3	7.4	3.63
1872	103.6	76.4	3.22	4.48	276.4	7.6	3.66
1873	101.3	77.4	4.69	4.40	272.3	6.1	3.69
1874	106.6	82.5	3.50	4.16	272.7	4.2	3.74
1875	113.3	73.4	3.19	3.99	259.8	6.0	3.79
1876	112.2	75.9	2.75	3.98	271.7	5.8	3.84
1877	121.2	82.4	2.75	3.91	271.4	5.9	3.89

1878	129.8		82.2		3.31	3.95	273.3	6.0	3.95	
1879	120.2		74.5		3.00	3.88	279.3	6.2	4.01	
1880	132.1		77.8		2.50	3.83	291.1	6.6	4.05	
1881	129.1		80.6		3.07	3.77	316.4	6.5	4.09	
1882	131.1		78.0		4.03	3.72	312.4	6.4	4.14	
1883	151.3		78.5		3.81	3.81	312.4	6.8	4.20	
1884	148.2		75.3		2.80	3.75	304.5	5.2	4.25	
1885	152.8		66.4		2.37	3.70	285.1	5.4	4.31	
1886	154.8		64.9		1.92	3.39	283.6	4.7	4.36	
1887	158.2		64.9		2.13	3.40	284.6	6.0	4.42	
1888	149.2		62.7		2.13	3.29	293.2	6.0	4.48	
1889	161.7		64.7		2.18	3.15	314.3	5.2	4.53	
1890	148.2		66.0		2.59	3.09	311.9	8.4	4.54	
1891	155.6		72.2		2.75	3.17	317.1	4.9	4.59	
1892	166.6		76.3		2.00	3.13	216.1	4.4	4.65	
1893	151.8		66.3		2.88	3.00	222.8	6.7	4.70	
1894	168.9		64.3		1.75	2.78	246.4	4.3	4.76	
1895	171.2		66.9		1.54	2.65	290.9	5.1	4.83	
1896	176.0		65.6		2.44	2.74	366.4	6.5	4.89	
1897	193.4		68.5		2.32	2.83	335.6	6.4	4.97	
1898	193.9		69.7		2.49	2.90	296.7	7.4	5.04	
1899	196.9		69.7		3.24	2.99	297.8	8.2	5.11	
1900	190.3		68.9		3.44	3.17	297.0	8.1	5.16	
1901	186.0		71.3		3.00	3.00	298.8	7.6	5.22	
1902	195.4		74.4		2.47	3.03	307.2	8.0	5.30	
1903	200.8		75.1		3.19	3.12	311.8	7.8	5.39	
1904	199.0		79.3		2.77	3.16	311.4	8.6	5.47	
1905	205.6		82.6		2.39	3.17	322.3	7.8	5.55	
1906	215.6		83.2		3.77	3.20	351.4	8.4	5.63	
1907	216.3		82.6		4.86	3.32	334.9	7.7	5.71	
1908	216.1		84.8		3.01	3.30	273.1	8.2	5.79	
1909	225.8		85.7		2.07	3.21	320.6	8.1	5.84	
1910	227.8		82.6		3.64	3.34	335.7	8.9	5.90	
1911	234.4	99.2	86.8		3.17	3.52	336.5	9.5	5.98	
1912	247.4	98.6	91.9		3.91	3.68	352.7	7.8	6.07	
1913	255.1	98.3	92.7		3.53	3.79	381.1	10.9	6.16	
1914	248.2	95.3	98.6		3.14	3.79	361.9	11.8	6.28	
1915	256.6	95.9	110.2		2.76	4.05	354.9	11.4	6.39	
1916	263.5	98.3	105.0		1.84	4.09	401.3	13.9	6.52	
1917	246.7	97.8	123.1		2.68	4.13	402.1	14.1	6.65	
1918	231.4	97.5	148.8		2.99	4.42	419.4	14.4	6.75	
1919	286.7	97.4	160.5		3.87	4.85	518.2	18.4	6.80	
1920	295.4	98.0	179.1		3.85	5.73	528.1	13.0	6.85	
1921	313.5	100.0	96.9	162.1	4.26	5.07	349.9	7.6	6.92	
1922	330.6	97.6	95.9	144.6	3.87	4.86	246.4	6.5	7.03	
1923	338.8	100.0	95.5	136.2	3.38	4.63	244.2	6.5	7.15	
1924	363.5	112.0	96.3	137.1	4.00	4.70	235.4	7.6	7.26	
1925	378.8	127.7	96.3	137.7	3.08	4.18	253.0	8.8	7.36	
1926	409.2	131.3	96.8	129.8	100.0	2.88	4.00	268.5	8.2	7.47
1927	426.3	145.8	96.8	129.6	100.0	3.76	4.01	292.7	8.1	7.58
1928	449.0	166.3	97.2	129.3	100.0	4.18	3.98	303.7	7.9	7.68
1929	452.6	169.9	97.3	130.0	100.0	4.82	3.94	301.4	7.7	7.78
1930	451.5	173.5	96.7	124.0	100.0	2.06	3.81	220.0	7.3	7.88

1931	424.0	163.9	93.9	120.1	100.0	1.42	3.86	154.0	7.3	8.00
1932	418.1	144.6	89.8	116.4	93.0	0.85	3.89	101.2	7.2	8.12
1933	417.3	154.2	88.7	112.5	93.0	1.08	3.68	114.4	6.7	8.24
1934	409.7	157.8	87.9	108.7	89.0	0.87	3.35	121.0	6.0	8.34
1935	425.0	154.2	86.5	112.4	25.0	3.15	3.43	121.0	6.0	8.43
1936	451.8	156.6	85.2	107.9	85.0	1.73	3.32	145.2	5.7	8.52
1937	477.6	175.9	87.5	112.8	85.0	0.28	3.13	228.8	5.3	8.60
1938	466.1	177.1	88.8	114.9	89.0	0.21	2.99	211.2	4.9	8.68
1939	497.7	192.8	91.2	115.7	89.0	1.22	3.56	187.8	5.8	8.78
1940	438.5	183.1	91.9	130.2	96.0	1.76	3.98	196.1	6.3	8.88
1941	415.3	156.6	92.7	144.6	100.7	2.02	3.58	256.5	5.6	8.96
1942	379.6	122.9	93.4	159.1	100.7	1.75	3.21	261.5	4.9	9.04
1943	370.4	112.0	94.1	159.1	100.7	1.17	3.11	305.1	4.6	9.10
1944	248.5	75.9	94.9	173.6	100.7	0.97	3.09	306.8	4.5	9.17
1945	254.3	54.2	95.6	188.0	119.4	1.71	3.04	270.7	4.3	9.26
1946	430.2	134.9	96.3	216.9	147.6	1.20	2.99	234.7	4.1	9.40
1947	496.9	172.3	97.1	231.4	152.2	1.35	3.06	273.3	4.1	9.63
1948	550.0	203.6	98.0	231.4	159.3	1.29	3.09	295.1	4.1	9.80
1949	598.5	226.5	98.2	246.3	163.9	1.27	3.12	266.6	4.0	9.95
1950	619.9	254.2	97.8	267.6	178.0	1.41	3.12	256.5	3.9	10.11
1951	632.9	265.1	97.4	296.0	194.4	1.36	3.42	253.1	4.1	10.26
1952	645.7	265.1	96.3	296.3	196.7	1.08	3.41	228.0	4.0	10.38
1953	701.8	291.6	97.1	294.5	201.4	0.57	3.17	267.7	3.6	10.49
1954	749.5	319.3	97.9	306.6	222.5	0.55	3.18	449.1	3.5	10.61
1955	805.1	345.8	98.5	313.1	234.2	0.96	3.19	528.5	3.4	10.75
1956	834.7	367.5	98.9	321.2	245.9	2.94	3.44	559.6	3.6	10.89
1957	858.2	384.3	98.6	338.2	274.0	5.47	4.22	487.1	4.2	11.03
1958	855.6	384.3	97.5	343.7	285.7	3.80	4.44	658.1	4.3	11.19
1959	897.4	426.5	98.0	347.8	292.8	2.29	4.40	839.5	4.1	11.35
1960	973.0	469.9	98.7	356.1	316.2	2.58	4.45	858.5	4.0	11.49
1961	975.8	486.7	99.0	363.9	327.9	1.72	4.18	860.2	3.1	11.64
1962	1042.6	513.3	99.1	373.4	349.0	2.51	4.24	770.4	3.5	11.81
1963	1080.4	541.0	99.1	387.2	372.4	2.82	4.31	832.5	3.4	11.97
1964	1169.9	595.2	99.2	413.5	430.9	4.35	5.09	946.5	3.4	12.13
1965	1231.1	627.7	99.1	430.0	473.1	4.73	5.50	841.2	3.9	12.29
1966	1265.1	665.1	98.9	453.3	519.9	6.44	6.58	678.8	4.7	12.46
1967	1331.6	697.6	98.0	466.9	555.1	5.67	6.42	875.7	4.5	12.60
1968	1417.1	773.5	98.1	479.1	576.2	5.19	6.52	1146.9	3.9	12.73
1969	1508.4	843.4	98.5	509.2	632.4	7.76	7.47	1062.3	4.1	12.88
1970	1594.1	903.6	98.6	531.7	707.3	7.96	8.01	963.8	4.6	13.04
1971	1661.5	939.8	98.1	573.7	810.2	5.26	7.63	860.2	4.9	13.19
1972	1716.6	975.9	97.0	621.2	900.2	2.96	7.35	1089.9	4.4	13.33
1973	1796.9	1024.1	96.9	674.1	1028.8	6.89	7.92	880.9	4.6	13.44
1974	1868.4	1060.2	96.3	738.0	1157.4	10.35	9.82	644.3	6.7	13.54
1975	1866.6	1060.2	94.7	812.7	1286.0	5.27	8.79	913.7	6.4	13.65
1976	1962.0	1108.4	94.4	885.8	1388.9	6.88	8.95	908.5	5.8	13.77
1977	2007.7	1144.6	94.5	939.8	1530.4	4.57	8.10	934.5	6.1	13.85
1978	2056.9	1168.7	94.5	982.1	1620.4	6.54	7.74	925.8	6.5	13.94
1979	2105.6	1192.8	94.4	1024.4	1684.7	9.47	8.78	965.5	6.8	14.03
1980	2123.7	1204.8	93.7	1096.0	1774.7	10.69	10.11	1074.4	7.4	14.14
1981	2108.9	1192.8	90.8	1165.0	1877.6	11.63	11.53	991.5	7.5	14.25
1982	2079.1	1168.7	87.6	1226.9	2019.1	8.32	9.88	1138.3	8.1	14.31

1983	2108.4	1192.8	85.0	1261.2	2057.6	5.62	8.26	1727.3	5.4	14.36
1984	2175.0	1228.9	84.6	1287.6	2083.4	6.10	8.18	2110.7	4.6	14.42
1985	2232.4	1265.1	85.8	1318.6	2147.7	6.32	7.34	2684.2	4.4	14.48
1986	2293.6	1301.2	86.8	1322.5	2186.2	5.61	6.35	2813.7	4.2	14.56
1987	2320.7	1313.3	87.4	1325.2	2263.4	5.31	6.39	2223.0	4.1	14.66
1988	2381.4	1349.4	91.7	1331.8	2302.0	4.77	6.35	2863.8	4.6	14.76
1989	2492.9	1409.6	92.6	1347.8	2346.3	7.33	7.21	3502.9	4.0	14.85
1990	2595.4	1469.9	93.6	1377.4	2412.7	8.62	8.93	2907.0	4.2	14.95
1991	2650.0	1526.5	95.6	1424.2	2501.2	9.21	8.74	3306.0	4.2	15.07
1992	2687.0	1491.9	94.6	1466.9	2612.9	9.29	8.10	3420.0	3.9	15.18
1993	2697.7	1472.8	93.5	1500.7	2699.8	6.75	6.69	4850.2	3.4	15.29
1994	2765.3	1516.9	92.5	1536.8	2748.0	5.12	7.20	4801.8	2.9	15.38
1995	2829.0	1552.2	93.0	1563.6	2779.4	4.32	7.20	5553.2	3.4	15.46
1996	2915.7	1591.0	93.4	1595.0	2832.2	2.98	6.49	7553.3	2.9	15.52
1997	3022.7	1640.7	94.5	1626.5	2912.8	3.31	5.81	10688.3	2.4	15.61
1998	3133.6	1664.0	95.9	1654.9	3007.3	3.41	4.87	12690.2	3.0	15.70
1999	3243.9	1671.7	96.8	1683.2	3099.0	2.92	4.94	16117.1	1.8	15.76

### 1. *National Product Index (NP)*

Real National Product index for the Netherlands starting at 100 in 1870.

- 1870-1994: Based on Gross Domestic Product (GDP) volume index from Maddison (1995). Corrected for boundary adjustments.
- 1995-1997: Based on GDP volume index from OECD (1999).
- 1998-1999: Based on annual volume growth of GDP at market prices from various monthly editions of the OECD Main Economic Indicators.

### 2. *Industrial Production Index (IP)*

Real Industrial Production index for the Netherlands starting at 100 in 1921.

- 1921-1990: Based on Industrial Production volume from Boeschoten (1992).
- 1991-1999: Based on annual volume growth of Industrial Production from various monthly editions of the OECD Main Economic Indicators.

### 3. *Employment (EM)*

100% minus employment rate for the Netherlands.

- 1911-1939 and 1947-1990: Based on Boeschoten (1992).
- 1940-1946: Linear interpolation of the employment rate between 1939 and 1947 values from Boeschoten (1992).
- 1991-1999: Based on unemployment rate from various monthly editions of the OECD Main Economic Indicators.

### 4. *Consumer Price Index (PI)*

Consumer Price Index for the Netherlands starting at 100 in 1813.

- 1813-1995: Based on long term data for the Netherlands from Poll (1996).
- 1996: Based on CBS national accounts consumer price index.
- 1997-1999: Based on CBS consumer price inflations for all households (annual average) from StatLine database at [www.cbs.nl](http://www.cbs.nl).

#### 5. *Wage Index (WI)*

Nominal wage index for the Netherlands starting at 100 in 1926.

- 1926-1988: Based on money wages in industry from table B4 in Mitchell (1992)
- 1989-1999: Based on hourly rates in manufacturing from various monthly editions of the OECD Main Economic Indicators.

#### 6. *Short Interest Rate (SR)*

Nominal short term interest rate for the Netherlands in percentage points.

- 1828-1995: From Poll (1996).
- 1996-1999: CBS “kasgeldleningen aan de lagere overheid” (annual average) from StatLine database at [www.cbs.nl](http://www.cbs.nl).

#### 7. *Long Interest Rate (LR)*

Nominal long term interest rate for the Netherlands in percentage points.

- 1814-1995: From Poll (1996).
- 1996-1999: CBS average yield on 5 governments bonds with the longest remaining time to maturity (annual average) from StatLine database at [www.cbs.nl](http://www.cbs.nl).

#### 8. *Equity Price Index (EP)*

Nominal index of equity prices in the Netherlands starting at 100 in 1816.

- 1816-1995: From Poll (1996).
- 1996-1999: Based on CBS equity price index (ultimo year) from StatLine database at [www.cbs.nl](http://www.cbs.nl).

#### 9. *Dividend Yield (DY)*

Nominal dividend yield on Dutch equities in percentage points.

- 1824-1995: From Poll (1996).
- 1996-1999: Based on CBS dividend price ratio of all listed companies from StatLine database at [www.cbs.nl](http://www.cbs.nl).

#### 10. *Population (POP)*

Population of the Netherlands in millions.

- 1839-1988: Mid-year population estimates from Mitchell (1992).
- 1989-1997: Mid-year population estimates from OECD (1999).
- 1998: From OECD Main Economic Indicators.
- 1999: CBS population (primo year) from StatLine database at [www.cbs.nl](http://www.cbs.nl).

Table F.2 Historical time series for the United Kingdom. Descriptions of the series and their sources are given at the end of the table.

<i>Year</i>	<i>11</i> <i>NP</i>	<i>12</i> <i>IP</i>	<i>13</i> <i>EM</i>	<i>14</i> <i>PI</i>	<i>15</i> <i>WI</i>	<i>16</i> <i>SR</i>	<i>17</i> <i>LR</i>	<i>18</i> <i>CY</i>	<i>19</i> <i>EP</i>	<i>20</i> <i>DY</i>	<i>21</i> <i>ET</i>	<i>22</i> <i>POP</i>
1600				100.0								
1601				116.8								
1602				102.6								
1603				97.6								
1604				88.0								
1605				97.6								
1606				102.0								
1607				97.8								
1608				110.5								
1609				121.8								
1610				109.6								
1611				100.9								
1612				114.2								
1613				119.6								
1614				123.5								
1615				122.2								
1616				122.4								
1617				117.0								
1618				114.2								
1619				107.6								
1620				105.7								
1621				100.4								
1622				113.9								
1623				128.1								
1624				118.3								
1625				116.3								
1626				120.3								
1627				108.1								
1628				101.5								
1629				111.1								
1630				129.6								
1631				148.6								
1632				126.4								
1633				123.1								
1634				133.1								
1635				130.1								
1636				129.2								
1637				135.3								
1638				154.0								
1639				132.2								
1640				119.0								
1641				127.7								
1642				121.4								
1643				120.5								
1644				115.7								
1645				125.1								
1646				124.0								
1647				145.3								
1648				167.8								
1649				178.9								
1650				182.8								

1651	153.4	
1652	141.2	
1653	126.1	
1654	118.3	
1655	115.7	
1656	121.8	
1657	133.3	
1658	140.7	
1659	152.5	
1660	149.0	
1661	141.2	
1662	167.5	
1663	147.1	
1664	143.1	
1665	134.2	
1666	144.7	
1667	125.7	
1668	131.2	
1669	124.6	
1670	125.7	
1671	129.6	
1672	121.4	
1673	127.5	
1674	141.6	
1675	150.5	
1676	142.0	
1677	129.0	
1678	137.9	
1679	133.8	
1680	123.7	
1681	123.5	
1682	130.7	
1683	127.9	
1684	124.2	
1685	141.8	
1686	121.8	
1687	126.4	
1688	120.0	
1689	116.6	
1690	111.8	
1691	107.4	
1692	118.1	
1693	142.0	
1694	151.0	
1695	140.5	
1696	151.9	
1697	151.0	
1698	167.1	
1699	168.4	
1700	146.2	7.62
1701	127.7	9.92
1702	126.8	7.87
1703	120.0	6.86
1704	127.9	7.10
1705	119.4	7.16



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1706	127.0	7.69
1707	115.7	7.34
1708	124.4	8.00
1709	151.9	8.14
1710	173.9	9.09
1711	193.7	9.76
1712	139.0	8.66
1713	129.4	7.14
1714	138.3	6.40
1715	140.7	6.49
1716	140.5	5.58
1717	131.2	4.10
1718	125.3	4.10
1719	132.7	3.94
1720	138.3	2.86
1721	131.6	4.76
1722	120.7	5.49
1723	114.4	4.90
1724	128.3	4.55
1725	132.9	4.31
1726	138.8	5.05
1727	129.8	4.90
1728	141.4	3.81
1729	148.4	3.16
1730	130.5	3.06
1731	120.5	3.08
1732	121.4	2.96
1733	118.5	3.26
1734	112.9	3.17
1735	115.3	3.01
1736	117.4	2.85
1737	126.6	2.81
1738	122.7	2.83
1739	119.2	3.00
1740	140.3	3.03
1741	155.1	3.00
1742	137.5	2.94
1743	126.1	2.93
1744	112.9	3.21
1745	115.0	3.70
1746	129.4	3.54
1747	125.1	3.64
1748	130.5	3.35
1749	132.7	3.01
1750	128.5	3.00
1751	125.1	2.94
1752	130.9	2.82
1753	127.5	2.89
1754	134.0	2.89
1755	125.9	3.23
1756	131.2	3.39
1757	159.7	3.31
1758	159.3	3.34
1759	146.6	3.55
1760	140.1	3.97
1761	133.8	4.51

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1762	139.0	3.24		
1763	142.7	3.51		
1764	155.3	3.40		
1765	160.8	3.27		
1766	162.7	3.33		
1767	172.1	3.24		
1768	170.2	3.33		
1769	156.2	3.47		
1770	155.6	3.83		
1771	168.8	3.39		
1772	186.9	3.38		
1773	186.3	3.43		
1774	188.0	3.29		
1775	177.6	3.35		
1776	173.6	3.66		
1777	173.0	3.99		
1778	180.0	4.88		
1779	164.7	4.84		
1780	159.0	4.98		
1781	165.6	5.19		
1782	169.1	4.62		
1783	189.3	5.26		
1784	190.4	5.32		
1785	182.8	4.19		
1786	182.8	3.98		
1787	181.7	3.88		
1788	188.9	4.05		
1789	186.5	3.81		
1790	189.8	3.70		
1791	189.5	3.31		
1792	192.4	3.80		
1793	197.8	4.07		
1794	213.1	4.62		
1795	237.7	4.28		
1796	252.9	5.44		
1797	227.7	6.06		
1798	222.7	5.49		
1799	250.1	4.76		
1800	341.4	4.77	100.0	1.0
1801	381.5	4.36	111.0	1.2
1802	293.7	4.09	112.6	1.3
1803	276.3	5.35	87.0	1.0
1804	285.2	5.04	94.5	1.2
1805	331.4	4.98	102.0	1.5
1806	316.8	4.97	104.1	1.7
1807	310.9	4.67	108.7	1.9
1808	321.6	4.49	112.6	2.1
1809	352.7	4.24	124.9	2.6
1810	363.8	4.47	114.5	2.4
1811	353.4	4.72	96.0	2.3
1812	400.0	5.00	87.1	2.4
1813	409.8	4.98	90.1	2.6
1814	357.7	4.54	79.9	2.9
1815	319.6	4.91	78.8	2.9
1816	292.8	4.76	76.0	2.8

1817				332.5			3.60	100.8	3.8	
1818				333.3			3.79	106.9	3.6	
1819				325.1			4.48	97.8	3.0	
1820				294.8		5.00	4.31	100.2	3.3	
1821				259.3		5.00	3.88	107.1	3.8	
1822				224.2		4.00	3.77	115.4	3.9	
1823				239.4		4.00	3.47	125.9	3.8	
1824				259.9		3.50	3.16	241.5	3.9	
1825				305.0		3.88	3.69	186.0	3.9	
1826				288.2		4.50	3.72	148.1	3.7	
1827				269.5		3.25	3.59	155.6	4.0	
1828				261.7		3.05	3.43	133.1	4.2	
1829				259.0	100.0	3.35	3.21	137.7	4.5	
1830				249.7	100.0	2.81	3.69	117.0	4.2	
1831				274.5	100.0	3.69	3.55	98.7	4.3	
1832				254.2	100.0	3.12	3.49	100.5	4.4	
1833				238.8	100.0	2.75	3.33	118.1	5.1	
1834				220.3	101.8	3.37	3.26	106.4	5.6	
1835				224.0	101.8	3.70	3.25	112.5	5.5	
1836				248.6	105.3	4.25	3.35	119.3	5.6	
1837				254.7	105.3	4.45	3.30	107.4	5.8	
1838				256.4	105.3	3.00	3.19	103.6	5.9	
1839				275.2	107.0	5.10	3.26	90.3	5.4	
1840				280.2	107.0	5.00	3.38	93.5	5.0	
1841				273.6	105.3	4.90	3.36	84.5	5.5	
1842				252.9	105.3	3.33	3.16	90.5	5.9	
1843				224.4	103.5	2.18	3.08	101.4	6.6	
1844				224.2	105.3	2.12	2.96	118.3	7.8	
1845				235.1	107.0	2.96	3.16	115.9	7.8	
1846				244.4	108.8	3.80	3.20	114.2	8.1	
1847				273.9	108.8	5.88	3.52	98.0	7.7	
1848				240.7	105.3	3.20	3.38	85.1	8.2	
1849				225.5	105.3	2.31	3.12	78.9	9.1	
1850				211.1	105.3	2.25	3.11	90.1	9.9	
1851				209.4	107.0	3.06	3.10	89.6	10.4	
1852				213.1	107.0	1.89	2.96	98.2	11.1	
1853				247.3	110.5	3.68	3.15	91.8	11.3	
1854				275.6	115.8	4.95	3.24	89.2	11.3	
1855	100.0	100.0	96.3	277.6	119.3	4.73	3.34	85.4	11.5	
1856	104.2	106.8	96.8	275.4	121.1	6.01	3.17	90.2	12.8	
1857	105.3	110.6	95.8	280.4	119.3	6.60	3.27	85.3	13.3	
1858	103.6	108.4	92.7	259.3	117.5	2.60	3.06	90.1	14.4	
1859	110.9	114.1	97.4	264.5	117.5	2.61	3.09	89.0	15.1	
1860	110.1	120.5	98.2	286.3	121.1	4.10	3.20	98.5	16.2	
1861	114.7	120.5	96.3	283.7	122.8	5.00	3.31	101.6	17.1	
1862	115.0	123.2	94.0	281.0	122.8	2.49	3.31	90.2	17.6	
1863	120.4	123.6	95.3	249.2	122.8	4.40	3.30	88.4	18.4	
1864	122.4	133.1	98.1	261.4	124.6	7.29	3.36	137.9	19.6	
1865	128.5	141.8	98.2	269.7	128.1	4.79	3.43	142.8	21.0	
1866	129.5	147.1	97.4	282.4	135.1	6.60	3.37	111.4	22.1	
1867	129.5	138.4	93.7	293.2	131.6	2.16	3.23	107.4	22.2	
1868	133.1	138.4	93.3	281.3	129.8	1.88	3.25	115.1	23.2	
1869	136.6	136.1	94.1	271.0	129.8	3.06	3.26	123.1	23.4	
1870	148.7	152.9	96.3	270.4	131.6	3.06	3.27	133.8	24.0	29.31
1871	151.6	165.4	98.4	287.6	136.8	2.62	3.25	159.0	25.6	29.65
1872	150.2	170.3	99.1	300.2	145.6	3.90	3.27	164.5	27.2	29.76

1873	152.0	172.2	98.9	313.1	154.4	4.49	3.26	171.1	29.4	30.04
1874	161.5	176.4	98.4	310.0	154.4	3.41	3.27	161.8	30.5	30.35
1875	160.3	177.6	97.8	285.4	150.9	2.89	3.20	150.5	32.3	30.66
1876	160.1	180.6	96.6	298.5	150.9	1.89	3.19	146.7	33.4	31.00
1877	158.3	180.2	95.6	289.8	150.9	2.35	3.15	132.5	34.3	31.35
1878	168.4	179.8	93.8	279.1	145.6	2.23	3.17	118.0	34.9	31.68
1879	158.2	173.4	89.3	263.6	142.1	1.76	3.08	132.4	39.1	32.03
1880	178.5	191.3	94.8	255.8	140.4	2.30	3.04	139.0	42.1	32.33
1881	174.3	203.4	96.5	264.3	143.9	2.90	3.02	138.0	44.2	32.62
1882	175.9	211.8	97.7	248.4	147.4	3.38	2.97	129.4	46.4	32.87
1883	186.3	214.8	97.4	257.5	147.4	3.03	2.99	122.9	49.2	33.10
1884	187.1	206.8	91.9	233.3	145.6	2.40	3.02	120.3	52.9	33.36
1885	183.9	198.1	90.7	223.5	143.9	2.14	3.00	120.3	52.9	33.63
1886	186.3	193.9	89.8	202.8	142.1	2.12	2.99	121.3	54.4	33.91
1887	192.2	209.5	92.4	208.1	142.1	2.39	2.95	116.7	57.5	34.17
1888	194.3	221.7	95.1	207.0	147.4	2.36	2.85	123.7	62.6	34.44
1889	200.3	237.3	97.9	206.5	149.1	2.69	2.83	139.7	67.2	34.71
1890	203.2	240.7	97.9	206.3	164.9	3.74	2.88	131.1	69.3	35.00
1891	206.7	243.7	96.5	217.4	166.7	2.51	2.89	132.2	73.5	35.30
1892	205.6	231.9	93.7	217.0	166.7	1.53	2.82	131.7	75.7	35.61
1893	201.5	228.1	92.5	199.1	164.9	2.19	2.80	134.0	75.4	35.94
1894	217.8	241.4	93.1	213.9	166.7	0.99	2.67	142.1	77.1	36.28
1895	221.8	252.9	94.2	210.9	166.7	0.81	2.58	158.0	78.3	36.62
1896	232.0	271.5	96.7	206.3	170.2	1.50	2.47	192.7	80.8	36.97
1897	227.3	279.1	96.7	209.8	171.9	1.80	2.44	202.8	88.2	37.34
1898	243.1	292.8	97.2	213.9	173.7	2.62	2.49	203.7	92.5	37.70
1899	259.9	304.6	98.0	207.0	177.2	3.26	2.73	199.2	90.4	38.07
1900	249.9	304.6	97.5	216.6	175.4	4.00	2.82	197.7	88.8	38.43
1901	268.4	305.3	96.7	214.8	175.4	3.48	2.96	187.9	92.6	38.78
1902	264.3	310.6	96.0	209.8	175.4	2.96	2.97	185.4	94.0	39.12
1903	261.7	304.2	95.3	218.7	175.4	3.45	2.82	175.0	93.3	39.45
1904	267.3	308.0	94.0	214.6	175.4	2.90	2.84	179.3	91.3	39.79
1905	276.1	325.9	95.0	215.5	177.2	2.20	2.80	190.7	91.9	40.13
1906	276.4	339.5	96.4	221.4	180.7	3.00	2.90	189.8	92.6	40.49
1907	277.5	346.0	96.3	224.6	184.2	3.76	3.02	161.6	91.3	40.84
1908	271.9	318.3	92.2	227.2	180.7	2.25	2.99	175.0	92.0	41.20
1909	281.1	320.5	92.3	230.5	182.5	2.14	3.02	183.2	92.6	41.57
1910	286.3	325.1	95.3	216.6	184.2	3.06	3.15	178.8	94.1	41.94
1911	297.1	347.9	97.0	214.4	189.5	2.83	3.24	179.3	95.9	42.27
1912	294.1	357.0	96.8	217.6	193.0	3.04	3.34	177.5	96.7	42.42
1913	311.7	380.2	97.9	222.4	198.2	3.48	3.48	165.8	97.1	42.62
1914	318.0	356.3	96.7	249.9	222.7	3.17	3.65	154.2	102.1	43.00
1915	350.8	363.7	98.9	286.9	259.2	3.71	4.29	146.4	100.2	43.27
1916	353.5	343.9	99.6	359.9	295.6	5.18	4.57	147.4	94.1	43.43
1917	358.3	320.9	99.4	428.1	376.6	4.99	4.57	131.9	91.7	43.52
1918	360.3	310.3	99.2	544.0	469.7	3.59	4.20	146.1	107.3	43.49
1919	325.9	342.0	96.6	491.1	538.6	3.89	4.94	149.8	115.5	43.45
1920	312.5	379.9	98.0	564.5	615.5	6.34	5.66	130.0	105.7	43.72
1921	294.7	294.0	88.7	446.2	542.6	5.21	5.00	122.8	106.1	44.07
1922	306.9	339.7	90.2	364.3	421.1	2.61	4.47	144.5	131.5	44.37
1923	316.8	360.1	91.9	376.0	400.9	2.64	4.45	147.4	5.00 140.8	44.60
1924	326.2	399.8	92.8	379.1	413.0	3.38	4.34	161.2	4.63 161.4	44.92
1925	343.6	415.4	92.1	372.1	417.1	4.11	4.54	168.5	4.39 176.0	45.06
1926	326.5	392.6	91.2	343.6	417.1	4.51	4.63	172.6	4.60 189.0	45.23
1927	350.4	452.7	93.2	325.9	409.0	4.26	4.50	186.9	4.58 213.7	45.39

1928	356.7	452.7	92.5	323.5	404.9	4.13	4.46		202.1	4.18	240.7	45.58
1929	365.2	462.3	92.7	329.2	404.9	5.25	4.75	5.73	187.2	4.11	232.4	45.67
1930	364.7	442.5	88.8	277.8	400.9	2.54	4.35	5.39	150.6	4.51	196.5	45.87
1931	344.8	414.2	84.9	249.7	396.8	3.49	4.55	5.73	115.5	5.26	158.9	46.07
1932	347.2	412.4	84.4	232.0	388.7	1.64	3.38	4.62	121.8	5.22	176.1	46.34
1933	350.5	439.5	85.9	241.2	384.7	0.59	3.37	4.00	154.8	4.13	232.1	46.52
1934	375.5	483.3	88.1	239.0	388.7	0.73	2.68	3.99	167.7	3.26	259.3	46.67
1935	389.5	520.6	89.0	250.3	392.8	0.52	2.89	3.89	180.9	3.50	290.4	46.87
1936	401.4	567.5	90.6	263.8	400.9	0.58	2.96	3.93	205.9	3.71	342.9	47.08
1937	418.8	601.2	92.2	277.8	413.0	0.56	3.28	4.09	166.1	3.93	289.0	47.29
1938	431.9	584.9	90.7	277.6	425.2	0.61	3.55	3.98	142.5	5.12	262.4	47.49
1939	449.6		94.2	263.4	425.2	1.22	3.68	4.32	143.6	5.96	280.9	47.99
1940	522.9		96.7	342.9	490.6	1.03	3.27	4.44	124.8	6.44	261.8	48.23
1941	555.7		98.8	388.7	523.3	1.01	3.05	4.16	153.1	6.57	340.1	48.22
1942	562.0		99.5	464.1	566.9	1.01	3.06	3.96	181.5	5.21	423.7	48.40
1943	572.8		99.6	467.3	588.7	1.00	3.14	3.91	196.2	4.55	478.2	48.79
1944	544.4		99.6	482.8	621.4	1.00	3.06	3.82	217.1	4.34	552.2	49.02
1945	505.9		99.5	497.2	643.2	0.90	2.75	3.80	216.3	4.20	572.6	49.18
1946	501.6	599.4	98.1	515.0	697.7	0.51	2.54	3.52	255.3	4.28	706.1	49.22
1947	488.6	631.2	98.6	562.1	719.5	0.51	3.01	3.76	248.3	4.38	719.0	49.52
1948	503.3	685.3	98.7	605.9	675.9	0.51	3.14	3.74	238.3	4.81	724.4	50.01
1949	519.4	725.6	98.8	685.2	697.7	0.52	3.51	4.10	205.3	5.05	658.4	50.31
1950	538.6	769.6	98.4	687.4	741.3	0.51	3.52	4.07	218.3	5.58	739.6	50.36
1951	557.8	791.6	98.7	796.5	806.7	0.58	4.05	4.70	223.6	5.32	799.0	50.57
1952	557.8	780.6	97.8	868.6	872.2	2.15	4.21	5.08	212.3	6.00	809.0	50.74
1953	583.5	824.6	98.2	813.7	926.7	2.31	3.89	4.70	246.2	6.28	996.9	50.88
1954	605.9	868.5	98.5	833.3	1003.0	1.80	3.76	4.44	331.1	5.72	1415.0	51.07
1955	625.2	912.5	98.8	877.1	1090.2	3.70	4.39	5.36	336.9	5.32	1517.4	51.22
1956	638.0	912.5	98.7	903.7	1166.5	4.93	4.90	5.85	306.1	5.87	1469.7	51.43
1957	650.8	934.5	98.4	943.6	1231.9	4.80	5.41	6.47	288.8	6.21	1511.9	51.66
1958	647.6	923.5	97.8	959.0	1253.7	4.56	4.89	6.02	394.1	6.49	2142.9	51.87
1959	670.0	967.5	97.7	959.0	1319.1	3.37	4.99	5.96	565.9	5.42	3224.3	52.16
1960	702.1	1044.4	98.3	979.7	1417.3	4.95	5.63	6.54	539.2	4.50	3216.8	52.37
1961	734.2	1055.4	98.4	1021.4	1504.5	5.24	6.56	7.30	524.5	4.84	3299.9	52.81
1962	753.4	1066.4	97.9	1047.1	1548.1	4.13	5.60	6.69	515.8	5.52	3425.4	53.29
1963	775.8	1099.4	97.4	1068.0	1635.3	3.67	5.79	6.40	569.4	4.95	3958.9	53.63
1964	823.9	1187.4	98.3	1119.8	1766.1	4.61	6.25	6.74	514.9	4.41	3744.9	53.99
1965	865.6	1231.3	98.5	1171.7	1918.7	5.91	6.44	7.14	547.6	5.16	4204.5	54.35
1966	859.2	1242.3	98.5	1213.1	1984.2	6.11	6.67	7.60	488.7	5.42	3967.2	54.64
1967	884.0	1253.3	97.6	1244.2	2093.2	5.82	7.13	7.93	644.8	5.46	5510.3	54.96
1968	921.2	1319.3	97.6	1316.8	2245.8	7.04	7.99	9.00	892.7	4.22	7883.5	55.21
1969	937.7	1352.3	97.7	1378.9	2431.1	7.63	8.94	10.71	778.8	3.54	7157.7	55.46
1970	958.3	1363.3	97.5	1487.8	2747.3	7.02	9.69	10.82	721.3	4.17	6915.5	55.63
1971	983.1	1374.3	96.6	1617.4	3033.5	5.58	8.62	9.19	1022.9	4.21	10191.3	55.91
1972	999.6	1396.2	96.3	1741.8	3491.3	5.51	9.71	10.37	1152.1	3.28	11859.2	56.08
1973	1078.1	1517.2	97.4	1928.5	4006.5	9.29	12.38	13.51	792.5	3.43	8460.0	56.21
1974	1074.0	1495.2	97.5	2293.7	4693.3	11.37	17.10	19.36	353.2	5.38	4079.1	56.22
1975	1065.7	1418.2	96.1	2866.9	5723.5	10.16	14.86	15.88	836.9	7.88	10278.0	56.22
1976	1102.9	1462.2	94.6	3300.9	6524.8	11.12	14.82	16.06	804.2	5.78	10499.8	56.21
1977	1115.3	1539.2	94.3	3703.9	7039.9	7.69	10.90	12.01	1131.3	6.09	15661.5	56.18
1978	1144.2	1572.2	94.4	4013.7	8127.4	8.50	12.38	13.47	1166.4	5.45	16990.3	56.17
1979	1169.0	1638.1	96.1	4703.5	9443.8	12.98	11.85	14.92	1216.6	5.52	18775.6	56.23
1980	1148.4	1528.2	95.0	5416.1	10703.0	15.11	11.94	13.85	1546.3	6.26	25411.9	56.31
1981	1140.1	1484.2	92.0	6067.1	11790.4	12.99	13.63	16.23	1653.1	6.02	28900.0	56.38
1982	1156.6	1506.2	90.6	6392.6	12877.9	11.41	10.65	12.38	2025.5	5.84	37341.9	56.34
1983	1197.9	1572.2	89.6	6733.6	14079.8	9.60	9.90	12.65	2487.5	5.19	48227.0	56.38

1984	1214.4	1572.2	89.4	7043.4	15224.5	9.30	9.94	12.50	3124.0	4.65	63643.3	56.49
1985	1264.0	1660.1	89.2	7438.6	16483.7	11.55	9.89	10.70	3620.5	4.55	76646.2	56.62
1986	1305.3	1693.1	89.1	7717.6	17571.2	10.34	10.28	11.59	4428.8	4.23	97599.7	56.76
1987	1359.0	1748.1	90.2	8004.4	19116.5	9.25	9.31	11.11	4602.5	3.69	105279	56.93
1988	1412.7	1814.0	92.2	8546.6	20604.6	9.87	9.10	10.96	4895.4	3.95	117037	57.07
1989	1443.5	1817.3	93.7	9205.2	22400.6	13.28	9.80	12.00	6306.8	4.32	158749	57.24
1990	1449.2	1805.8	94.2	10065.4	24506.8	14.09	10.41	12.15	5446.1	4.56	143024	57.41
1991	1420.5	1752.8	91.9	10514.8	26515.0	10.85	9.74	10.95	6125.0	5.11	172863	57.65
1992	1413.0	1740.7	90.3	10786.1	28256.3	8.94	8.56	10.08	7411.5	4.88	208333	57.85
1993	1442.1	1776.9	89.7	10995.2	29530.7	5.25	6.64	7.72	8951.8	4.39	267485	58.35
1994	1504.3	1865.3	90.7	11312.9	30927.6	5.15	8.42	9.31	8028.1	3.69	251840	58.70
1995	1545.7	1912.3	91.7	11678.6	32324.5	6.33	7.75	8.33	9526.1	3.96	312000	58.61
1996	1581.5	1931.4	92.8	11963.6	33488.1	5.77	7.71	8.23	10700.5	3.84	364002	58.80
1997	1634.9	1952.5	94.5	12397.6	34910.4	6.48	6.38	6.77	12700.8	3.63	455375	59.01
1998	1678.1	1965.8	95.3	12738.6	36688.3	6.82	4.49	5.19	14161.7	3.05	511683	59.24
1999	1713.5	1975.4	95.7	12963.2	38466.1	5.19	4.98	5.14	17150.4	2.74	635523	59.24

### 11. National Product Index (NP)

Real National Product index for the United Kingdom starting at 100 in 1855.

- 1830-1988: Based on Gross Domestic Product (GDP) at factor costs from Mitchell (1992). Aberrant observation for 1868 corrected by interpolating between 1867 and 1869. All values prior to 1920 lowered by 6.6% to correct for the exclusion of Southern Ireland in 1920. The 6.6% is based on the 1920 values with and without Southern Ireland. Note that Maddison (1995) applies a similar correction. However in that case the jump in the series in 1920 remains substantial while this corrected series also shows a remarkable resemblance with the *uncorrected* series from Mitchell (1992).
- 1989-1999: Based on GDP volume index from OECD (1999).
- 1998-1999: Based on annual volume growth of GDP at market prices from various monthly editions of the OECD Main Economic Indicators.

### 12. Industrial Production Index (IP)

Real Industrial Production index for the United Kingdom starting at 100 in 1855.

- 1855-1988: Based on Industrial Production volume index from Mitchell (1992). Missing data 1939-1945. Where necessary the natural logarithm of the series is linearly interpolated across this period. All values prior to 1920 lowered by 2% to correct for the exclusion of Southern Ireland in 1920. The 2% is based on the 1920 values with and without Southern Ireland. The original series in Mitchell (1992) consists of three sources for respectively the 1801-1854, 1855-1949 and 1950-1999 periods. The data for the 1801-1854 is excluded here because of a clearly very different (smoother) behaviour of the series compared to the other periods.
- 1989-1999: Based on annual volume growth of Industrial Production from various monthly editions of the OECD Main Economic Indicators.

### 13. *Employment (EM)*

100% minus employment rate for the United Kingdom.

- 1855-1988: Based on unemployment rate from Mitchell (1992).
- 1989-1999: Based on unemployment rate from various monthly editions of the OECD Main Economic Indicators.

### 14. *Consumer Price Index (PI)*

Consumer Price Index for the United Kingdom starting at 100 in 1600. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- Phelps Brown and Hopkins (1956, 1957 and 1959).
- Central Statistical Office, Annual Abstract of Statistics, London: CSO and Monthly Digest of Statistics.

### 15. *Wage Index (WI)*

Nominal wage index for the United Kingdom starting at 100 in 1926.

- 1829-1988: Based on money wages in industry from Mitchell (1992). Missing values based on consumer price inflation from previous series.
- 1989-1999: Based on weakly earnings employees from various monthly editions of the OECD Main Economic Indicators.

### 16. *Short Interest Rate (SR)*

Nominal short term interest rate for the United Kingdom in percentage points.

- 1820-1899: Open market discount rate from Table 23 on page 208 and Table 61 on page 455 in Homer and Sylla (1991). Annual averages.
- 1900-1999: December values from monthly closing yields on three Month Treasury-Bills from [www.GlobalFindata.com](http://www.GlobalFindata.com).

### 17. *Long Interest Rate (LR)*

Nominal long term interest rate for the United Kingdom in percentage points.

Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- 1700-1823: Neal (1990).
- 1824-1844: The Times (London)
- 1844-1852: Banker's Magazine
- 1853-1999: Central Statistical Office, Annual Abstract of Statistics, London: CSO and the Financial Times.

Notes: Several sources are used for the yield on British government securities. From July 1700 until June 1729, the dividend yield on Million Bank stock is used, assuming a dividend of 6% through 1716, and 5% from 1717 through 1729. This is used because the Million Bank had been taken over by the government and the bank acted as a mutual fund that owned a large portion of government paper. It provides the best proxy for a risk-free government bond during that period, according to Larry Neal. From July 1729 through July 1753, the 3% annuities' yield is used, and from August 1753 on, the British consol is used. The British consol paid 3% from August 1753 until December 1888, 2 3/4% from 1889 through 1906, and 2 1/2% beginning in 1907.



18. *Corporate bond yield (CY)*

Nominal yield on long term high rated corporate bonds for the United Kingdom in percentage points.

1929-1999: UK Debenture Yield from [www.GlobalFindata.com](http://www.GlobalFindata.com), end of year year closing yields.

19. *Equity Price Index (EP)*

Nominal index of equity prices in the United Kingdom starting at 100 in 1800.

Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- 1811-1850: Rostow and Schwartz (1953) p. 368.
- 1851-1867: Hayek as given in Rostow and Schwartz (1953) p. 456.
- 1867-1906: Smith and Horne (?).
- 1907-1933: Banker's Magazine
- 1933-1962: Economist.
- 1962-1999: Financial Times.

Notes: Rostow's Total Index of Share Prices is used from 1811 to 1850. Hayek's index was taken from Rostow and excludes banks, insurance and bridge stocks, but includes industrial stocks. This index is linked to the London and Cambridge Economic Service index, which begins in July 1867 and continues until 1906. The L&CES index consisted of 25 stocks in 1867 and had grown to 75 stocks by 1914. The Banker's Magazine kept a capitalization-weighted index of 287 stocks, which gave the total capital values of the companies that were included. This was the broadest index of London shares at the time and the index is used beginning in 1907. Although this index was calculated beginning in 1887, the Banker's Magazine usually omitted calculating the index for one month during the summer, and for this reason it is excluded until 1907 when calculations were made for every month. The London market closed in August 1914 and reopened in January 1915. The Banker's Magazine Index is used through May 1933. Beginning in June 1933, The Actuaries General Index. This index included financial stocks, commodities and utilities, but excluded debentures and preferred shares. Beginning in April 1962, the Financial Times-Actuaries All-Share Index is used. All indexes have been chain linked to one another to create a continuous index.

20. *Dividend Yield (DY)*

Nominal dividend yield on United Kingdom equities in percentage points.

1923-1999: UK FT-Actuaries Dividend Yield from [www.GlobalFindata.com](http://www.GlobalFindata.com), annual average of monthly closing prices.

21. *Equity Total Return Index (ET)*

Nominal total return index of equities in the United Kingdom starting at 1 in 1800.

1800-1999: UK FT-Actuaries All-Share Total Return Index Series from [www.GlobalFindata.com](http://www.GlobalFindata.com).

Note: This series is used instead of a combination of the equity price index (19) and dividend yield (20) as for the other countries. The reason is that this series starts in 1800 while the dividend yield starts only in 1923.



## 22. Population (POP)

Population of the United Kingdom in millions.

1870-1994: Mid-year population estimates from Maddison (1995).

1995-1997: Mid-year population estimates from OECD (1999).

1998-1999: From OECD Main Economic Indicators.

Table F.3 Historical time series for the United States. Descriptions of the series and their sources are given at the end of the table.

<i>Year</i>	<i>23 NP</i>	<i>24 IP</i>	<i>25 EM</i>	<i>26 PI</i>	<i>27 WI</i>	<i>28 SR</i>	<i>29 LR</i>	<i>30 CY</i>	<i>31 EP</i>	<i>32 DY</i>	<i>33 POP</i>
1780											
1781											
1782											
1783											
1784											
1785											
1786					100.0						
1787					100.0						
1788					96.6						
1789					100.0						
1790					100.0						3.93
1791					105.2						4.05
1792					100.0						4.19
1793					124.1						4.33
1794					137.9						4.47
1795					165.5						4.61
1796					174.1						4.75
1797					182.8						4.88
1798					156.9		7.56				5.02
1799					162.1		7.29				5.16
1800					163.8		6.94		100.0		5.30
1801					155.2		6.44		106.9		5.49
1802					131.0		6.02		116.8		5.68
1803					143.1		6.16		105.5		5.87
1804					158.6		6.29		101.1		6.07
1805					156.9		6.38		96.7		6.26
1806					165.5		6.14		101.1		6.45
1807					167.2		6.08		105.5		6.64
1808					146.6		5.96		105.5		6.84
1809					155.2		5.85		106.6		7.03
1810					170.7		5.82		104.4		7.22
1811					175.9		5.95		96.7		7.46
1812					156.9		6.12		100.0		7.70
1813					151.7		6.30		101.1		7.94
1814					162.1		7.64		84.3		8.18
1815					189.7		6.25		86.5		8.42
1816					187.9		6.75		83.2		8.66
1817					170.7		5.86		93.1		8.90
1818					186.2		5.78		90.1		9.14
1819					162.1		5.90		83.2		9.38
1820				100.0	155.2		5.71		87.2		9.62
1821				95.3	136.2		5.49		92.7		9.94
1822				98.6	163.8		5.63		88.3		10.27
1823				93.9	146.6		5.64		88.3		10.60

1824			87.8	155.2		5.52		92.7		10.92
1825			89.2	174.1		5.62		87.2		11.25
1826			84.6	169.0		5.79		87.2		11.58
1827			87.8	172.4		5.77		84.7		11.91
1828			87.8	174.1		5.80		84.7		12.24
1829			89.2	179.3		5.85		83.6		12.57
1830			83.1	172.4		4.72		89.8		12.90
1831			86.2	172.4	6.12	4.76		92.7		13.32
1832			87.8	137.9	6.25	4.80		97.1		13.74
1833			89.2	172.4	7.83	4.92		96.4		14.16
1834			78.5	172.4	14.70	4.97		108.8		14.58
1835			92.3	172.4	7.00	4.83		112.0		15.00
1836			104.7	172.4	18.00	4.96		98.9		15.42
1837			110.8	172.4	14.25	4.95		87.6		15.84
1838			109.3	172.4	9.04	5.01		89.1		16.26
1839			109.3	206.9	12.58	5.21		78.1		16.68
1840			92.3	206.9	7.75	5.07		82.5		17.12
1841			92.3	206.9	6.80	4.99		71.5		17.73
1842			84.6	206.9	8.08	4.95		58.4		18.35
1843			78.5	172.4	4.41	5.17		85.0		18.96
1844			80.1	172.4	4.87	5.11		98.2		19.57
1845			83.1	137.9	4.71	5.47		105.8		20.18
1846			89.2	137.9	8.33	5.83		90.5		20.79
1847			89.2	172.4	9.59	5.80		91.6		21.41
1848			78.5	189.7	15.10	5.80		88.3		22.02
1849			83.1	224.1	10.25	5.41		88.3		22.63
1850			78.5	206.9	8.04	5.31		104.7		23.26
1851			92.3	206.9	9.66	5.17		101.5		24.09
1852			92.3	206.9	6.33	5.01		121.5		24.91
1853			98.6	206.9	10.25	4.97		105.8		25.74
1854			98.6	241.4	10.37	5.61		74.1		26.56
1855			103.1	241.4	8.92	5.45		75.2		27.39
1856			104.7	241.4	8.83	5.71		78.5		28.21
1857			107.7	241.4	11.56	5.77	10.66	54.0		29.04
1858			106.1	206.9	4.81	5.29	8.77	61.7		29.86
1859			96.9	206.9	6.14	5.49	8.94	55.1		30.69
1860	100.0		93.9	241.4	7.31	5.86	9.12	62.8		31.51
1861	100.0		96.9	244.8	6.70	6.58	8.95	61.7		32.35
1862	93.8		106.1	255.0	5.32	5.80	6.74	96.0		33.19
1863	106.3		120.1	306.0	5.65	5.49	6.62	132.5		34.03
1864	112.5		146.1	336.6	7.36	5.20	6.66	141.2		34.86
1865	106.3		156.9	340.0	7.77	5.59	8.05	129.2		35.70
1866	131.3		158.5	326.4	6.33	5.38	7.79	133.9		36.54
1867	137.5		156.9	319.6	7.32	5.35	7.99	135.8		37.38
1868	143.8		150.8	329.8	7.28	5.24	8.03	150.4		38.21
1869	156.3		146.1	329.8	9.66	5.07	8.35	153.3		39.05
1870	100.0	156.3	140.0	326.4	7.23	5.30	7.96	161.7		39.91
1871	104.7	162.5	137.0	319.6	6.98	5.10	7.79	173.4	5.23	40.94
1872	108.9	193.8	138.4	323.0	8.63	5.39	7.74	185.0	6.03	41.97
1873	114.2	187.5	135.4	309.4	10.27	5.39	7.95	161.7	6.71	43.01
1874	113.7	181.3	135.4	292.4	5.98	5.30	7.34	166.1	6.68	44.04
1875	119.5	175.0	129.3	282.2	5.44	5.11	6.87	159.1	6.57	45.07
1876	121.1	175.0	124.4	268.6	5.13	5.31	6.65	130.7	7.11	46.11
1877	124.7	187.5	116.6	258.4	5.01	5.44	6.55	118.6	4.73	47.14
1878	130.0	200.0	108.7	251.6	4.82	3.98	6.32	125.5	4.90	48.17

1879	146.3	225.0		116.6	248.2	5.14	3.88	5.88	179.6	4.26	49.21
1880	163.7	262.5		121.4	255.0	5.23	3.57	5.24	213.1	4.43	50.26
1881	169.5	287.5		127.7	272.0	5.36	3.39	5.25	219.7	5.06	51.54
1882	180.0	306.3		124.4	285.6	5.64	3.33	5.25	213.1	5.42	52.82
1883	184.2	312.5		121.4	292.4	5.62	3.22	5.19	195.3	6.00	54.10
1884	187.9	293.8		116.6	292.4	5.21	3.25	5.13	158.4	6.39	55.38
1885	188.9	293.8		115.0	295.8	4.05	3.23	4.72	189.8	4.22	56.66
1886	194.7	356.3		115.0	299.2	4.77	3.11	4.59	205.8	3.70	57.94
1887	203.7	375.0		118.2	306.0	4.73	3.19	4.72	192.3	4.67	59.22
1888	202.6	387.5		118.2	309.4	4.91	3.12	4.55	187.6	3.91	60.47
1889	215.3	418.8		116.6	312.8	4.85	3.14	4.49	194.2	4.04	61.78
1890	218.4	450.0	96.0	118.2	316.2	5.62	3.27	4.73	167.9	4.37	63.06
1891	227.9	456.3	94.6	118.2	319.5	5.46	3.40	4.63	197.4	4.01	64.36
1892	250.0	493.8	97.0	116.6	322.8	4.10	3.53	4.56	201.1	4.09	65.67
1893	237.9	437.5	88.3	116.6	326.2	6.78	3.49	4.55	160.9	4.89	66.97
1894	231.1	425.0	81.6	113.5	319.5	3.04	3.47	4.30	156.9	4.58	68.28
1895	258.9	506.3	86.3	111.9	319.5	2.83	3.33	4.27	157.7	4.06	69.58
1896	253.7	462.5	85.6	111.9	326.2	5.82	3.34	4.28	154.0	4.01	70.89
1897	277.9	500.0	85.5	111.9	322.8	3.50	3.10	4.02	173.4	3.55	72.13
1898	283.7	568.8	87.6	113.5	322.8	3.83	3.12	3.91	206.2	3.48	73.49
1899	309.5	625.0	93.5	118.2	332.8	4.15	2.99	3.96	219.7	3.47	74.80
1900	317.9	631.3	95.0	119.8	342.8	4.38	2.90	3.84	250.7	3.82	76.09
1901	353.7	700.0	96.0	122.8	349.5	4.28	2.86	3.82	290.1	3.72	77.58
1902	357.4	806.3	96.3	126.0	359.4	4.09	2.95	3.93	293.8	3.84	79.16
1903	374.7	806.3	96.1	126.0	376.1	5.44	2.98	4.05	239.8	4.57	80.63
1904	370.0	775.0	94.6	127.7	376.1	4.21	3.05	3.92	301.1	3.57	82.17
1905	397.4	887.5	95.7	130.7	379.4	4.40	3.07	3.92	348.2	3.31	83.82
1906	443.2	943.8	98.3	137.1	392.7	5.68	3.06	4.05	359.1	3.82	85.45
1907	450.0	956.3	97.2	141.8	409.4	6.34	3.38	4.47	239.8	6.13	87.01
1908	413.2	768.8	92.0	141.8	396.0	4.37	3.28	4.08	329.6	4.27	88.71
1909	463.7	987.5	94.9	149.7	399.4	3.98	3.45	4.12	375.9	4.17	90.49
1910	468.4	1025.0	94.1	149.7	412.7	5.02	3.46	4.18	330.3	5.03	92.41
1911	483.7	968.8	93.3	152.9	419.3	4.02	3.52	4.20	332.5	4.95	93.86
1912	506.3	1168.8	95.4	157.6	436.0	4.75	3.51	4.29	342.3	4.85	95.34
1913	526.3	1225.0	95.7	160.8	452.6	5.58	3.58	4.53	293.4	5.38	97.23
1914	485.8	1162.5	92.1	162.4	456.0	4.89	3.63	4.66	268.2	5.50	99.11
1915	499.5	1365.5	91.5	165.6	456.0	3.50	3.61	4.49	346.0	4.32	100.55
1916	568.4	1623.8	94.9	186.5	509.2	3.43	3.61	4.44	357.7	5.73	101.96
1917	554.2	1605.4	95.4	220.3	579.1	4.73	4.11	5.25	248.2	9.30	103.27
1918	604.2	1586.9	98.6	265.3	712.2	5.86	4.27	4.98	288.3	7.16	103.21
1919	609.5	1383.9	98.6	303.9	842.0	5.40	4.90	5.73	325.5	5.26	104.51
1920	603.7	1513.1	94.8	311.9	985.1	7.38	5.40	6.26	248.5	7.27	106.46
1921	590.0	1217.9	88.3	278.1	911.9	6.53	4.47	5.50	266.8	5.77	108.54
1922	622.6	1550.0	93.3	271.7	858.7	4.41	4.32	5.08	320.4	5.63	110.05
1923	704.7	1753.0	97.6	278.1	931.9	4.98	4.35	5.09	312.0	5.93	111.95
1924	726.3	1679.2	95.0	278.1	965.2	3.90	3.96	4.95	370.8	5.08	114.11
1925	743.2	1863.7	96.8	287.8	965.2	3.99	3.80	4.85	454.7	4.38	115.83
1926	791.6	1974.4	98.2	284.6	965.2	4.23	3.56	4.68	492.3	4.96	117.40
1927	799.5	1992.9	96.7	278.1	965.2	4.02	3.17	4.46	637.2	4.30	119.04
1928	808.4	2048.2	95.8	274.9	1001.8	4.81	3.45	4.61	888.7	3.48	120.51
1929	857.9	2288.1	96.8	276.5	1001.8	5.78	3.36	4.67	782.8	4.56	121.77
1930	781.6	1956.0	91.3	258.8	985.1	3.56	3.22	4.47	559.9	5.47	123.19
1931	721.6	1642.3	84.1	234.7	911.9	2.63	3.93	5.34	296.4	9.08	124.15
1932	626.3	1236.3	76.4	210.6	788.8	2.72	3.35	4.53	251.5	5.66	124.95
1933	613.2	1439.3	75.1	212.2	788.8	1.66	3.53	4.45	368.6	3.47	125.69
1934	660.5	1568.5	78.3	215.4	948.5	0.89	2.99	3.80	346.7	4.13	126.49

1935	711.1	1882.1	79.9	221.9	965.2	0.29	2.82	3.43	490.1	3.34	127.36
1936	812.1	2214.3	83.1	225.1	985.1	0.15	2.52	3.09	627.0	4.41	128.18
1937	846.8	2361.9	85.7	231.5	1108.3	0.44	2.68	3.18	385.0	8.34	128.96
1938	813.2	1845.2	81.0	225.1	1108.3	0.07	2.48	3.05	482.1	3.41	129.97
1939	877.9	2343.5	82.8	225.1	1128.2	0.00	2.30	2.91	455.8	4.48	131.03
1940	945.8	2712.5	85.4	226.7	1181.5	0.00	1.88	2.72	386.1	6.25	132.12
1941	1117.9	3598.2	90.1	249.2	1304.6	0.07	2.50	2.84	317.2	7.92	133.40
1942	1341.6	4502.4	95.3	271.7	1521.0	0.36	2.49	2.81	356.6	5.69	134.86
1943	1608.4	5425.0	98.1	279.7	1717.3	0.36	2.49	2.74	425.9	4.97	136.74
1944	1743.2	5295.8	98.8	286.2	1807.2	0.36	2.47	2.70	484.7	4.67	138.40
1945	1673.2	4481.1	98.1	292.6	1823.8	0.36	2.32	2.60	633.6	3.69	139.93
1946	1327.9	3666.3	96.1	345.7	1933.6	0.43	2.22	2.60	558.4	4.44	141.39
1947	1307.9	4015.5	96.1	376.2	2183.3	0.57	2.41	2.86	558.4	5.48	144.13
1948	1357.4	4190.1	96.2	387.5	2379.6	0.99	2.43	2.76	554.7	6.25	146.63
1949	1362.6	3957.3	94.1	379.4	2469.5	1.12	2.18	2.57	611.7	6.68	149.19
1950	1481.1	4597.5	94.7	401.9	2576.0	1.25	2.39	2.66	744.9	6.86	151.68
1951	1626.8	5004.9	96.7	426.0	2792.3	1.51	2.74	3.03	867.5	5.91	154.29
1952	1696.3	5179.4	97.0	429.3	2952.1	1.76	2.79	2.98	969.7	5.39	156.95
1953	1759.5	5586.8	97.1	432.5	3111.8	1.93	2.74	3.12	905.5	5.84	159.57
1954	1747.4	5295.8	94.5	429.3	3185.0	0.98	2.60	2.92	1313.1	4.31	162.39
1955	1845.8	5936.0	95.6	430.9	3328.1	1.68	2.92	3.16	1659.9	4.13	165.28
1956	1882.6	6227.0	95.9	443.7	3487.9	2.66	3.44	3.79	1703.3	4.21	168.22
1957	1918.9	6227.0	95.7	456.6	3667.6	3.34	3.26	3.73	1459.5	4.55	171.27
1958	1909.5	5819.6	93.2	464.6	3774.1	1.79	3.84	4.09	2015.0	3.23	174.14
1959	2013.7	6517.9	94.5	472.7	3917.2	3.35	4.32	4.61	2185.8	3.16	177.07
1960	2054.7	6634.3	94.5	479.1	4043.7	3.13	3.82	4.35	2120.8	3.37	180.67
1961	2106.3	6750.7	93.3	482.3	4150.2	2.32	4.07	4.44	2611.3	2.86	183.69
1962	2235.8	7274.5	94.5	488.7	4276.7	2.80	3.85	4.23	2302.9	3.30	186.54
1963	2333.7	7740.1	94.3	496.8	4403.1	3.20	4.16	4.37	2738.0	3.11	189.24
1964	2473.7	8263.8	94.8	501.6	4526.3	3.56	4.14	4.43	3093.1	3.04	191.89
1965	2628.9	9078.6	95.5	511.3	4669.4	4.02	4.44	4.73	3373.4	3.06	194.30
1966	2799.5	10009.7	96.2	528.9	4865.7	4.90	4.54	5.40	2931.8	3.64	196.56
1967	2872.1	10184.3	96.2	545.0	5065.4	4.49	5.35	6.24	3520.8	3.08	198.71
1968	3001.6	10824.5	96.4	570.7	5384.9	5.42	5.82	6.53	3790.5	2.99	200.71
1969	3094.7	11348.2	96.5	606.1	5707.8	6.81	6.92	7.84	3359.9	3.48	202.68
1970	3094.7	10824.5	95.1	639.9	5994.0	6.68	6.16	7.48	3363.1	3.36	205.05
1971	3198.4	10999.0	94.1	660.8	6386.7	4.51	5.60	7.22	3725.9	2.99	207.66
1972	3379.5	12163.0	94.4	683.3	6836.0	4.04	5.70	7.11	4308.4	2.71	209.90
1973	3575.8	13268.7	95.1	742.8	7318.6	6.98	6.47	7.73	3560.2	3.64	211.91
1974	3555.3	13152.3	94.4	834.4	7907.7	8.09	6.77	8.90	2502.2	5.37	213.85
1975	3524.2	11813.8	91.7	892.3	8643.2	6.04	7.09	8.66	3291.6	4.08	215.97
1976	3715.8	13210.5	92.4	935.7	9338.7	5.16	7.23	7.93	3922.6	3.87	218.04
1977	3907.4	14025.2	93.1	998.4	10164.1	5.26	8.04	8.30	3470.8	5.08	220.24
1978	4108.9	14956.4	94.0	1088.4	11039.4	7.23	8.95	9.28	3507.7	5.44	222.59
1979	4227.9	15712.9	94.2	1233.1	11987.9	10.32	10.18	10.91	3939.4	5.55	225.06
1980	4227.9	14025.2	93.0	1387.5	13009.7	12.04	11.87	13.49	4954.7	4.67	227.76
1981	4342.1	15596.5	92.7	1511.3	14297.7	15.21	13.78	14.35	4472.6	5.65	230.14
1982	4261.6	14490.8	90.5	1569.1	15192.9	11.28	10.45	11.83	5132.8	4.87	232.52
1983	4424.7	15305.5	90.5	1628.6	15798.7	8.89	11.84	12.58	6019.3	4.29	234.80
1984	4719.5	16993.2	92.5	1692.9	16444.3	10.04	11.45	12.10	6103.6	4.64	237.00
1985	4874.7	17400.6	92.8	1757.2	17070.0	7.70	9.31	9.95	7710.9	3.82	239.28
1986	5012.1	17575.2	93.0	1776.5	17409.5	6.18	7.45	8.47	8838.3	3.47	241.63
1987	5175.3	18855.5	93.8	1855.3	17732.3	5.87	8.95	10.07	9017.5	3.66	243.94
1988	5371.6	19262.9	94.5	1937.3	18214.9	6.73	9.00	9.60	10135.8	3.70	246.31
1989	5506.3	19745.7	94.7	2027.3	18725.6	8.48	7.98	8.88	12897.8	3.31	247.34

1990	5552.6	19952.7	94.5	2151.1	19406.5	7.85	8.31	9.03	12051.8	3.74	249.91
1991	5485.8	19556.0	93.3	2217.0	19917.2	5.71	7.52	8.21	15222.3	2.91	252.64
1992	5598.9	20271.9	92.6	2281.4	20570.9	3.57	7.39	7.90	15901.8	2.87	255.41
1993	5772.6	20990.2	93.2	2344.1	20959.0	3.08	6.28	7.00	17023.7	2.71	258.12
1994	5997.9	22227.3	93.9	2406.8	21541.2	4.15	7.83	8.45	16761.7	2.89	260.68
1995	6153.3	22945.6	94.4	2467.8	22123.4	5.64	6.00	6.74	22507.7	2.31	263.17
1996	6369.6	23955.2	94.6	2549.8	22787.1	5.12	6.64	7.23	27024.5	2.01	265.56
1997	6624.9	25492.5	95.1	2593.2	23450.8	5.22	5.95	6.68	35417.2	1.61	266.79
1998	6914.0	26548.0	95.5	2635.0	24114.5	5.06	5.09	6.24	44862.4	1.34	269.09
1999	7206.5	27488.8	95.8	2705.8	24778.2	4.85	6.48	7.42	53622.3	1.14	272.88

### 23. National Product Index (NP)

Real National Product index for the United States starting at 100 in 1870.

- 1870-1994: Based on Gross Domestic Product (GDP) volume index from Maddison (1995). Corrected for boundary adjustments.
- 1995-1997: Based on GDP volume index from OECD (1999).
- 1998-1999: Based on annual volume growth of GDP at market prices from various monthly editions of the OECD Main Economic Indicators.

### 24. Industrial Production Index (IP)

Real Industrial Production index for the United States starting at 100 in 1855.

- 1855-1988: Based on manufacturing index of industrial production from Mitchell (1993).
- 1989-1999: Based on annual volume growth of Industrial Production from various monthly editions of the OECD Main Economic Indicators.

### 25. Employment (EM)

100% minus employment rate for the United States.

- 1890-1988: Based on unemployment rate from Mitchell (1993).
- 1989-1999: Based on unemployment rate from various monthly editions of the OECD Main Economic Indicators.

### 26. Consumer Price Index (PI)

Consumer Price Index for the United States starting at 100 in 1820. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- U.S. Government, Statistical Abstract of the United States.
- Snyder (1924).
- Bureau of Labor Statistics.

Notes: This index is based on a combination of three indices. From 1820 through 1874, the annual cost-of-living index calculated by the Federal Reserve Bank is used. From 1875 until 1912, it uses a monthly Index of General Prices calculated by the Federal Reserve Bank of New York, which was weighted between wholesale commodity prices (20%) Wage payments (35%), the Cost of Living (35%) and Rents (10%). From 1913 on, the Bureau of Labor's Consumer Price Index is used.

27. *Wage Index (WI)*

Nominal wage index for the United States starting at 100 in 1786.

- 1786-1988: Based on money wages in industry from Mitchell (1993).
- 1989-1999: Based on hourly earnings in manufacturing from various monthly editions of the OECD Main Economic Indicators.

28. *Short Interest Rate (SR)*

Nominal short term interest rate for the United States in percentage points. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- 1831-1900: Annual average commercial paper US short term interest rate from Homer (1963) Table 44 on page 318.
- 1901-1934: 60-90 days annual average prime commercial paper from Homer (1963) Table 51 on page 364.
- 1935-1999: Yield on 3-month Treasury Bills from Federal Reserve Bulletin of the Board of Governors of the Federal Reserve System.

29. *Long Interest Rate (LR)*

Nominal long term interest rate for the United States in percentage points.

Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- Homer (1963).
- 1800-1862: Martin (1886).
- 1843-1853: Hunt's Merchants Magazine.
- 1854-1861: The Economist.
- 1862-1921: William B. Dana Co., *The Financial Review*, New York: William B. Dana Co. which reprinted data published by The Commercial and Financial Chronicle.
- Federal Reserve Bank (1941, 1970).
- Salomon Brothers (1995).

Notes: From February 1862 until December 1877, the 6% U.S. Government bonds of 1881 are used. From January 1878 until January 1895, the 4% U.S. Government Bonds of 1907 are used, and from February 1895 until September 1917, the 4% U.S. Government Bonds of 1925 are used. Where no trades were recorded during a given month, the previous month's yield was used. The 4% Liberty Bonds are used from October 1917 through December 1918, and beginning in 1919, the Federal Reserve Board's 10-15 year Treasury Bond index is used.

30. *Corporate bond yield (CY)*

Nominal yield on long term high rated corporate bonds for the United States in percentage points. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- 1857-1899: High-grade railroad bonds as found in F. R. Macaulay (1938), *The Movements of Interest Rates, Bond Yield, and Stock Prices in the United States since 1856*, New York: National Bureau of Economic Research.
- 1900-1999: Moody's index of yields on AAA Corporate bonds is used.

### 31. *Equity Price Index (EP)*

Nominal index of equity prices in the United States starting at 100 in 1800. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- 1800-1801: The actual prices of United States 3% stock, drawn from Smith and Cole (1935).
- 1802-1870: Schwert (1990) methodology to provide an index of United States stocks dating back to 1802. This index combines the price indexes of bank stocks (1802-1815), bank and insurance stocks (February 1815-December 1845), and Rails (1834-1862) from Smith and Cole (1935) and Railroads (1863-1870) from Macaulay (1938). Where these indices overlap, the indices have been weighted according to the number of stocks included in the indices.
- 1871-1999: The Cowles/Standard and Poor's Composite index of stocks. The Standard and Poor's indices were first calculated in 1918, and the Cowles Commission back-calculated the data to 1871 using the Commercial and Financial Chronicle. For more information, see Standard and Poors (1996).

### 32. *Dividend Yield (DY)*

Nominal dividend yield on United States equities in percentage points. Sources used by [www.GlobalFindata.com](http://www.GlobalFindata.com):

- Standard Statistics Corp. (1931).
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